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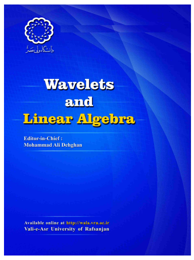

On Some Properties of K-g-Riesz Bases in Hilbert Spaces

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Abstract

In this paper, we study the K-Riesz bases and the K-g-Riesz bases in Hilbert spaces. We show that for $K \in B(\mathcal{H})$, a K-Riesz basis is precisely the image of an orthonormal basis under a bounded left-invertible operator such that the range of this operator includes the range of K. Also, we show that $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a K-g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ if and only if there exists a g-orthonormal basis $\{Q_i\}_{i\in I}$ for $\mathcal{H}$ and a bounded right-invertible operator U on $\mathcal{H}$ such that $\Lambda_i = Q_i U$ for all $i \in I$, and $R(K) \subset R(U^*)$.

1. Introduction

The concept of frame was introduced by Duffin and Schaeffer [8] in 1952 in the context of nonharmonic Fourier series. Frames have many nice properties which make them very useful in

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the characterization of function spaces, signal processing and many other fields.
A sequence $\{f_i\}_{i\in I}$ in a Hilbert space $\mathcal{H}$ is called a frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that for all $f \in \mathcal{H}$,

$$A\|f\|^2 \leqslant \sum_{i\in I} |\langle f, f_i\rangle|^2 \leqslant B\|f\|^2. \tag{1.1}$$

$\{f_i\}_{i\in I}$ is called a Bessel sequence if the right hand side of (1.1) holds for all $i \in I$.
We say that a sequence $\{f_i\}_{i\in I}$ is a Riesz basis for the Hilbert space $\mathcal{H}$, if it is complete in $\mathcal{H}$ (i.e., $\overline{span}\{f_i\}_{i\in I} = \mathcal{H}$) and there exist constants $A, B > 0$ such that for any finite scalar sequence $\{c_i\}$,

$$A\sum |c_i|^2 \leqslant \Big\|\sum c_i f_i\Big\|^2 \leqslant B\sum |c_i|^2.$$

Throughout this paper, $\mathcal{H}$ is a separable Hilbert space on complex field $\mathbb{C}$, I is a countable set, $\{\mathcal{H}_i\}_{i\in I}$ is a sequence of separable Hilbert spaces, $B(\mathcal{H}_1, \mathcal{H}_2)$ is the set of all bounded linear operators from the Hilbert space $\mathcal{H}_1$ to the Hilbert space $\mathcal{H}_2$, $B(\mathcal{H})$ is the set of all bounded linear operators on $\mathcal{H}$, $K \in B(\mathcal{H})$, and $R(K)$ and $N(K)$ are the range and the kernel of the operator K, respectively. $l^2(I)$ represents an infinite-dimensional complex Hilbert space consisting of sequences $\{c_i\}_{i\in I}$ such that $\|\{c_i\}_{i\in I}\|^2 = \sum_{i\in I} |c_i|^2 < +\infty$. For all $a = \{a_i\}_{i\in I}$ and $b = \{b_i\}_{i\in I}$ in $l^2(I)$, the inner product is specified as follow:

$$\langle a, b\rangle = \sum_{i\in I} a_i\overline{b_i}.$$

Generalized frame, or simply g-frame was introduced by Sun [13] in 2006. Despite of the fact that the members of discrete frames are vectors, the members of g-frames are bounded linear operators.
A sequence $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is called a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$, if there exist constants $A, B > 0$ such that for all $f \in \mathcal{H}$,

$$A\|f\|^2 \leqslant \sum_{i\in I} \|\Lambda_i f\|^2 \leqslant B\|f\|^2. \tag{1.2}$$

If the right hand inequality of (1.2) holds for all $f \in \mathcal{H}$ then $\{\Lambda_i\}_{i\in I}$ is called a g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$. We define the space

$$l^2(\{\mathcal{H}_i\}_{i\in I}) = \Big\{\{f_i\}_{i\in I} \ : \ f_i \in \mathcal{H}_i,\ i \in I \ and \ \sum_{i\in I} \|f_i\|^2 < +\infty\Big\}$$

with the inner product defined by

$$\langle \{f_i\}_{i\in I}, \{g_i\}_{i\in I}\rangle = \sum_{i\in I} \langle f_i, g_i\rangle.$$

It is clear that $l^2(\{\mathcal{H}_i\}_{i\in I})$ is a Hilbert space with the pointwise operations. We define the synthesis operator for a g-Bessel sequence $\Lambda = \{\Lambda_i\}_{i\in I}$ as:

$$T_\Lambda : l^2(\{\mathcal{H}_i\}_{i\in I}) \to \mathcal{H}, \ \ T_\Lambda(\{f_i\}_{i\in I}) = \sum_{i\in I} \Lambda_i^*(f_i). \tag{1.3}$$

It is easy to show that the adjoint operator of T_Λ is

$$T_\Lambda^* : \mathcal{H} \to l^2(\{\mathcal{H}_i\}_{i\in I}), \ \ T_\Lambda^*(f) = \{\Lambda_i f\}_{i\in I}. \tag{1.4}$$

T_Λ^* is called the analysis operator for $\{\Lambda_i\}_{i\in I}$.

