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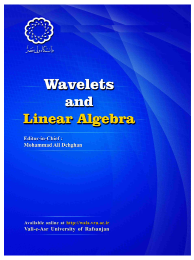

Decomposability of Weak Majorization

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Abstract

Let $x, y \in \mathbb{R}^n$. We use the notation $x \prec_w y$ when x is weakly majorized by y. We say that $x \prec_w y$ is decomposable at k $(1 \leq k < n)$ if $x \prec_w y$ has a coincidence at k and $y_k \neq y_{k+1}$. Corresponding to this majorization we have a doubly substochastic matrix P. The paper presents $x \prec_w y$ is decomposable at some k $(1 \leq k < n)$ if and only if P is of the form $D \oplus Q$ where D and Q are doubly stochastic and doubly substochastic matrices, respectively. Also, we write some algorithms to obtain x from y when $x \prec_w y$.

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1. Introduction

Let $\mathbf{M}_n$ be the set of all real matrices of order n. A matrix $D \in \mathbf{M}_n$ of nonnegative real numbers for which the sums of the entries in each row and each column are all one is said to be *doubly stochastic*. We denote the set of all doubly stochastic matrices of order n by Ω_n.
Let $\mathbb{R}^n$ be the set of all $n-$tuples $(x_1, \ldots, x_n)$ of real numbers. Set $\mathbb{R}^n_+$ the set of all $x \in \mathbb{R}^n$ with nonnegative entires. For vector $x \in \mathbb{R}^n$, the notations $x \geq 0$ and $x > 0$ mean that $x_i \geq 0$ and $x_i > 0$ for $i = 1, \ldots, n$, respectively. Also, for $x,\ y \in \mathbb{R}^n$, $x \geq y$ means that $x - y \geq 0$. Let $x^{\downarrow}$ and $x^{\uparrow}$ be the vectors obtained by rearranging the coordinates of $x \in \mathbb{R}^n$ in the decreasing and the increasing orders, respectively. Thus, $x^{\downarrow} = (x^{\downarrow}_1, \ldots, x^{\downarrow}_n)$, where $x^{\downarrow}_1 \geq \cdots \geq x^{\downarrow}_n$. Similarly, $x^{\uparrow} = (x^{\uparrow}_1, \ldots, x^{\uparrow}_n)$, where $x^{\uparrow}_1 \leq \cdots \leq x^{\uparrow}_n$.
Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. We say that x is *majorized* by y, in symbols $x \prec y$, if

$$\sum_{i=1}^{k} x^{\downarrow}_i \leq \sum_{i=1}^{k} y^{\downarrow}_i, \qquad 1 \leq k \leq n,$$

and

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i.$$

For further information about majorization, we refer the reader to [2, 3], [5]–[7] and [9] . Let $x, y \in \mathbb{R}^n$, we say that x is (*weakly*) *submajorized* by y, in symbols $x \prec_w y$, if

$$\sum_{i=1}^{k} x^{\downarrow}_i \leq \sum_{i=1}^{k} y^{\downarrow}_i, \qquad 1 \leq k \leq n.$$

The following concepts are defined in [1, 4] and [8].
Let $x, y \in \mathbb{R}^n$, $x \prec y$, and

$$\delta_k = \sum_{i=1}^{k} (y^{\downarrow}_i - x^{\downarrow}_i) \qquad 1 \leq k \leq n-1, \tag{1.1}$$

then $\delta_k \geq 0$.
If $\delta_k = 0$, we say that $x \prec y$ has a *coincidence* at k. If $x \prec y$ has a coincidence at k and $y_k \neq y_{k+1}$, we say that $x \prec y$ is *decomposable* at k.

Definition 1.1. A matrix $P \in \mathbf{M}_n$ with nonnegative entries is called *doubly substochastic* if the sums of the entries in each row and each column are less than or equal one.

Theorem 1.2. *[9] The following assertions are true for weakly majorization.*

(i) *[9, A.4] Let $x, y \in \mathbb{R}^n_+$, then $x \prec_w y$ if and only if $x = Py$ for some doubly substochastic matrix $P \in \boldsymbol{M}_n$.*

(ii) *[9, A.9] Let $x, y \in \mathbb{R}^n$, then $x \prec_w y$ if and only if there exists a vector $u \in \mathbb{R}^n$ such that $x \leq u \prec y$.*

The following lemma involves a special kind of linear transformation called a $T-$transform. The matrix of a $T-$transform has the form

$$T = \lambda I + (1-\lambda)Q,$$

where $0 \le \lambda \le 1$ and Q is a permutation matrix that just interchanges two coordinates.

