Wavelets and Linear Algebra 5(1) (2018) 1- 9

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Wavelets and Linear Algebra

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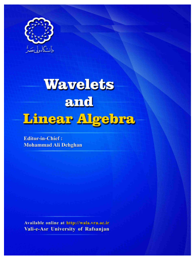

On a New G-Frame and Duality

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Article Info

Article history:
Received 31 October 2016
Accepted 13 May 2017
Available online 24 June 2017
Communicated by Asghar Rahimi

Keywords:
Singleton g-frame operator, Dual of a singleton g-frame, Singleton g-Riesz basis, Singleton g-orthonormal basis.

2000 MSC:
42C15
41A58.

Abstract

We introduce a new g-frame (singleton g-frame), g-orthonormal basis and g-Riesz basis and study corresponding notions in some other generalizations of frames. Also, we investigate duality for some kinds of g-frames. Finally, we illustrate an example which provides a suitable translation from discrete frames to Sun's g-frames.

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http://doi.org/10.22072/wala.2017.51870.1089

1. Introduction and Preliminaries

Let H be a Hilbert space. The collection $(f_i)_{i\in I} \subset H$ is called a frame for H if there exist $A, B > 0$ such that $A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2$ for all $f \in H$. The constants A, B are called lower and upper bounds, respectively. We refer to [4, 8, 9, 10, 11, 12] for an introduction to the frame theory and its applications.

Whereas several generalizations of frames exist in the literature [1, 2, 6, 7, 14], in [3] we defined a new g-frame, a singleton g-frame, with the aid of which we have shown that other extensions of frames are equivalent to continuous frames defined by Ali et al. in [1]. In this note we introduce singleton g-orthonormal and g-Riesz bases and compare them with the corresponding notions in [14] and [1]. Also, we continue our research on frame operator of a singleton g-frame and obtain a relation between Sun [14] and Ali et al. [1] frame operators. Then we study duality in singleton g-frames and establish relations between dual of singleton g-frames and dual of continuous frames. Finally, we illustrate an example which provides a suitable translation from discrete frames to Sun's g-frames. Our results generalize some of the results appearing in the literature on frames. Such a unified approach seems to be useful, since it describes the basic features and includes most of the special cases. Also, it helps us compare several generalizations of frames with each other.

Here we recall some definitions and preliminaries that are required in the sequel.

Let $(K_j)_{j\in J}$ (J is at most countable) and H be Hilbert spaces and $\Lambda_j : H \to K_j$, $j \in J$ be bounded linear operators. The set $\{\Lambda_j : j \in J\}$ is called a Sun g-frame if there exist $A, B > 0$ such that $A\|x\|^2 \leq \sum_{j\in J} \|\Lambda_j x\|^2 \leq B\|x\|^2$ for all $x \in H$ [14].

Let (Ω, μ) be a measure space and H be a Hilbert space. The mapping $F : (\Omega, \mu) \to H$ is called a continuous frame with bounds A, B, if $\omega \to \langle f, F(\omega)\rangle$ is a measurable function on Ω for every $f \in H$ and

$$A\|f\|^2 \leq \int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega) \leq B\|f\|^2,\ f \in H.$$

The operator $T_F : H \to L^2(\Omega, \mu)$ given by $T_F(f)(\omega) = \langle f, F(\omega)\rangle$, $f \in H$, $\omega \in \Omega$ is a bounded (above) linear operator. This operator is called the frame transform. It is 1-1 and bounded below if and only if F is a continuous frame.

Let $F : (\Omega, \mu) \to H$ be a continuous frame. A continuous frame $G : (\Omega, \mu) \to H$ is called a dual of F if $f = \int_\Omega \langle f, G(\omega)\rangle F(\omega) d\mu(\omega)$ for all $f \in H$. Consider the frame operator S of F defined by $Sf = \int_\Omega \langle f, F(\omega)\rangle F(\omega) d\mu(\omega) = T_F^* T_F f$ for $f \in H$. Then the continuous frame $S^{-1}F$ is a dual of F called the standard dual of F.

Now we recall our new generalization of frames [3].

Definition 1.1. Let H and K be two Hilbert spaces. A linear operator $\Lambda : H \longrightarrow K$ is called a singleton g-frame for H, with respect to K if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \|\Lambda f\|^2 \leq B\|f\|^2,$$

for all $f \in H$. The constants A, B are called lower and upper bounds of Λ, respectively.

