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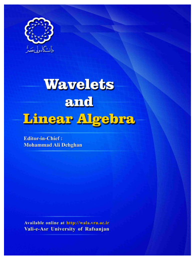

Linear Preservers of Doubly stochastic matrices and permutation matrices from M_m to M_n

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Abstract

Chi-Kwang Li, Bit-Shun Tam and Nam-Kiu Tsing have obtained necessary and sufficient condition for a linear operator on linear space of generalized doubly stochastic matrices to be strong preserver of doubly stochastic matrices and permutation matrices.
We show if a linear operator $T : M_m \to M_n$ is a (strong) preserver of doubly stochastic matrices, then T is a (strong) preserver of the linear manifold of r-generalized doubly stochastic matrices and the linear space of generalized doubly stochastic matrices. Also we give necessary and sufficient condition for a linear operator $T : M_m \to M_n$ to be (strong) preserver of doubly stochastic matrices and permutation matrices.

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1. Introduction and Preliminaries

A matrix D over a field $\mathbb{F}$ ($\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$) having every row and column sum equal to r, is said to be an r-generalized doubly stochastic matrix. It can easily be shown that the only square matrices have such property when $r \neq 0$. The set of all r-generalized doubly stochastic matrices with entries in $\mathbb{F}$ is denoted by $\Omega^r(n, \mathbb{F})$. A generalized doubly stochastic matrix is an element of $\Omega(n, \mathbb{F}) = \cup_{r \in \mathbb{F}} \Omega^r(n, \mathbb{F})$.
A matrix with nonnegative entries in $\Omega^1(n, \mathbb{R})$ is called a doubly stochastic matrix. The set of all doubly stochastic matrices is denoted by Δ_n and the set of all doubly stochastic matrices with positive entries is shown by DP_n. The convex hull of the finitely many points $x_1, \ldots, x_k \in \mathbb{R}^n$ is called a polytope and is denoted by $\mathrm{co}\{x_1, \ldots, x_k\}$. A point x of a polytope P is called vertex or extreme point of P, if $P - \{x\}$ is convex.

Theorem 1.1. *(Birkhoff,1946) [4] The permutation matrices constitute the extreme points of the set of doubly stochastic matrices. Moreover the set of all doubly stochastic matrices is the convex hull of the permutation matrices.*

A linear co-ordinate ψ on a set V is a bijection $\psi : V \to \mathbb{R}^n$. Two sets of linear co-ordinates ψ and ϕ are linearly equivalent if $\psi o \phi^{-1}$ is a linear isomorphism. A linear atlas on a set V is an equivalence class of linear co-ordinates. A linear manifold is a set V with linear atlas, see [6].

Theorem 1.2. *([5], Theorem 4.7) $\Omega^0(n, \mathbb{R})$ is a vector space of dimension* $(n-1)^2$.

Theorem 1.3. *([5], P.108) The linear manifold $\Omega^1(n, \mathbb{R})$ is diffeomorphic to $\Omega^0(n, \mathbb{R})$.*

By Theorems 1.2 and 1.3 we get the following result.

Corollary 1.4. *For any real number r, $\Omega^r(n, \mathbb{R})$ is a linear manifold of dimension* $(n-1)^2$.

Theorem 1.5. $\Omega(n, \mathbb{R})$ *is a linear space of dimension* $(n-1)^2 + 1$.

Proof. Suppose $\beta = \{D_1, D_2, \ldots, D_{(n-1)^2}\}$ is a basis of linear space $\Omega^0(n, \mathbb{R})$ and I is the $n \times n$ identity matrix. We show that the set $\beta \cup \{I\}$ is a basis for $\Omega(n, \mathbb{R})$. If $A \in \Omega^r(n, \mathbb{R})$, then $A - rI \in \Omega^0(n, \mathbb{R})$. Therefore,

$$A - rI = \sum_{i=1}^{(n-1)^2} \alpha_i D_i.$$

Hence

$$A = rI + \sum_{i=1}^{(n-1)^2} \alpha_i D_i$$

and the proof is completed. □

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Δ_n consist of all $n \times n$ matrices $D = [d_{ij}]$ which satisfies the following constraints

$$d_{ij} \geq 0, \quad (i, j = 1, \ldots, n), \tag{1.1}$$

$$\sum_{k=1}^{n} d_{ik} = 1 = \sum_{k=1}^{n} d_{ki}, \quad (i = 1, \ldots, n). \tag{1.2}$$