Proposition 1.1. *[12] $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-Bessel sequence for $\mathcal{H}$ with bound B, if and only if the operator T_Λ defined in (1.3) is a well-defined and bounded operator with $\|T_\Lambda\| \leqslant \sqrt{B}$.*

Lemma 1.2. *Let $\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)$. Suppose that for each $i \in I$, $\{e_{i,j} : j \in J_i\}$ is an orthonormal basis for $\mathcal{H}_i$, where J_i is a subset of $\mathbb{Z}$. Then consider*

$$u_{i,j} = \Lambda_i^* e_{i,j};\ i \in I,\ j \in J_i. \tag{1.5}$$

We call $\{u_{i,j} : i \in I, j \in J_i\}$, the sequence induced by $\{\Lambda_i\}_{i\in I}$ with respect to $\{e_{i,j} : i \in I, j \in J_i\}$. Also, we have the following relations:

$$\Lambda_i f = \sum_{j\in J_i} \langle f, u_{i,j}\rangle e_{i,j}, \quad f \in \mathcal{H},$$

$$\Lambda_i^* g_i = \sum_{j\in J_i} \langle g, e_{i,j}\rangle u_{i,j}, \quad g_i \in \mathcal{H}_i.$$

For more study on g-frames, we can refer to [3, 4, 1, 2, 14].
Sun also introduced g-Riesz bases in [13]. We say that $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$, if it is g-complete, i.e., $\{f \in \mathcal{H} : \Lambda_i f = 0, i \in I\} = \{0\}$, and there exist constants $A, B > 0$ such that for any finite subset $I_1 \subset I$ and $g_i \in \mathcal{H}_i$, $i \in I_1$,

$$A \sum_{i\in I_1} \|g_i\|^2 \leqslant \Big\| \sum_{i\in I_1} \Lambda_i^* g_i \Big\|^2 \leqslant B \sum_{i\in I_1} \|g_i\|^2.$$

We say that $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-orthonormal basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ if it satisfies in the following:

$$\langle \Lambda_{i_1}^* g_{i_1}, \Lambda_{i_2}^* g_{i_2}\rangle = \delta_{i_1,i_2}\langle g_{i_1}, g_{i_2}\rangle, \quad i_1, i_2 \in I,\ g_{i_1} \in \mathcal{H}_{i_1},\ g_{i_2} \in \mathcal{H}_{i_2},$$

$$\sum_{i\in I} \|\Lambda_i f\|^2 = \|f\|^2, \quad f \in \mathcal{H}.$$

K-frames for Hilbert spaces were introduced by L. Găvruta [9] to study the atomic decomposition systems. Some properties of K-frames were discussed in [17]. A sequence $\{f_i\}_{i\in I}$ in a Hilbert space $\mathcal{H}$ is called a K-frame for $\mathcal{H}$, if there exist constants $A, B > 0$ such that for each $f \in \mathcal{H}$,

$$A\|K^* f\|^2 \leqslant \sum_{i\in I} |\langle f, f_i\rangle|^2 \leqslant B\|f\|^2.$$

We see that every K-frame is a Bessel sequence.
Y. Huang and D. Hua [11], introduced the concept of K-Riesz bases in Hilbert spaces (see also [20]). For $K \in B(\mathcal{H})$, a sequence $\{f_i\}_{i\in I}$ in $\mathcal{H}$ is called $\overline{R(K)}$-complete, if

$$\{f \in \mathcal{H} : \langle f, f_i\rangle = 0,\ i \in I\} \subset \overline{R(K)}^{\perp} = N(K^*).$$

A sequence $\{f_i\}_{i\in I}$ is a K-Riesz basis for $\mathcal{H}$, if it is $\overline{R(K)}$-complete and there exist constants $A, B > 0$ such that for all finite scalar sequence $\{c_i\}$,

$$A\sum|c_i|^2 \leqslant \Big\|\sum c_i f_i\Big\|^2 \leqslant B\sum|c_i|^2.$$

In [19] and [18], Y. Zhou and Y.C. Zhu studied K-g-frames in Hilbert spaces. We call a sequence $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ if there exist constants $A, B > 0$ such that

$$A\|K^* f\|^2 \leqslant \sum_{i\in I}\|\Lambda_i f\|^2 \leqslant B\|f\|^2, \quad f \in \mathcal{H}.$$

Lemma 1.3. *[7] Let $\mathcal{H}$, $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. Let $L_1 \in B(\mathcal{H}_1, \mathcal{H})$ and $L_2 \in B(\mathcal{H}_2, \mathcal{H})$ be two bounded operators. The following statements are equivalent:*
(i) $R(L_1) \subset R(L_2)$;
(ii) $L_1L_1^* \leqslant \lambda^2 L_2L_2^*$ for some $\lambda \geqslant 0$, and
(iii) there exists a bounded operator $X \in B(\mathcal{H}_1, \mathcal{H}_2)$ such that $L_1 = L_2X$.

Remark 1.4. In Lemma 1.3, if $L_1 \neq 0$ then $\lambda > 0$.

Definition 1.5. [5] Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. We say that $T \in B(\mathcal{H}_1, \mathcal{H}_2)$ is left-invertible (or, respectively, right-invertible) if there exists a bounded linear operator $G \in B(\mathcal{H}_2, \mathcal{H}_1)$ such that

$$GT = I_{\mathcal{H}_1}, \quad (\text{or } \; TG = I_{\mathcal{H}_2}).$$

We say that such an operator G is a bounded linear left (or right) inverse of T.

Proposition 1.6. *[5] A linear operator $T \in B(\mathcal{H}_1, \mathcal{H}_2)$ is left-invertible if and only if its adjoint $T^* \in B(\mathcal{H}_2, \mathcal{H}_1)$ is right invertible. Moreover, G is a left inverse of T if and only if G^* is a right inverse of T^*.*

Proposition 1.7. *[5] Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. The following assertions are equivalent:*
(i) $T \in B(\mathcal{H}_1, \mathcal{H}_2)$ is injective and $R(T)$ is closed in $\mathcal{H}_2$.
(ii) T is left-invertible.

Lemma 1.8. *[15] Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces and $T \in B(\mathcal{H}_1, \mathcal{H}_2)$. Then*

$$R(T)^\perp = N(T^*), \; R(T^*)^\perp = N(T), \; \overline{R(T)} = N(T^*)^\perp, \; \overline{R(T^*)} = N(T)^\perp.$$

2. Some properties of K-Riesz bases

In [10], the authors proved that a Riesz basis is precisely the image of an orthonormal basis under a bounded invertible operator. In the following theorem, we show that for $K \in B(\mathcal{H})$, a K-Riesz basis is precisely the image of an orthonormal basis under a bounded left-invertible operator such that the range of this operator includes the range of K.