Remark 1.2. Let $\Lambda : H \to K$ be a singleton g-frame with bounds A, B. Put $J = \{1\}, \Lambda_1 = \Lambda$ and $K_1 = K$ in the definition of Sun g-frame. It is easy to verify that $\{\Lambda_i\}_j$ is a Sun g-frame. We call it the Sun g-frame corresponding to Λ. Let $\{\Lambda_j : H \longrightarrow K_j; j \in J\}$ (J is at most countable) be a Sun g-frame. We put $K = \oplus_{j\in J} K_j$ and define $\Lambda : H \longrightarrow K$ by $f \longrightarrow (\Lambda_j f)_{j\in J}$. We call Λ the singleton g-frame corresponding to the Sun g-frame $\{\Lambda_j : j \in J\}$. Also, let $(e_i)_{i\in I}$ be an orthonormal basis for K and μ be the counting measure on I. We define $F : (I, \mu) \longrightarrow H$ by $i \longrightarrow \Lambda^* e_i$ and call it the continuous frame corresponding to Λ. Now, let $F : (\Omega, \mu) \longrightarrow H$ be a continuous frame. We call $T_F : H \longrightarrow L^2(\Omega, \mu)$ the singleton g-frame corresponding to F (see [3]).

Here we recall the definitions of Sun g-Riesz basis, Sun g-orthonormal basis, continuous Riesz basis and continuous orthonormal basis (see [14, 1]).

Let $\Lambda_j : H \to K_j,\ j \in J$ (J is at most countable) be bounded linear operators.

- If $\{f : \Lambda_j f = 0,\ j \in J\} = \{0\}$, then $\{\Lambda_j : j \in J\}$ is called Sun g-complete.
- If $\{\Lambda_j : j \in J\}$ is Sun g-complete and there are A and B such that for any finite subset $J_1 \subset J$ and $g_j \in K_j,\ j \in J_1$,
$$A \sum_{j\in J_1} \|g_j\|^2 \le \| \sum_{j\in J_1} \Lambda_j^* g_j\|^2 \le B \sum_{j\in J_1} \|g_j\|^2, \tag{1.1}$$
then $\{\Lambda_j : j \in J\}$ is said to be a Sun g-Riesz basis for H with respect to $\{K_j : j \in J\}$.
- The set $\{\Lambda_j : j \in J\}$ is called a Sun g-orthonormal basis for H with respect to $\{K_j : j \in J\}$ if it satisfies the following:
$$\langle \Lambda_j^* g_j, \Lambda_i^* g_i\rangle = \delta_{j,i}\langle g_j, g_i\rangle,\ j, i \in J,\ g_j \in K_j,\ g_i \in K_i, \tag{1.2}$$
$$\sum_{j\in J} \|\Lambda_j f\|^2 = \|f\|^2,\ f \in H. \tag{1.3}$$

Let $F : (\Omega, \mu) \to H$ be a continuous frame. Then F is called a continuous orthonormal basis if

$$\int_\Omega |\langle x, F(\omega)\rangle|^2 d\mu(\omega) = \|x\|^2,$$

for all $x \in H$ and

$$\Big\| \int_\Omega \varphi(\omega) F(\omega) d\mu(\omega)\Big\|^2 = \Big\|\varphi\Big\|^2,$$

for all $\varphi \in L^2(\Omega, \mu)$. Also, F is called a continuous Riesz basis if there exist $A, B > 0$ such that

$$A \int_E |\varphi(\omega)|^2 d\mu(\omega) \le \Big\| \int_E \varphi(\omega) F(\omega) d\mu(\omega)\Big\|^2 \le B \int_E |\varphi(\omega)|^2 d\mu(\omega), \tag{1.4}$$

for all measurable sets $E \subset \Omega$ with $\mu(E) < \infty$ and $\varphi \in L^2(\Omega, \mu)$.

If G is a continuous Riesz basis then T_G is an invertible operator. As a result, if G is a continuous orthonormal basis then T_G is a unitary operator. In fact we have the following proposition.

Proposition 1.3. *Let G be a continuous Riesz basis with bounds A, B. Then T_G is onto. If G is a continuous orthonormal basis then T_G is unitary.*

Proof. Put $X = \{\varphi \in L^2(\Omega, \mu), \ \mu(\varphi \neq 0) < \infty\}$. It is easy to check that $T_G^*\varphi = \int_\Omega \varphi(\omega)G(\omega)d\mu(\omega)$. So $A\|\varphi\|^2 \leq \|T_G^*\varphi\|^2 \leq B\|\varphi\|^2$ for all $\varphi \in X$. Since T_G^* is continuous, X is dense in $L^2(\Omega, \mu)$ ([13, Ch.3]) and T_G^* is bounded below on X, we have $KerT_G^* = \{0\}$. Since $imT_G = (KerT_G^*)^\perp$, T_G is onto. The other implication is obvious.