Lemma 1.3. *[9, A.3] Let $x, y \in \mathbb{R}^n$, then $x \prec y$ if and only if x can be derived from y by successive applications of a finite number of $T-$transforms.*

Here, we study weak majorization and related doubly substochastic matrices. We also present that decomposability of $x \prec_w y$ is a necessary and sufficient condition for P to be a direct sum of $D \oplus Q$, where $x = Py$, $D \in \Omega_n$ and Q is a doubly substochastic matrix.
We find an algorithm to construct a vector u such that $x \le u \prec y$. Using this algorithm we write an algorithm to find linear transformations which transform y to x where $x \prec_w y$.

2. Decomposability and weak majorization

Let $x,\ y \in \mathbb{R}^n_+$ and $x \prec_w y$, then by Theorem 1.2 (i) there exists some doubly substochastic matrix P such that $x = Py$. In this section, we find some necessary and sufficient conditions for construction of the matrix P as a direct sum of a doubly stochastic and a doubly substochastic matrices.

Definition 2.1. Let $x, y \in \mathbb{R}^n$, $x \prec_w y$, and

$$\delta_k = \sum_{i=1}^{k} (y_i^{\downarrow} - x_i^{\downarrow}) \qquad 1 \le k \le n-1, \tag{2.1}$$

then $\delta_k \ge 0$. If $\delta_k = 0$, we state that $x \prec_w y$ has a *coincidence* at k. If $x \prec_w y$ has a coincidence at k and $y_k \ne y_{k+1}$, we say that $x \prec_w y$ is *decomposable* at k.

In the following two theorems, we prove that $P = D \oplus Q$ for some $D \in \Omega_k$ and doubly substochastic matrix $Q \in \mathbf{M}_{n-k}$ if and only if there exists some $1 \le k \le n-1$ such that $x \prec_w y$ is decomposable at k.

Theorem 2.2. *Let $x, y \in \mathbb{R}^n_+$. Suppose that $x \prec_w y$ is decomposable at k and $x = Py$ for some doubly substochastic matrix $P \in \boldsymbol{M}_n$. Then there exist matrices $D \in \Omega_k$ and doubly substochastic matrix $Q \in \boldsymbol{M}_{n-k}$ such that $P = D \oplus Q$.*

Proof. Without loss of generality, we can assume that $x,\ y$ are vectors with entries in nonincreasing order. Let $P = (p_{ij})$. Then

$$\begin{aligned}
\textstyle\sum_{i=1}^{k}(x_i - y_k) &= \textstyle\sum_{i=1}^{k}(\sum_{j=1}^{n} p_{ij}y_j - y_k) \\
&\le \textstyle\sum_{i=1}^{k}(\sum_{j=1}^{n} p_{ij}(y_j - y_k)) \\
&= \textstyle\sum_{j=1}^{n}(\sum_{i=1}^{k} p_{ij}(y_j - y_k)) \\
&= \textstyle\sum_{j=1}^{k}\sum_{i=1}^{k} p_{ij}(y_j - y_k) + \sum_{j=k+1}^{n}\sum_{i=1}^{k} p_{ij}(y_j - y_k)
\end{aligned}$$

$y_j - y_k < 0$ for $j > k$, thus

$$\begin{aligned} \sum_{i=1}^{k}(x_i - y_k) &\leq \sum_{j=1}^{k}(\sum_{i=1}^{k} p_{ij}(y_j - y_k)) \\ &= \sum_{j=1}^{k}(y_j - y_k)\sum_{i=1}^{k} p_{ij} \\ &\leq \sum_{j=1}^{k}(y_j - y_k). \end{aligned}$$