Theorem 2.1. *Let $\mathcal{H}_0$ be a separable Hilbert space. Then $\{f_i\}_{i\in I}$ is a K-Riesz basis for $\mathcal{H}$ if and only if there exists a bounded left-invertible operator $\Theta \in B(\mathcal{H}_0, \mathcal{H})$ such that $R(K) \subset R(\Theta)$, and $\Theta e_i = f_i$ for all $i \in I$, where $\{e_i\}_{i\in I}$ is an orthonormal basis for $\mathcal{H}_0$.*

Proof. Suppose that $\{f_i\}_{i\in I}$ is a K-Riesz basis for $\mathcal{H}$. Then

$$\{f \in \mathcal{H} : \langle f, f_i\rangle = 0,\ i \in I\} \subset N(K^*) \tag{2.1}$$

and there exist $A, B > 0$ such that for all finite scalar sequence $\{c_i\}$,

$$A \sum |c_i|^2 \leqslant \Big\| \sum c_i f_i \Big\|^2 \leqslant B \sum |c_i|^2. \tag{2.2}$$

Then, for all $\{c_i\}_{i\in I} \in l^2(I)$,

$$A \sum_{i\in I} |c_i|^2 \leqslant \Big\| \sum_{i\in I} c_i f_i \Big\|^2 \leqslant B \sum_{i\in I} |c_i|^2. \tag{2.3}$$

Define $\Theta : \mathcal{H}_0 \to \mathcal{H}$ by

$$\Theta(x) = \sum_{i\in I} \langle x, e_i\rangle f_i.$$

We have $\{\langle x, e_i\rangle\}_{i\in I} \in l^2(I)$, for all $x \in \mathcal{H}_0$. Then by (2.3),

$$\|\Theta(x)\|^2 = \Big\| \sum_{i\in I} \langle x, e_i\rangle f_i \Big\|^2 \leqslant B \sum_{i\in I} |\langle x, e_i\rangle|^2 = B\|x\|^2.$$

Then, Θ is well-defined and bounded and $\Theta e_i = f_i$, for all $i \in I$. Also, (2.3) implies that

$$A\|x\|^2 \leqslant \|\Theta(x)\|^2 \leqslant B\|x\|^2, \quad x \in \mathcal{H}_0. \tag{2.4}$$

By (2.4) we conclude that Θ is injective and $R(\Theta)$ is a closed subspace of $\mathcal{H}$. Therefore, by Proposition 1.7, Θ is left-invertible. We have

$$\Theta^* : \mathcal{H} \to \mathcal{H}_0, \quad \Theta^*(f) = \sum_{i\in I} \langle f, f_i\rangle e_i. \tag{2.5}$$

Now, if $f \in \mathcal{H}$ and $\Theta^*(f) = 0$ then for all $i \in I$, $\langle f, f_i\rangle = 0$. Thus by (2.1), $f \in N(K^*)$. It means that $N(\Theta^*) \subset N(K^*)$, and $N(K^*)^\perp \subset N(\Theta^*)^\perp$. By Lemma 1.8, $R(K) \subset \overline{R(K)} \subset \overline{R(\Theta)} = R(\Theta)$.
Conversely, let $\Theta \in B(\mathcal{H}_0, \mathcal{H})$ be a left-invertible operator such that for all $i \in I$, $\Theta e_i = f_i$ and $R(K) \subset R(\Theta)$. Let Θ_l^{-1} be the left inverse of Θ. Then for all finite scalar sequence $\{c_i\}$, we have

$$\Big\| \sum c_i f_i \Big\|^2 = \Big\| \sum c_i \Theta e_i \Big\|^2 \leqslant \|\Theta\|^2 . \sum |c_i|^2,$$

and

$$\sum |c_i|^2 = \Big\| \sum c_i e_i \Big\|^2 = \Big\| \Theta_l^{-1} \Theta \Big(\sum c_i e_i \Big) \Big\|^2 \leqslant \|\Theta_l^{-1}\|^2 . \Big\| \sum c_i f_i \Big\|^2.$$

This shows that (2.2) holds for $\{f_i\}_{i\in I}$ with $\|\Theta_l^{-1}\|^{-2}$ and $\|\Theta\|^2$. Let $f \in \mathcal{H}$ and for all $i \in I$, $\langle f, f_i\rangle = 0$, then

$$0 = \langle f, \Theta e_i\rangle = \langle \Theta^* f, e_i\rangle, \quad i \in I,$$

thus $\Theta^* f = 0$. Since $R(K) \subset R(\Theta)$, by Lemma 1.8, $N(\Theta^*) \subset N(K^*)$. Therefore,

$$\{f \in \mathcal{H} : \langle f, f_i\rangle = 0, i \in I\} \subset N(K^*),$$

and so $\{f_i\}_{i\in I}$ is $\overline{R(K)}$-complete. □

Corollary 2.2. *For $0 \neq K \in B(\mathcal{H})$, every K-Riesz basis is a K-frame.*

Proof. Let $\{f_i\}_{i\in I}$ be a K-Riesz basis for $\mathcal{H}$, then by Theorem 2.1, there exists a bounded left-invertible operator $\Theta \in B(\mathcal{H})$ such that for all $i \in I$, $f_i = \Theta e_i$, and $R(K) \subset R(\Theta)$, where $\{e_i\}_{i\in I}$ is an orthonormal basis for $\mathcal{H}$. Then, for all $f \in \mathcal{H}$, we have

$$\sum_{i\in I} |\langle f, f_i\rangle|^2 = \sum_{i\in I} |\langle f, \Theta e_i\rangle|^2 = \sum_{i\in I} |\langle \Theta^* f, e_i\rangle|^2 = \|\Theta^* f\|^2 \leqslant \|\Theta\|^2.\|f\|^2. \tag{2.6}$$

Since $R(K) \subset R(\Theta)$, by Lemma 1.3, there exists $\lambda > 0$ such that $KK^* \leqslant \lambda^2 \Theta\Theta^*$. Then,

$$\|K^* f\|^2 \leqslant \lambda^2 \|\Theta^* f\|^2 = \lambda^2 \sum_{i\in I} |\langle f, f_i\rangle|^2, \quad f \in \mathcal{H}.$$

Therefore, $\{f_i\}_{i\in I}$ is a K-frame for $\mathcal{H}$ with bounds $\frac{1}{\lambda^2}$ and $\|\Theta\|^2$, respectively. □

3. K-g-Riesz bases

The concept of K-g-Riesz bases was introduced in [18]. In this section, we prove some new results about K-g-Riesz bases in Hilbert spaces.