□

The rest of this paper is organized as follows. In Section 2 we introduce singleton g-orthonormal and g-Riesz bases and compare them with the corresponding notions in [14] and [1]. In Section 3 we study the frame operator of a singleton g-frame which helps us compare Sun and Ali et al. [14, 1] frame operators. Then, we introduce dual of a singleton g-frame and investigate corresponding notions in continuous frames. Finally, we bring an example through which, we illustrate a suitable translation of [5, Theorem 3.1.13] to the Sun g-frame setting.

2. G-Riesz Bases and G-Orthonormal Bases

In this section we define a singleton g-Riesz basis (singleton g-orthonormal basis) and compare it with a Sun g-Riesz basis (Sun g-orthonormal basis) and a continuous Riesz basis (continuous orthonormal basis).

Definition 2.1. A singleton g-frame $\Lambda : H \to K$ is called a singleton g-Riesz basis if it is invertible. Also Λ is called a singleton g-orthonormal basis if it is unitary.

The following theorem establishes a relation between singleton g-Riesz bases (singleton g-orthonormal bases) and continuous Riesz bases (continuous orthonormal bases).

Theorem 2.2. *Let Λ be a singleton g-frame and F be the continuous frame corresponding to Λ as in Remark 1.2. If Λ is a singleton g-Riesz basis (singleton g-orthonormal basis), then F is a continuous Riesz basis (continuous orthonormal basis). Conversely, let F be a continuous frame and Λ be the singleton g-frame corresponding to F as in Remark 1.2. If F is a continuous Riesz basis (continuous orthonormal basis), then Λ is a singleton g-Riesz basis (singleton g-orthonormal basis).*

Proof. Assume that $\Lambda : H \to K$ is a singleton g-Riesz basis with the corresponding continuous frame F as in Remark 1.2. Since Λ is invertible, Λ^* is also invertible. So there exist α, β such that

$$\alpha\|f\|^2 \leq \|\Lambda^* f\|^2 \leq \beta\|f\|^2, \ \ f \in K. \tag{2.1}$$

Let E be a subset of I with finite measure (so finite) and $\varphi \in L^2(I, \mu)$. We show that

$$\alpha \int_E |\varphi(i)|^2 d\mu(i) \leq \left\| \int_E \varphi(i)F(i)d\mu(i)\right\|^2 \leq \beta \int_E |\varphi(i)|^2 d\mu(i),$$

which is equivalent to

$$\alpha\Big\|\sum_{i\in E}\varphi(i)e_i\Big\|^2 \le \Big\|\Lambda^*\sum_{i\in E}\varphi(i)e_i\Big\|^2 \le \beta\Big\|\sum_{i\in E}\varphi(i)e_i\Big\|^2. \tag{2.2}$$

Substituting $y = \sum_{i\in E}\varphi(i)e_i$ into (2.1) gives (2.2). Thus, F is a continuous Riesz basis.

Now, let $\Lambda : H \to K$ be a singleton g-orthonormal basis with the corresponding continuous frame F as in Remark 1.2. For $f \in H$,

$$\begin{aligned}\int_I |\langle f, F(i)\rangle|^2 d\mu(i) &= \sum_{i\in I}|\langle f, F(i)\rangle|^2 = \sum_{i\in I}|\langle f, \Lambda^* e_i\rangle|^2 \\ &= \sum_{i\in I}|\langle \Lambda f, e_i\rangle|^2 = \|\Lambda f\|^2 = \|f\|^2.\end{aligned}$$

Moreover,

$$\begin{aligned}\Big\|\int_I \varphi(i)F(i)d\mu(i)\Big\|^2 &= \Big\|\sum_{i\in I}\varphi(i)\Lambda^*(e_i)\Big\|^2 \\ &= \Big\|\Lambda^*\sum_{i\in I}\varphi(i)e_i\Big\|^2 \\ &= \Big\|\sum_{i\in I}\varphi(i)e_i\Big\|^2 = \Big\|\varphi\Big\|^2.\end{aligned}$$

Therefore F is a continuous orthonormal basis.