But

$$\sum_{i=1}^{k}(x_i - y_k) = \sum_{j=1}^{k}(y_j - y_k).$$

Thus,

$$\sum_{j=k+1}^{n}\sum_{i=1}^{k} p_{ij}(y_j - y_k) = 0.$$

Since $y_j - y_k$ is negative, for all $j = k+1, \ldots, n$, then $p_{ij} = 0$, where $1 \leq i \leq k$ and $k+1 \leq j \leq n$ so that

$$P = \left[\begin{array}{c|c} D & 0 \\ \hline C & Q \end{array}\right].$$

Now, we claim that $C = 0$. We know that $x_i = \sum_{j=1}^{k} p_{ij}y_j$ for $i = 1, \ldots, k$. Then

$$\begin{aligned} \sum_{i=1}^{k} x_i = \sum_{i=1}^{k}\sum_{j=1}^{k} p_{ij}y_j &\implies \sum_{i=1}^{k} y_i = \sum_{i=1}^{k}\sum_{j=1}^{k} p_{ij}y_j \\ &\implies \sum_{i=1}^{k} y_i - \sum_{i=1}^{k}\sum_{j=1}^{k} p_{ij}y_j = 0 \\ &\implies \sum_{i=1}^{k}(1-\alpha_i)y_i = 0, \end{aligned}$$

where $\alpha_i = \sum_{j=1}^{k} p_{ji}$, for $i = 1, \ldots, k$.
So, $1 - \alpha_i = 0$ for $i = 1, \ldots, k$, because $y_i > 0$ and $1 - \alpha_i \geq 0$ for all $i = 1, \ldots, k$ (if $y_k = 0$, then $y_{k+1} = 0$. But we know that $y_k \neq y_{k+1}$). It follows that $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 1$.
As D is a doubly substochastic matrix of order k, we deduce that $D \in \Omega_k$ and $C = 0$.
So that

$$P = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & Q \end{array}\right].$$

□

Here, we state a corollary which one can prove it with the same argument in Theorem 2.2. This corollary omit the decomposable assumption of Theorem 2.2.

Corollary 2.3. *Suppose $x, y \in \mathbb{R}^n_+$, $x \prec_w y$, and $x = Py$ where P is a doubly substochastic matrix. If there is a coincidence at k and $y_1 \geq \ldots \geq y_k = y_{k+1} = \cdots = y_l > y_{l+1}$ where $k < l < n$, then $p_{ij} = 0$, where $1 \leq i \leq k$ and $l+1 \leq j \leq n$.*

Now, we state the converse of Theorem 2.2 which is some necessary conditions for decomposability of $x \prec_w y$.

Theorem 2.4. *Let $x, y \in \mathbb{R}^n_+$ with $x \prec_w y$. If there exists some k $(1 \leq k \leq n)$ that for every $P \in \boldsymbol{M}_n$ such that $x = Py$, we have $P = D \oplus Q$ where $D \in \Omega_k$ and Q is a doubly substochastic matrix of order $n-k$, then $x \prec_w y$ is decomposable at k.*

Proof. Let $x \prec_w y$. Then $x = Py$ for some doubly substochastic matrix P. The hypothesis ensures that $P = D \oplus Q$ where $D \in \Omega_k$ and Q is a doubly substochastic matrix of order $n-k$.

The relation $x = Py$ ensures that $(x_1, \ldots, x_k) = D(y_1, \ldots, y_k)$ where $D \in \Omega_k$, and so $(x_1, \ldots, x_k) \prec (y_1, \ldots, y_k)$. It follows that $\sum_{i=1}^k x_i = \sum_{i=1}^k y_i$. Therefore, $x \prec y$ has a coincidence at k.
Now, we claim that $y_k \neq y_{k+1}$. If not; $y_k = y_{k+1}$. We will construct a matrix P' such that $x = P'y$, but P' is not as a direct sum of some doubly stochastic matrix and a doubly substochastic matrix. Set $P = [P^1 P^2 \ldots P^n]$, where P^i is the ith column of the matrix P. Now, define

$$P' = [P^1 \ldots P^{k-1} P^{k+1} P^k P^{k+2} \ldots P^n].$$

We observe that $x = P'y$. As $y_k = y_{k+1}$ and P has the form given in the hypothesis, we see that P' has the same form which we wanted to create. It is a contradiction. Hence $y_k \neq y_{k+1}$, and so $x \prec_w y$ is decomposable at k. □

3. Algorithms for weak majorization

Let $x, y \in \mathbb{R}^n_+$. This section is devided into two parts. In the first part, with an algorithm, we obtain the vector structure u which $x \leq u \prec y$ when $x \prec_w y$. In the second part, we obtain the structure of linear transformations that convert vector y into vector x in the relation $x \prec_w y$.

3.1. Some middle vector for $x \prec_w y$

Consider $x, y \in \mathbb{R}^n_+$ assuming that $x \prec_w y$. Here, we present some vector u such that $x \leq u \prec_w y$. We see $x \prec_w u \prec_w y$.