Definition 3.1. Let $\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)$, for all $i \in I$ and $K \in B(\mathcal{H})$.
(i) If $\{f \in \mathcal{H} : \Lambda_i f = 0, i \in I\} \subset \overline{R(K)}^{\perp} = N(K^*)$ then we say that $\{\Lambda_i\}_{i\in I}$ is $\overline{R(K)}$-g-complete.
(ii) If $\{\Lambda_i\}_{i\in I}$ is $\overline{R(K)}$-g-complete and there are positive constants A and B such that for any finite subset $I_1 \subset I$ and $g_i \in \mathcal{H}_i$, $i \in I_1$,

$$A \sum_{i\in I_1} \|g_i\|^2 \leqslant \Big\| \sum_{i\in I_1} \Lambda_i^* g_i \Big\|^2 \leqslant B \sum_{i\in I_1} \|g_i\|^2,$$

then we say that $\{\Lambda_i\}_{i\in I}$ is a K-g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$.

Example 3.2. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. For all $i \in \mathbb{N}$, define $\Lambda_i : \mathcal{H} \to \mathbb{C}$ by $\Lambda_i f = \langle f, e_i\rangle$ and $K : \mathcal{H} \to \mathcal{H}$ by $Kf = \sum_{i=2}^{\infty} \langle f, e_i\rangle e_i$. We see that $K^* f = \sum_{i=2}^{\infty} \langle f, e_i\rangle e_i$, and for $c \in \mathbb{C}$, $\Lambda_i^* c = c e_i$. For all $f \in \mathcal{H}$, we have

$$\|K^* f\|^2 = \Big\| \sum_{i=2}^{\infty} \langle f, e_i\rangle e_i \Big\|^2 = \sum_{i=2}^{\infty} |\langle f, e_i\rangle|^2 = \sum_{i=2}^{\infty} |\Lambda_i f|^2.$$

If $\Lambda_i f = 0$, for all $i \in \mathbb{N}$, then $K^* f = 0$, i.e., $\{\Lambda_i\}_{i\in I}$ is $\overline{R(K)}$-g-complete. Also,

$$\Big\|\sum_{i=1}^{n} \Lambda_i^* c_i\Big\|^2 = \Big\|\sum_{i=1}^{n} c_i e_i\Big\|^2 = \sum_{i=1}^{n} |c_i|^2.$$

Therefore, $\{\Lambda_i\}_{i\in I}$ is a K-g-Riesz basis for $\mathcal{H}$ with respect to $\mathbb{C}$.

Theorem 3.3. *Let $\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)$ and $u_{i,j}$ be defined as in (1.5). Then $\{\Lambda_i\}_{i\in I}$ is a K-g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ if and only if $\{u_{i,j} : i \in I, j \in J_i\}$ is a K-Riesz basis for $\mathcal{H}$.*

Proof. First we assume that $\{\Lambda_i\}_{i\in I}$ is a K-g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$. Then

$$\{f \in \mathcal{H} \ : \ \Lambda_i f = 0,\ i \in I\} \subset N(K^*),$$

and there exist constants $A, B > 0$ such that for any finite subset $I_1 \subset I$, we have

$$A \sum_{i\in I_1} \|g_i\|^2 \leqslant \Big\|\sum_{i\in I_1} \Lambda_i^* g_i\Big\|^2 \leqslant B \sum_{i\in I_1} \|g_i\|^2. \tag{3.1}$$

Since $\{e_{i,j} \ : \ j \in J_i\}$ is an orthonormal basis for $\mathcal{H}_i$, every $g_i \in \mathcal{H}_i$ has an expansion of the form $g_i = \sum_{j\in J_i} c_{i,j} e_{i,j}$, where $\{c_{i,j} : \ j \in J_j\} \in l^2(J_i)$. It follows that (3.1) is equivalent to

$$A \sum_{i\in I_1} \sum_{j\in J_i} |c_{i,j}|^2 \leqslant \Big\|\sum_{i\in I_1} \sum_{j\in J_i} c_{i,j} u_{i,j}\Big\|^2 \leqslant B \sum_{i\in I_1} \sum_{j\in J_i} |c_{i,j}|^2.$$

On the other hand, we see from $\Lambda_i f = \sum_{j\in J_i} \langle f, u_{i,j}\rangle e_{i,j}$ that

$$\{f \in \mathcal{H} \ : \Lambda_i f = 0,\ i \in I\} = \{f \in \mathcal{H} \ : \langle f, u_{i,j}\rangle = 0,\ i \in I, j \in J_i\}.$$

Hence $\{\Lambda_i\}_{i\in I}$ is $\overline{R(K)}$-g-complete if and only if $\{u_{i,j}\}_{i\in I, j\in J_i}$ is $\overline{R(K)}$-complete. Therefore, $\{\Lambda_i\}_{i\in I}$ is a K-g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ if and only if $\{u_{i,j} : \ i \in I, j \in J_i\}$ is a K-Riesz basis for $\mathcal{H}$. □

Theorem 3.4. *A sequence $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) \ : \ i \in I\}$ is a K-g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ if and only if there exists a g-orthonormal basis $\{Q_i \in B(\mathcal{H}, \mathcal{H}_i) \ : \ i \in I\}$ for $\mathcal{H}$ and a bounded right-invertible operator $U \in B(\mathcal{H})$ such that $\Lambda_i = Q_i U$ for all $i \in I$, and $R(K) \subset R(U^*)$.*