By Birkhoff's theorem and Corollary 1.4, Δ_n is a polytope in the $(n-1)^2$-dimensional linear manifold $\Omega^1(n, \mathbb{R})$ and so is called Birkhoff's polytope. Permutation matrices are vertices (extreme points) of Δ_n, see [2].
DP_n is an embedded manifold of $\mathbb{R}^{n \times n}$, see [1]. Let $X \in DP_n$, the tangent space $T_X DP_n$ is given by the following theorem:

Theorem 1.6. *([1] P.6 Proposition 1) The tangent space $T_X DP_n$ is*

$$\{Z \in \mathbb{R}^{n \times n} | Z1 = 0, Z^T 1 = 0\},$$

wherein 0 *is all zeros vector and* 1 *is all ones vector.*

From the above theorem, the tangent space of DP_n is equal to linear space $\Omega^0(n, \mathbb{R})$ and by Theorem 1.2, the dimension of DP_n is $(n-1)^2$.
The Riemannian metric g on $T_X DP_n$ is defined by

$$g(\eta_X, \lambda_X) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{(\eta_X)_{ij} (\lambda_X)_{ij}}{(X)_{ij}}, \textit{for all } \eta_X, \lambda_X \in T_X DP_n \tag{1.3}$$

see [1].

Given a subset S of vector space V, a linear preserver of S is a linear operator ϕ on$V$ such that $\phi(S) \subseteq S$ and is called strong linear preserver of S if $\phi(S) = S$.
Linear preserver problem is one of the most active and continuing subjects in the matrix theory during the past century that leaves certain properties invariant.

Theorem 1.7. *([3], Theorem 2.2) Let T be a linear map on linear space $\Omega(n, \mathbb{R})$. The following conditions are equivalent:*

(i) $T(\Delta_n) = \Delta_n$,

(ii) $T(\pi_n) = \pi_n$ *(π_n is the set of all permutation matrices),*

(iii) *T is of the form $X \to PXQ$ or $X \to PX^tQ$ for some $P, Q \in \pi_n$.*

2. Linear preservers of doubly stochastic matrices

In this section, we give some examples of linear preservers of doubly stochastic matrices. We also give a necessary and sufficient condition for that a linear operator preserves doubly stochastic matrices and r-generalized doubly stochastic matrices.

Definition 2.1. Suppose that $m, n \in \mathbb{N}$ and $T : M_m \to M_n$ is a linear operator. Then T is called a preserver of doubly stochastic matrices if $T(\Delta_m) \subseteq \Delta_n$ and a strong preserver of doubly stochastic matrices if $T(\Delta_m) = \Delta_n$.

We denote the set of all preservers of doubly stochastic matrices by $P_{m,n}$ and the set of all strong preservers of doubly stochastic matrices by $SP_{m,n}$.
It is clear that

- $SP_{m,n} \subseteq P_{m,n}$.
- If $T_1 \in P_{m,n}$ and $T_2 \in P_{n,k}$, then $T_2 o T_1 \in P_{m,k}$.
- If $T_1 \in SP_{m,n}$ and $T_2 \in SP_{n,k}$, then $T_2 o T_1 \in SP_{m,k}$.
- $P_{m,n}$ is convex.

Example 2.2. Let $T : M_2 \to M_4$ be defined by

$$TX = \begin{bmatrix} x_{11} & 0 & x_{12} & 0 \\ 0 & x_{11} & 0 & x_{12} \\ x_{21} & 0 & x_{22} & 0 \\ 0 & x_{21} & 0 & x_{22} \end{bmatrix},$$

where $X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$.

If $X \in \Delta_2$, then $TX \in \Delta_4$. Therefore, $T \in P_{2,4}$ but $T \notin SP_{2,4}$, because for example let

$$D = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix}.$$

Thus $D \notin \mathrm{I}m(T)$ and therefore, $D \notin T(\Delta_2)$.