Proposition 3.1. *Let $x, y \in \mathbb{R}^n_+$ with $x \prec_w y$. Then there is a vector u such that $x \leq u \prec_w y$ and there exists some l $(1 \leq l \leq n)$ such that $u \prec_w y$ has a coincidence at l. Furthermore, $x_{l+1} \leq y_{l+1}$ whenever $l \neq n$.*

Proof. We consider the various possible cases separately.

Case 1. If $x \prec_w y$ has no coincidence.
Define $\delta = \min_{1 \leq k \leq n} \delta_k$, where δ_k is as in the relation (2.1). So $\delta = \delta_l$ for some $1 \leq l \leq n$. Put $u = x + \delta e_1$ $(e_1 \in \mathbb{R}^n)$. As $\delta > 0$, it follows that $x \leq u$. We observe that $u \prec_w y$, because $x \prec_w y$ and $x_1 + \delta \leq y_1$. We see $u \prec_w y$ has a coincidence at l. Now, we claim that $x_{l+1} \leq y_{l+1}$ whenever $l \neq n$. Otherwise, $x_{l+1} > y_{l+1}$, and it shows that $\delta_{l+1} < \delta_l$, that is a contradiction. Therefore, $x_{l+1} \leq y_{l+1}$.

Case 2. If $x \prec_w y$ has a coincidence at k.
Set $l = k$ and $u = x$. Since $x \prec_w y$ has a coincidence at l, we have $x_{l+1} \leq y_{l+1}$ when $l \neq n$. □

Let $x, y \in \mathbb{R}^n_+$ and $x \prec_w y$. In the following algorithm, we present the vector u which obtained in the Theorem 1.2(ii) of [9] and in the Proposition 3.1 with a stronger condition, see Theorem 3.2.

Algorithm $\mathcal{A}$

Input: Two vectors $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ with $x \prec_w y$.

1. Initialize: Let $u = 0$.

2. for $x \prec_w y$ do

 (a) If $x \prec_w y$ doesn't have any coincidences;

 let $\delta = \min_{1 \le t \le n} \delta_t = \delta_k$, and let $v = (x_1 + \delta, x_2, \ldots, x_k) \in \mathbb{R}^k$.

 set $u = u \oplus v$.

 let $x = (x_{k+1}, \ldots, x_n)$ and $y = (y_{k+1}, \ldots, y_n)$.

 Go on 2.

 (b) If k is the greatest index such that $x \prec_w y$ has a coincidence at k.

 (a') If $k = n$; let $u = u \oplus x$.

 (b') If $k < n$; let $v = (x_1, \ldots, x_k)$.

 set $u = u \oplus v$.

 let $x = (x_{k+1}, \ldots, x_n)$ and $y = (y_{k+1}, \ldots, y_n)$.

 Go on 2.

Output: u.

In the following theorem, we find the relations between x and y with u obtained from the Algorithm $\mathcal{A}$.

Theorem 3.2. *Suppose $x, y \in \mathbb{R}^n_+$ with $x \prec_w y$. Then Algorithm $\mathcal{A}$ offers some $u \in \mathbb{R}^n$ such that $x \le u \prec y$.*

Proof. First, we claim that

$$(x_1, \ldots, x_k) \le v \prec (y_1, \ldots, y_k),$$

where $v = (x_1 + \delta, x_2, \ldots, x_k)$.

As $\delta > 0$, we see $x \le v$. By the definition of δ,

$$\sum_{i=1}^{l} v_i = \delta + \sum_{i=1}^{l} x_i \le \sum_{i=1}^{l} y_i, \quad 1 \le l \le k.$$

Also, as $\delta = \delta_k$, we observe that

$$\sum_{i=1}^{k} v_i = \delta + \sum_{i=1}^{k} x_i = \sum_{i=1}^{k} y_i.$$

It implies that $v \prec y$. Set $X_1 = (x_1, \ldots, x_k), Y_1 = (y_1, \ldots y_k)$ and $U_1 = v$. Now, find v_1 such that $X_2 = (x_{k+1}, \ldots, x_{k_1}) \le v_1 \prec Y_2 = (y_{k+1}, \ldots, y_{k_1})$ then set $U_2 = v_1$. By continuing this process, we can find $U_3, \ldots, U_m$.