Conversely, let F be a continuous Riesz basis with the corresponding singleton g-frame T_F as in Remark 1.2. By Proposition 1.3 T_F is onto. Also T_F is trivially 1-1. Thus by the open mapping theorem it is invertible i.e. singleton a g-Riesz basis.

If F is a continuous orthonormal basis, then T_F is unitary by Proposition 1.3, i.e. it is a singleton g-orthonormal basis.

□

In the following theorem we obtain relations between Sun g-Riesz bases (Sun g-orthonormal bases) and singleton g-Riesz bases (singleton g-orthonormal bases).

Theorem 2.3. *Let $\{\Lambda_j : H \to K_j,\ j \in J\}$ (J is at most countable) be a Sun g-frame and Λ be the singleton g-frame corresponding to it. If $\{\Lambda_j : H \to K_j,\ j \in J\}$ is a Sun g-Riesz basis (Sun g-orthonormal basis), then Λ is a singleton g-Riesz basis (singleton g-orthonormal basis). Conversely, if Λ is a singleton g-Riesz singleton (singleton g-orthonormal basis) basis, then the Sun g-frame corresponding to it is a Sun g-Riesz basis (Sun g-orthonormal basis) .*

Proof. Let $\{\Lambda_j : H \to K_j,\ j \in J\}$ be a Sun g-Riesz basis with the corresponding singleton g-frame $\Lambda : H \to \bigoplus_{j\in J} K_j$ as in Remark 1.2. Let $X = \{(f_j)_{j\in J} \in \bigoplus_{j\in J} K_j,\ f_j \neq 0, \text{for finitely many j}\}$. It is trivial that X is dense in $\bigoplus_{j\in J} K_j$, and (1.1) can be rewritten as follows

$$A\|f\|^2 \le \|\Lambda^* f\|^2 \le B\|f\|^2, f \in X.$$

Since Λ^* is continuous and bounded below and X is dense in $\bigoplus_{j\in J} K_j$, Λ^* is 1-1 on $\bigoplus_{j\in J} K_j$. So Λ is onto. Also, Λ is trivially 1-1, so is invertible, i.e. Λ is a singleton g-Riesz basis.

Now, Let $\{\Lambda_j : H \to K_j,\ j \in J\}$ be a Sun g-orthonormal basis with the corresponding singleton g-frame $\Lambda : H \to \bigoplus_{j\in J} K_j$ as in Remark 1.2. Then

$$\|\Lambda f\|^2 = \|(\Lambda_j f)_{j\in J}\|^2 = \sum_{j\in J} \|\Lambda_j f\|^2 = \|f\|^2,$$

i.e. Λ is an isometry. It is enough to show that Λ is onto. Since $Im\Lambda$ is closed, and $Im\Lambda = (Ker\Lambda^*)^\perp$, then it enough to show that $Ker\Lambda^* = \{0\}$. Let $(f_j)_{j\in J} \in \bigoplus_{j\in J} K_j$ and $\Lambda^*(f_j)_{j\in J} = 0$. Since $\Lambda^*(f_j)_{j\in J} = \sum_{j\in J} \Lambda_j^* x_j$, we have

$$\begin{aligned} 0 = \Big\| \sum_{j\in J} \Lambda_j^* f_j \Big\|^2 &= \Big\langle \sum_{i\in J} \Lambda_i^* f_i, \sum_{j\in J} \Lambda_j^* f_j \Big\rangle \\ &= \sum_{i\in J} \sum_{j\in J} \langle \Lambda_i^* f_i, \Lambda_j^* x_j \rangle \\ &= \sum_{j\in J} \langle \Lambda_j^* f_j, \Lambda_j^* f_j \rangle \\ &= \sum_{j\in J} \big\| f_j \big\|^2. \end{aligned}$$

Hence, $f_j = 0,\ j \in J$.

For the converse, let Λ be a singleton g-Riesz basis i.e. invertible. So Λ^* is also invertible, and (1.1) is obviously satisfied. Then Λ is a Sun g-Riesz basis.

Finally, let $\Lambda : H \to K$ be a singleton g-orthonormal basis, then obviously (1.2) and (1.3) hold. Hence Λ as a Sun g-frame is a Sun g-orthonormal basis.

□

3. Frame Operators and Duality

In this section we introduce duality of singleton g-frames and compare it with the duality in continuous frames. Then we define the frame operator of a singleton g-frame and determine its relation with the corresponding notions in Sun [14] and Ali et al. [1].