Proof. Let $\{e_{i,j} : \ j \in J_i\}$ be an orthonormal basis for $\mathcal{H}_i$, for every $i \in I$. We assume that $\{\Lambda_i\}_{i\in I}$ is a K-g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$. By Theorem 3.3, we can find a K-Riesz basis $\{u_{i,j} : i \in I, j \in J_i\}$ for $\mathcal{H}$ such that

$$\Lambda_i f = \sum_{j\in J_i} \langle f, u_{i,j}\rangle e_{i,j}, \quad i \in I,\ f \in \mathcal{H}.$$

Take an orthonormal basis $\{v_{i,j} : i \in I, j \in J_i\}$ for $\mathcal{H}$. Since $\{u_{i,j} : i \in I, j \in J_i\}$ is a K-Riesz basis for $\mathcal{H}$, by Theorem 2.1, there exists a bounded left-invertible operator $\Theta \in B(\mathcal{H})$ such that

$$\Theta v_{i,j} = u_{i,j}, \quad i \in I,\ j \in J_i,$$

and $R(K) \subset R(\Theta)$. Put $U = \Theta^*$, then by Proposition 1.6, $U : \mathcal{H} \to \mathcal{H}$ is a bounded right-invertible operator and $R(K) \subset R(U^*)$. Let $Q_i \in B(\mathcal{H}, \mathcal{H}_i)$ be such that

$$Q_i g = \sum_{j \in J_i} \langle g, v_{i,j} \rangle e_{i,j}, \quad i \in I, \ g \in \mathcal{H}.$$

By Theorem 3.1 in [13], $\{Q_i\}_{i \in I}$ is a g-orthonormal basis for $\mathcal{H}$. Moreover, for any $f \in \mathcal{H}$,

$$Q_i U f = \sum_{j \in J_i} \langle U f, v_{i,j} \rangle e_{i,j} = \sum_{j \in J_i} \langle f, \Theta v_{i,j} \rangle e_{i,j} = \sum_{j \in J_i} \langle f, u_{i,j} \rangle e_{i,j} = \Lambda_i f.$$

Hence for all $i \in I$, $\Lambda_i = Q_i U$.
Conversely, let $\{Q_i\}_{i \in I}$ be a g-orthonormal basis for $\mathcal{H}$ and U be a bounded right-invertible operator on $\mathcal{H}$ such that $\Lambda_i = Q_i U$ for all $i \in I$, and $R(K) \subset R(U^*)$. Then

$$\langle Q_{i_1}^* g_{i_1}, Q_{i_2}^* g_{i_2} \rangle = \delta_{i_1, i_2} \langle g_{i_1}, g_{i_2} \rangle, \quad i_1, i_2 \in I, \ g_{i_1} \in \mathcal{H}_{i_1}, \ g_{i_2} \in \mathcal{H}_{i_2},$$

and

$$\sum_{i \in I} \|Q_i f\|^2 = \|f\|^2, \quad f \in \mathcal{H}.$$

If $\Lambda_i f = 0$, for all $i \in I$, then

$$0 = \sum_{i \in I} \|\Lambda_i f\|^2 = \sum_{i \in I} \|Q_i U f\|^2 = \|U f\|^2,$$

i.e., $f \in N(U)$. From $R(K) \subset R(U^*)$ and by Lemma 1.8, $f \in N(K^*)$. Thus, $\{\Lambda_i\}_{i \in I}$ is $\overline{R(K)}$-g-complete. By Theorem 3.1 in [13], we can find an orthonormal basis $\{v_{i,j} : i \in I, j \in J_i\}$ for $\mathcal{H}$ such that $Q_i g = \sum_{j \in J_i} \langle g, v_{i,j} \rangle e_{i,j}$, for all $g \in \mathcal{H}$. Hence,

$$\Lambda_i f = Q_i U f = \sum_{j \in J_i} \langle U f, v_{i,j} \rangle e_{i,j} = \sum_{j \in J_i} \langle f, U^* v_{i,j} \rangle e_{i,j}, \quad f \in \mathcal{H}.$$

By Proposition 1.6, U^* is a bounded left-invertible operator and $R(K) \subset R(U^*)$, then by Theorem 2.1, $U^* v_{i,j}$ is a K-Riesz basis for $\mathcal{H}$. Thus, by Theorem 3.3 we conclude that $\{\Lambda_i\}_{i \in I}$ is a K-g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in I}$. □

Theorem 3.5. *For $0 \neq K \in B(\mathcal{H})$, every K-g-Riesz basis is a K-g-frame.*

Proof. Let $\{\Lambda_i\}_{i \in I}$ be a K-g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in I}$, then by Theorem 3.4, there exist a bounded right-invertible operator U on $\mathcal{H}$ and a g-orthonormal basis $\{Q_i\}_{i \in I}$ for $\mathcal{H}$ such that $\Lambda_i = Q_i U$, for all $i \in I$, and $R(K) \subset R(U^*)$. For all $f \in \mathcal{H}$, we have

$$\sum_{i \in I} \|\Lambda_i f\|^2 = \sum_{i \in I} \|Q_i U f\|^2 = \|U f\|^2 \leqslant \|U\|^2 . \|f\|^2.$$

Also, since $R(K) \subset R(U^*)$, by Lemma 1.3, there exists $\lambda > 0$ such that $KK^* \leqslant \lambda^2 U^* U$. Then,

$$\|K^* f\|^2 \leqslant \lambda^2 \|U f\|^2 = \lambda^2 \sum_{i \in I} \|Q_i U f\|^2 = \lambda^2 \sum_{i \in I} \|\Lambda_i f\|^2, \quad f \in \mathcal{H}.$$

Therefore, $\{\Lambda_i\}_{i \in I}$ is a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in I}$. □

Definition 3.6. [16] Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g-Bessel sequence in $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$. For $\{g_i\}_{i\in I} \in l^2(\{\mathcal{H}_i\}_{i\in I})$, if $\sum_{i\in I} \Lambda_i^* g_i = 0$ we can get $g_i = 0$ for any $i \in I$, then $\{\Lambda_i\}_{i\in I}$ is called $l^2(\{\mathcal{H}_i\}_{i\in I})$-linear independent.