Now, since $u = U_1 \oplus U_2 \oplus \cdots \oplus U_m$ and $X_i \le U_i \prec Y_i$ for $i = 1, \ldots, m$ where $x = X_1 \oplus X_2 \oplus \cdots \oplus X_m$ and $y = Y_1 \oplus Y_2 \oplus \cdots \oplus Y_m$, we deduce that $x \le u \prec y$. □

3.2. Linear transformations and weak majorization

The following theorems are existential theorems which are in the book Marshall [9]. Here, for $x, y \in \mathbb{R}^n_+$ with $x \prec_w y$ we write an algorithm that presents the structure of linear transformations expressed in these theorems.

Theorem 3.3. *[9] Let $x, y \in \mathbb{R}^n_+$. The following conditions are equivalent.*
(i) $x \prec_w y$;
(ii) x can be derived from y by successive applications of a finite number of T-transforms followed by a finite number of transformations of the form

$$T_\alpha(z) = (z_1, \ldots, z_{i-1}, \alpha z_i, z_{i+1}, \ldots, z_n), \quad 0 \leq \alpha \leq 1.$$

Theorem 3.4. *[9] Let $x, y \in \mathbb{R}^n_+$ with $x \prec_w y$. Then there exist some T-transforms $T_1, \ldots, T_m$ and some linear transformations $T_{\alpha_1}, \ldots, T_{\alpha_l}$ such that*

$$x = T_{\alpha_1} \ldots T_{\alpha_l} T_1, \ldots, T_m y.$$

Proof. Let $x \prec_w y$. Theorem 2.4 ensures that there is some u such that $x \leq u \prec y$. As $u \prec y$, there exist T-transforms $T_1, \ldots, T_m$ such that $u = T_1 \ldots T_m y$.
Also, since $x \leq u$, there exist transformations $T_{\alpha_1}, \ldots, T_{\alpha_l}$ such that $x = T_{\alpha_1} \ldots T_{\alpha_l} u$. Now, we conclude that $x = T_{\alpha_1} \ldots T_{\alpha_l} T_1 \ldots T_m y$. □

If $x,\ y \in \mathbb{R}^n$ and $x \prec y$, then by Lemma 1.3 there exist $T-$transforms $T_1, \ldots, T_k$ such that $x = T_1 \cdots T_k y$. Next, Algorithms $\mathcal{B}$ and C express the structures of $T_1, \ldots, T_k$.

Algorithm $\mathcal{B}$

Input: Two vectors $x^\downarrow = (x^\downarrow_1, \ldots, x^\downarrow_n), y^\downarrow = (y^\downarrow_1, \ldots, y^\downarrow_n) \in \mathbb{R}^n$ with $x \prec y$.
1. Initialize: Let $Q = I$.
2. for $x \prec y$ do
let $j = \max\{i : x_i < y_i\}$,
let $k = \min\{i : j < i, x_i > y_i\}$,
let $\delta = \min\{y_j - x_j, x_k - y_k\}$,
let $\lambda = 1 - \frac{\delta}{y_j - y_k}$,
set $Q \longmapsto$ interchange the ith and jth rows of Q.
Output: Q.

The $\mathcal{A}(x, y)$ symbol used for the subalgorithm means to put the inputs x and y in the algorithm $\mathcal{A}$.

Algorithm C

Input: Two vectors $x^\downarrow = (x^\downarrow_1, \ldots, x^\downarrow_n), y^\downarrow = (y^\downarrow_1, \ldots, y^\downarrow_n) \in \mathbb{R}^n$ with $x \prec y$.
1. $1 \longmapsto i$.
2. for $x \prec y$ do
(a) If $x = y$, then Go on 2.
(b) $Q = \mathcal{B}(x, y)$,
set $T_i = \lambda I + (1 - \lambda) Q$,

set $y = T_i y$.
3. Go on 2.
Output: $T_1, T_2, \dots, T_{i-1}$.

Algorithm $\mathcal{D}$

Input: Two vectors $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$ with $x \prec_w y$.
1. $u = \mathcal{A}(x, y)$
2. $T_1, \dots, T_{i-1} = C(u, y)$
3. $(i : 1, n)$
4. for $x \leq u$ do
   (a) If $x_i = u_i$, then $T_{\alpha_i} = I$ and Go on 4.
   (b) $T_{\alpha_i}(u_1, \dots, u_n) = (x_1, \dots, x_i, u_{i+1}, \dots, u_n)$, Go on 4.
   Output: $T_{\alpha_1}, \dots, T_{\alpha_n}$

Output: $T_{\alpha_1}, \dots T_{\alpha_n}, T_1, \dots, T_{i-1}$

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