Definition 3.1. Let $\Lambda : H \to K$ be a singleton g-frame with bounds A, B. We define $S = \Lambda^*\Lambda$ as the frame operator of Λ.

Note that S is positive. Since $A\|f\|^2 \le \|\Lambda f\|^2 = \langle \Lambda^*\Lambda f, f\rangle = \langle S f, f\rangle$, S is 1-1. Also, $A\|f\|^2 \le \|\Lambda f\|^2 = \langle S f, f\rangle \le \|S f\|\ \|f\|$, so $A\|f\| \le \|S f\|$. Thus $S(H)$ is closed. If g is orthogonal to $S(H)$, we have $A\|g\|^2 \le \langle S g, g\rangle = 0$. So $g = 0$ which implies that S is onto. Therefore, S is invertible.

In the following proposition we establish a relation between the frame operators of a singleton g-frame and a continuous frame.

Proposition 3.2. *If F is a continuous frame and Λ is the singleton g-frame corresponding to F as in Remark 1.2, then the frame operators of F and Λ are the same. Similarly, if Λ is a singleton g-frame and F is the continuous frame corresponding to Λ, then the frame operator of Λ is equal to the frame operator of F.*

Proof. Let $F : (\Omega, \mu) \to H$ be a continuous frame with the corresponding singleton g-frame $\Lambda = T_F$ as in Remark 1.2. Then $S_\Lambda f = T_F^* T_F f = S_F f$ in which S_Λ and S_F are the frame operators of Λ and F, respectively. The proof of the other part is easy.

□

The Sun g-frame operator of a Sun g-frame $\{\Lambda_j : H \to K_j,\ j \in J\}$ is defined in [14] as

$$S f = \sum_{j \in J} \Lambda_j^* \Lambda_j f,\ f \in H. \tag{3.1}$$

In the following proposition we give a relation between the frame operator of a singleton g-frame and a Sun g-frame.

Proposition 3.3. *Let Λ be the singleton g-frame corresponding to the Sun g-frame $\{\Lambda_j : H \to K_j,\ j \in J\}$. Then the frame operator of Λ and $\{\Lambda_j : H \to K_j,\ j \in J\}$ are the same.*

Proof. Let $\{\Lambda_j : H \to K_j,\ j \in J\}$ be a Sun g-frame and Λ be the singleton g-frame corresponding to it as in Remark 1.2. Let S be the Sun g-frame operator of $\{\Lambda_j : H \to K_j,\ j \in J\}$ and S_Λ be the frame operator of Λ. Since, $\Lambda^*(f_j)_{j\in J} = \sum_{j\in J} \Lambda_j^* f_j$, then

$$S_\Lambda f = \Lambda^* \Lambda f = \Lambda^* (\Lambda_j f)_{j\in J} = \sum_{j\in J} \Lambda_j^* \Lambda_j f = S f.$$

Therefore, $S_\Lambda = S$.

□

Remark 3.4. If Λ is a singleton g-frame, the Sun g-frame corresponding to Λ is Λ itself. So the frame operator of Λ as a singleton g-frame and frame operator of Λ as a Sun g-frame are equal.

Definition 3.5. Let $\Phi, \Lambda : H \to K$ be two singleton g-frames. We call Φ a dual of Λ if $\Lambda^* \Phi = id_H$ (In this case obviously $\Phi^* \Lambda = id_H$ and then Λ is also a dual of Φ). We denote the standard dual ΛS^{-1} of Λ by $\tilde{\Lambda}$ where S is defined as in Definition 3.1.

The following proposition states that if Λ is a dual of P, then the corresponding continuous frames satisfy the same duality relation.

Proposition 3.6. *Let Λ be a singleton g-frame and $F(\Lambda)$ be the continuous frame corresponding to it as in Remark 1.2. If P is a dual of Λ, then $F(P)$ is a dual of $F(\Lambda)$.*

Proof. Let $F(\Lambda)$ and $F(P)$ be as in Remark 1.2. We have

$$\begin{aligned}\int_I \langle f, F(P)(i)\rangle F(\Lambda)(i)d\mu(i) &= \sum_{i\in I}\langle f, P^* e_i\rangle \Lambda^* e_i\\ &= \Lambda^* \sum_{i\in I}\langle Pf, e_i\rangle\\ &= \Lambda^* Pf = f.\end{aligned}$$

So $F(P)$ is a dual of $F(\Lambda)$. □

The following proposition establishes a relation between standard dual of singleton g-frames and standard dual of continuous frames.