Theorem 3.7. *Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$. Then $\{\Lambda_i\}_{i\in I}$ is a K-g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ if and only if $R(T_\Lambda)$ is closed and $\{\Lambda_i\}_{i\in I}$ is $l^2(\{\mathcal{H}_i\}_{i\in I})$-linear independent.*

Proof. First let $\{\Lambda_i\}_{i\in I}$ be a K-g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$, then it is $\overline{R(K)}$-g-complete and there exist $A, B > 0$ such that for any finite subset $I_1 \subset I$ and $g_i \in \mathcal{H}_i$, $i \in I_1$,

$$A \sum_{i\in I_1} \|g_i\|^2 \leqslant \Big\| \sum_{i\in I_1} \Lambda_i^* g_i \Big\|^2 \leqslant B \sum_{i\in I_1} \|g_i\|^2.$$

Then, we have

$$A \sum_{i\in I} \|g_i\|^2 \leqslant \Big\| \sum_{i\in I} \Lambda_i^* g_i \Big\|^2 \leqslant B \sum_{i\in I} \|g_i\|^2, \quad \{g_i\}_{i\in I} \in l^2(\{\mathcal{H}_i\}_{i\in I}). \tag{3.2}$$

Let $\{g_i\}_{i\in I} \in l^2(\{\mathcal{H}_i\}_{i\in I})$ and $\sum_{i\in I} \Lambda_i^* g_i = 0$, then by (3.2), for all $i \in I$, $g_i = 0$. Thus, $\{\Lambda_i\}_{i\in I}$ is $l^2(\{\mathcal{H}_i\}_{i\in I})$-linear independent. Also from (3.2) we conclude that $R(T_\Lambda)$ is closed.
Conversely, let $R(T_\Lambda)$ is closed and $\{\Lambda_i\}_{i\in I}$ is $l^2(\{\mathcal{H}_i\}_{i\in I})$-linear independent. Since $\{\Lambda_i\}_{i\in I}$ is a K-g-frame, then there exist $A, B > 0$ such that for all $f \in \mathcal{H}$,

$$A\|K^* f\|^2 \leqslant \sum_{i\in I} \|\Lambda_i f\|^2 \leqslant B\|f\|^2.$$

Thus if $\Lambda_i f = 0$ for all $i \in I$, then $A\|K^* f\|^2 = 0$, so $f \in N(K^*)$. Therefore, $\{\Lambda_i\}_{i\in I}$ is $\overline{R(K)}$-g-complete. Since $\{\Lambda_i\}_{i\in I}$ is a g-Bessel sequence, by Proposition 1.1, T_Λ is bounded and there exists $B > 0$ such that $\|T_\Lambda\| \leqslant \sqrt{B}$. It means that

$$\|T_\Lambda(\{g_i\}_{i\in I})\|^2 = \Big\| \sum_{i\in I} \Lambda_i^* g_i \Big\|^2 \leqslant B \sum_{i\in I} \|g_i\|^2, \quad \{g_i\}_{i\in I} \in l^2(\{\mathcal{H}_i\}_{i\in I}).$$

Since $\{\Lambda_i\}_{i\in I}$ is $l^2(\{\mathcal{H}_i\}_{i\in I})$-linear independent, $N(T_\Lambda) = 0$ and since $R(T_\Lambda)$ is closed, by Proposition 1.7, there exists a bounded operator $(T_\Lambda)_l^{-1} : \mathcal{H} \to l^2(\{\mathcal{H}_i\}_{i\in I})$ such that for any $\{g_i\}_{i\in I} \in l^2(\{\mathcal{H}_i\}_{i\in I})$, $(T_\Lambda)_l^{-1} T_\Lambda(\{g_i\}_{i\in I}) = \{g_i\}_{i\in I}$. Then

$$\sum_{i\in I} \|g_i\|^2 = \|\{g_i\}_{i\in I}\|^2 = \|(T_\Lambda)_l^{-1} T_\Lambda(\{g_i\}_{i\in I})\|^2 \leqslant \|(T_\Lambda)_l^{-1}\|^2 \|T_\Lambda(\{g_i\}_{i\in I})\|^2 = \|(T_\Lambda)_l^{-1}\|^2 \Big\| \sum_{i\in I} \Lambda_i^* g_i \Big\|^2.$$

Therefore,

$$\|(T_\Lambda)_l^{-1}\|^{-2} \sum_{i\in I} \|g_i\|^2 \leqslant \Big\| \sum_{i\in I} \Lambda_i^* g_i \Big\|^2, \quad \{g_i\}_{i\in I} \in l^2(\{\mathcal{H}_i\}_{i\in I}).$$

□

Example 3.8. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for the Hilbert space $\mathcal{H}$, and let $\mathcal{H}_i = \mathbb{C}^2$, for all $i \in \mathbb{N}$. We define bounded operators

$$\Lambda_i : \mathcal{H} \to \mathbb{C}^2, \quad \Lambda_i f = \big(\langle f, e_i\rangle, \langle f, e_{i+1}\rangle\big),$$

$$K : \mathcal{H} \to \mathcal{H}, \quad Kf = \sum_{i=1}^{\infty} \langle f, e_i\rangle e_{i+1} + \sum_{i=1}^{\infty} \langle f, e_{i+1}\rangle e_i.$$

We see that

$$\|\Lambda_i f\|^2 = \Big\|\big(\langle f, e_i\rangle, \langle f, e_{i+1}\rangle\big)\Big\|^2 = |\langle f, e_i\rangle|^2 + |\langle f, e_{i+1}\rangle|^2, \quad i \in \mathbb{N},$$

and

$$K^* f = \sum_{i=1}^{\infty} \langle f, e_{i+1}\rangle e_i + \sum_{i=1}^{\infty} \langle f, e_i\rangle e_{i+1}, \quad f \in \mathcal{H}.$$