Proposition 3.7. *Let Λ be a singleton g-frame and $\tilde{\Lambda}$ be its standard dual. We keep the notation in Proposition 3.6. Then $F(\tilde{\Lambda})$ is the standard dual of $F(\Lambda)$. Conversely, if $F, G : (\Omega, \mu) \to H$ are two continuous frames and G is the standard dual of F, then the singleton g-frame corresponding to G is the standard dual of the singleton g-frame corresponding to F.*

Proof. Let S_Λ and $S_{F(\Lambda)}$ be the frame operators of Λ and $F(\Lambda)$, respectively. Then

$$\begin{aligned}F(\tilde{\Lambda})(i) &= (\tilde{\Lambda})^* e_i = (\Lambda S_\Lambda^{-1})^* e_i\\ &= S_\Lambda^{-1}\Lambda^* e_i = S_\Lambda^{-1} F(\Lambda)(i).\end{aligned}$$

By Proposition 3.2, the frame operator of Λ is equal to the frame operator of $F(\Lambda)$. Then

$$F(\tilde{\Lambda}) = S_{F(\Lambda)}^{-1} F(\Lambda).$$

So $F(\tilde{\Lambda})$ is the standard dual of $F(\Lambda)$.

For the converse, let F be a continuous frame with the corresponding singleton g-frame T_F as in Remark 1.2. Consider the frame operator $S_F = T_F^* T_F$ of F and let G be the standard dual of F. Then $T_G f(\omega) = T_{S_F^{-1}F} f(\omega) = \langle f, S_F^{-1} F(\omega)\rangle = T_F(S_F^{-1} f)(\omega)$, i.e. $T_G = T_F S_F^{-1}$. Note that the equality $S_F = T_F^* T_F$ means that S_F is also the frame operator of singleton g-frame T_F. So (by Definition 3.5) T_G is the standard dual of T_F. □

Example 3.8. It is well known that a frame either remains a frame or an incomplete set whenever any one of its elements is removed [5, Theorem 3.1.13]. Sun claims that this theorem does not hold in his g-frames [14, Example 3.6]. We claim that the correct translation of the above theorem to Sun g-frames is the following.

Let $\{\Lambda_j : H \to K_j; j \in J\}$ (J is at most countable) be a Sun g-frame with bounds A, B. Let $\{e_{j,i} : i \in I_j\}$ be an orthonorml basis for K_j and $\pi_{j,i} : K_j \to \mathbb{C}$ defined by $\pi_{j,i}(\sum_{r\in I_j} \alpha_r e_{j,r}) = \alpha_i$ be the canonical projection. Define $\Lambda_{j,i} = \pi_{j,i}\Lambda_j$. The set $\{\Lambda_{j,i} : j \in J, i \in I_j\}$ is a Sun g-frame with the same bounds A, B. Now it is easily seen that the above theorem is true for this kind of Sun g-frames. In fact, the above theorem ([5, Theorem 3.1.13]) is true for any Sun g-frame $\{\Lambda_j : H \to \mathbb{C}; j \in J\}$.

Indeed, first let $\{\Lambda_j : H \longrightarrow \mathbb{C}; j \in J\}$ be a Sun g-frame (constructed as in the above argument). We define $f_j = \Lambda_j^* 1$. It is easy to verify that for a fixed r we have

$$\sum_{j\in J, j\neq r} |\langle f, f_j\rangle|^2 = \sum_{j\in J, j\neq r} \|\Lambda_j f\|^2.$$

This means that $\{\Lambda_j; j \in J, j \neq r\}$ is a Sun g-frame if and only if $(f_j)_{j\in J, j\neq r}$ is a discrete frame. Secondly, suppose $\{\Lambda_j; j \in J, j \neq r\}$ is not a Sun g-frame. By the above argument $(f_j)_{j\in J, j\neq r}$ is not a discrete frame. So it is incomplete and there is $0 \neq f \in H$ such that it is orthogonal to all f_j's. Thus $\Lambda_j f = 0, j \in J$. It implies that $\{\Lambda_j; j \in J, j \neq r\}$ is not Sun g-complete. In other words, in a Sun g-frame $\{\Lambda_j : j \in J\}$ any Λ_j is corresponding to probably more than one element of a discrete frame $(f_i)_{i\in I}$.

Acknowledgements. This research was supported by a grant from Ferdowsi University of mashhad, No. MP94333RRT.

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