For all $f \in \mathcal{H}$, we have

$$\begin{aligned} \|K^* f\|^2 = \Big\| \sum_{i=1}^{\infty} \langle f, e_{i+1}\rangle e_i + \sum_{i=1}^{\infty} \langle f, e_i\rangle e_{i+1} \Big\|^2 &\leqslant 2 \sum_{i=1}^{\infty} |\langle f, e_{i+1}\rangle|^2 + 2 \sum_{i=1}^{\infty} |\langle f, e_i\rangle|^2 \\ &= 2 \sum_{i=1}^{\infty} \|\Lambda_i f\|^2. \end{aligned} \tag{3.3}$$

Therefore,

$$\frac{1}{2}\|K^* f\|^2 \leqslant \sum_{i=1}^{\infty} \|\Lambda_i f\|^2 = \sum_{i=1}^{\infty} |\langle f, e_i\rangle|^2 + \sum_{i=1}^{\infty} |\langle f, e_{i+1}\rangle|^2 \leqslant 2\|f\|^2.$$

We conclude that $\{\Lambda_i\}_{i\in\mathbb{N}}$ is a K-g-frame for $\mathcal{H}$ with respect to $\mathbb{C}^2$. Also, if $\Lambda_i f = 0$, for all $i \in \mathbb{N}$ then by (3.3), $\|K^* f\|^2 = 0$, i.e., $\{\Lambda_i\}_{i\in\mathbb{N}}$ is $\overline{R(K)}$-g-complete. Moreover, for all $i \in \mathbb{N}$,

$$\Lambda_i^*(c, d) = ce_i + de_{i+1}, \quad (c, d) \in \mathbb{C}^2. \tag{3.4}$$

Let $\{(c_i, d_i)\}_{i=1}^{\infty}$ be a sequence with the property that

$$(c_1, d_1) = (0, -1), \ (c_2, d_2) = (1, 0), \ (c_i, d_i) = (0, 0), \ \forall\, i \geqslant 3.$$

Then, $\{(c_i, d_i)\}_{i=1}^{\infty} \in l^2(\{\mathcal{H}_i\}_{i\in\mathbb{N}})$ and by (3.4),

$$\sum_{i=1}^{\infty} \Lambda_i^*(c_i, d_i) = \Lambda_1^*(c_1, d_1) + \Lambda_2^*(c_2, d_2) = \Lambda_1^*(0, -1) + \Lambda_2^*(1, 0) = -e_2 + e_2 = 0,$$

but $(c_1, d_1) \neq (0, 0)$. Therefore, $\{\Lambda_i\}_{i\in\mathbb{N}}$ is not $l^2(\{\mathcal{H}_i\}_{i\in\mathbb{N}})$-lineary independent, thus by Theorem 3.7, $\{\Lambda_i\}_{i\in\mathbb{N}}$ is not a K-g-Riesz basis for $\mathcal{H}$ with respect to $\mathbb{C}^2$.

Theorem 3.9. *Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a K-g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ and let $\{w_{i,j}\}_{j\in M_i}$ be a Riesz basis for $\mathcal{H}_i$, for all $i \in I$ with bounds C_i and D_i such that $0 < \inf_i C_i$ and $\sup_i D_i < \infty$, where M_i is a subset of $\mathbb{Z}$. Then $\{\Lambda_i^* w_{i,j}\}_{i\in I, j\in M_i}$ is a K-Riesz basis for $\mathcal{H}$.*

Proof. Since $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a K-g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$, then

$$\{f \in \mathcal{H} : \Lambda_i f = 0, i \in I\} \subset N(K^*),$$

and there exist $A, B > 0$ such that for all finite subset $F \subseteq I$, $g_i \in \mathcal{H}_i$, $i \in F$,

$$A \sum_{i\in F} \|g_i\|^2 \leqslant \Big\| \sum_{i\in F} \Lambda_i^* g_i \Big\|^2 \leqslant B \sum_{i\in F} \|g_i\|^2. \tag{3.5}$$

Moreover, since $\{w_{i,j}\}_{j\in M_i}$ is a Riesz basis for $\mathcal{H}_i$, for all $i \in I$,

$$\{g_i \in \mathcal{H}_i : \langle g_i, w_{i,j}\rangle = 0, j \in M_i\} = \{0\},$$

and for each $\{c_{i,j}\}_{j\in M_i} \in l^2(M_i)$,

$$C_i \sum_{j\in M_i} |c_{i,j}|^2 \leqslant \Big\| \sum_{j\in M_i} c_{i,j} w_{i,j} \Big\|^2 \leqslant D_i \sum_{j\in M_i} |c_{i,j}|^2, \quad i \in I. \tag{3.6}$$

We have

$$\begin{aligned}\{f \in \mathcal{H} : \langle f, \Lambda_i^* w_{i,j}\rangle = 0, i \in I, j \in M_i\} &= \{f \in \mathcal{H} : \langle \Lambda_i f, w_{i,j}\rangle = 0, i \in I, j \in M_i\} \\ &= \{f \in \mathcal{H} : \Lambda_i f = 0, i \in I\} \\ &\subset N(K^*).\end{aligned}$$

Thus, $\{\Lambda_i^* w_{i,j}\}_{i\in I, j\in M_i}$ is $\overline{R(K)}$-complete.
If $\inf_i C_i = C$ and $\sup_i D_i = D$ then by (3.5) and (3.6), for all finite subset $F \subseteq I$ and scalar sequence $\{\beta_{i,j}\}_{i\in F, j\in M_i}$, we have

$$\begin{aligned}AC \sum_{i\in F}\sum_{j\in M_i} |\beta_{i,j}|^2 &\leqslant A \sum_{i\in F} \Big\| \sum_{j\in M_i} \beta_{i,j} w_{i,j} \Big\|^2 \leqslant \Big\| \sum_{i\in F} \Lambda_i^* \Big(\sum_{j\in M_i} \beta_{i,j} w_{i,j} \Big) \Big\|^2 \\ &\leqslant B \sum_{i\in F} \Big\| \sum_{j\in M_i} \beta_{i,j} w_{i,j} \Big\|^2 \leqslant BD \sum_{i\in F}\sum_{j\in M_i} |\beta_{i,j}|^2.\end{aligned}$$

So

$$AC \sum_{i\in F}\sum_{j\in M_i} |\beta_{i,j}|^2 \leqslant \Big\| \sum_{i\in F}\sum_{j\in M_i} \beta_{i,j} \Lambda_i^* w_{i,j} \Big\|^2 \leqslant BD \sum_{i\in F}\sum_{j\in M_i} |\beta_{i,j}|^2.$$

□

Theorem 3.10. *Let $\{\Lambda_i\}_{i\in I}$ be a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ and let T_Λ be defined as (1.3). Let there exists a finite subset σ of I for which $\{\Lambda_i\}_{i\in I\setminus\sigma}$ is a K-g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I\setminus\sigma}$. If $\sum_{i\in I} \Lambda_i^* g_i$ is converges, then $\{g_i\}_{i\in I} \in l^2(\{\mathcal{H}_i\}_{i\in I})$.*

Proof. Suppose that $\sum_{i\in I} \Lambda_i^* g_i$ converges, where $g_i \in \mathcal{H}_i$ for all $i \in I$. So $\sum_{i\in I\setminus\sigma} \Lambda_i^* g_i$ converges. Since $\{\Lambda_i\}_{i\in I\setminus\sigma}$ is a K-g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I\setminus\sigma}$, by Theorem 3.4, there exist a bounded right-invertible operator $U : \mathcal{H} \to \mathcal{H}$ and a g-orthonormal basis $\{Q_i\}_{i\in I\setminus\sigma}$ for $\mathcal{H}$ such that $\Lambda_i = Q_i U$ for all $i \in I \setminus \sigma$, and $R(K) \subset R(U^*)$. So

$$\sum_{i\in I\setminus\sigma} \Lambda_i^* g_i = \sum_{i\in I\setminus\sigma} (Q_i U)^* g_i = U^* \Big(\sum_{i\in I\setminus\sigma} Q_i^* g_i \Big).$$

Since $\{Q_i\}_{i\in I\setminus\sigma}$ is a g-orthonormal basis, we have

$$\sum_{i\in I\setminus\sigma} \|g_i\|^2 = \Big\| \sum_{i\in I\setminus\sigma} Q_i^* g_i \Big\|^2 < \infty.$$

Then $\{g_i\}_{i\in I\setminus\sigma} \in l^2(\{\mathcal{H}_i\}_{i\in I\setminus\sigma})$ and this implies that $\{g_i\}_{i\in I} \in l^2(\{\mathcal{H}_i\}_{i\in I})$. □

References

[1] M.R. Abdollahpour and F. Bagarello, On some properties of g-frames and g-coherent states, *Nuovo Cimento, X. Ser. B*, **125**(11), (2010), 1327–1342.

[2] M.R. Abdollahpour and A. Najati, Approximation of the inverse G-frame operator, *Proc. Indian Acad. Sci., Math. Sci.*, **121**(2), (2011), 143–154.

[3] M.R. Abdollahpour and A. Najati, Besselian G-frames and near g-Riesz bases, *Appl. Anal. Discrete Math.*, **5**, (2011), 259–270.

[4] M.R. Abdollahpour and A. Najati, G-frames and Hilbert-Schmidt operators, *Bull. Iran. Math. Soc.*, **37**(4), (2011), 141–155.

[5] J.P. Aubin, *Applied Functional Analysis*, A Wiley Series of Texts, Monographs and Tracts, 2nd Edition, 2000.

[6] O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, Boston, 2003.

[7] R.G. Douglas, On majorization, factorization, and range inclusion of operators on Hilbert space, *Proc. Am. Math. Soc.*, **17**(2), (1966), 413–415.

[8] J. Duffin and A.C. Schaeffer, A class of nonharmonic Fourier series, *Trans. Am. Math. Soc.*, **72**, (1952), 341–366.

[9] L. Găvruta, Frames for operators, *Appl. Comput. Harmon. Anal.*, **32**(1), (2012), 139–144.

[10] D. Han and D. Larson, Frames, bases and group representations, *Mem. Am. Math. Soc.*, **147**(697), (2000), 1–110.

[11] Y. Huang and D. Hua, K-Riesz frames and the stability of K-Riesz frames for Hilbert spaces, *Sci. Sin., Math.*, **47**(3), (2017), 383–396.

[12] A. Najati, M.H. Faroughi and A. Rahimi, G-frames and Stability of g-frames in Hilbert Spaces, *Methods Funct. Anal. Topol.*, **14**(3), (2008), 271–286.

[13] W. Sun, G-frames and g-Riesz bases, *J. Math. Anal. Appl.*, **322**(1), (2006), 437–452.

[14] W. Sun, Stability of g-frames, *J. Math. Anal. Appl.*, **326**, (2007), 858–868.

[15] A.E. Taylor and D. Lay, *Introduction to Functional Analysis*, New York, 1980.

[16] X. Xiao and Y. Zhu, Exact K-g-frames in Hilbert spaces, *Result. Math.*, **72**(3), (2017), 1329–1339.

[17] X. Xiao, Y. Zhu and L. Găvruta, Some properties of K-frames in Hilbert spaces, *Result. Math.*, **63**(3-4), (2013), 1243–1255.

[18] Y. Zhou and Y.C. Zhu, Characterization of K-g-frames in Hilbert spaces, *Acta Math. Sin., Chin. Ser.*, **57**(5), (2014), 1031–1040.

[19] Y. Zhou and Y.C. Zhu, K-g-frames and dual g-frames for closed subspaces, *Acta Math. Sin., Chin. Ser.*, **56**(5), (2013), 799–806.

[20] Y. Zhu, Z. Shu and X. Xiao, K-frames and K-Riesz bases in complex Hilbert spaces, *Sci. Sin., Math.*, **48**, (2018), 609–622.