Theorem 2.3. *Suppose that $m, n \in \mathbb{N}$ and $T : M_m \to M_n$ be a linear operator. If T is a (strong) preserver of doubly stochastic matrices, then T is a (strong) preserver of linear manifold of r-generalized doubly stochastic matrices for all real number r and also T is a (strong) preserver of generalized doubly stochastic matrices.*

Proof. As a consequence of the Birkhoff's theorem, the linear space generated by doubly stochastic matrices is equal to the linear space generated by the permutation matrices and this is the linear space of generalized doubly stochastic matrices.
Suppose that $A \in \Omega^r(m,\mathbb{R})$. As the above, $A = \alpha_1 D_1 + \alpha_2 D_2 + \cdots + \alpha_k D_k$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_k = r$ and $D_i \in \Delta_m (i = 1, 2, \ldots, k)$.
We have

$$T(A) = \alpha_1 T(D_1) + \cdots + \alpha_k T(D_k).$$

Since $T \in P_{m,n}$, we have $T(D_i) \in \Delta_n$ and therefore $T(A) \in \Omega^r(n,\mathbb{R})$.
Now if T is a strong preserver of Δ_m and $B \in \Omega^r(n,\mathbb{R})$, there exist $H_i \in \Delta_n (i = 1, \ldots, k)$ and $\alpha_i \in \mathbb{R}(i = 1, \ldots, k)$ such that $\alpha_1 + \cdots + \alpha_k = r$ and $B = \alpha_1 H_1 + \cdots + \alpha_k H_k$. Since $T \in SP_{m,n}$, then for any H_i there exist $D_i \in \Delta_n$ such that $T(D_i) = H_i$, thus

$$B = \alpha_1 T(D_1) + \cdots + \alpha_k T(D_k).$$

We put $A = \alpha_1 D_1 + \alpha_2 D_2 + \cdots + \alpha_k D_k$, then we have $T(A) = B$. This means that T is a strong linear preserver of the linear manifold of r-generalized doubly stochastic matrices and as a result T is a strong preserver of the linear space of generalized doubly stochastic matrices. □

Example 2.4. Suppose that $3 \leq n \in \mathbb{N}, 1 \leq i \leq n$ and $D_1, D_2, \ldots, D_n \in \Delta_n$. Consider the linear operator $T : M_n \to M_n$ is defined by

$$TX = x_{i1} D_1 + x_{i2} D_2 + \cdots + x_{in} D_n,$$

where $(x_{i1}, \ldots, x_{in})$ is the i-th row of X.
If $X \in \Delta_n$, then TX is a convex linear combination of $D_1, \ldots, D_n$ and therefore, $TX \in \Delta_n$ and $T \in P_{n,n}$.
Now we prove $T \notin SP_{n,n}$. By contradiction suppose that T is a strong presever of Δ_n. By Theorem 2.3, T is a strong preserver of linear space $\Omega(n,\mathbb{R})$. So by Theorems 1.5, $(n-1)^2+1 = \dim\Omega(n,\mathbb{R}) = \dim\langle D_1, \ldots, D_n\rangle \leq n$, as a result $n = 1$ or $n = 2$ that is in contradiction by $n \geq 3$.

Remark 2.5. In the above example for $n = 1$ the only doubly stochastic matrix is $D = [1]$. Therefore, in this case $T \in SP_1$ and $SP_1 = \{T : \mathbb{R} \to \mathbb{R} | T(1) = 1$ and T is linear $\} = \{Id : \mathbb{R} \to \mathbb{R}\}$.
But if $n = 2$, then $TX = x_{i1} D_1 + x_{i2} D_2$, then $T \in SP_2$ iff $\mathrm{co}(D_1, D_2) = \Delta_2$ iff $\{D_1, D_2\} = \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\}$.

Example 2.6. Suppose that $P_1, P_2 : \mathbb{R}^n \to \mathbb{R}^n$ are two permutations and

$$T : M_n \to M_n$$

is defined by

$$TX = P_1 X P_2.$$

Since Δ_n is closed under multiplication operation, then for all $X \in \Delta_n$, $P_1 X P_2 \in \Delta_n$. Therefore, T is a linear preserver of doubly stochastic matrices. In fact T is strong linear preserver of doubly stochastic matrices, because if $D \in \Delta_n$, then $P_1^{-1} D P_2^{-1} \in \Delta_n$ and $T(P_1^{-1} D P_2^{-1}) = D$. Therefore, $T(\Delta_n) = \Delta_n$ that means $T \in SP_{n,n}$.

Theorem 2.7. *Suppose that $m, n \in \mathbb{N}$.*

(i) *If $D \in \Delta_n$, then there exists $T \in P_{m,n}$ such that $T(\Delta_m) = \{D\}$.*

(ii) $P_{m,n} \neq \emptyset$.

Proof. It is enough to define the operator $T : M_m \to M_n$ by

$$TX = (x_{11} + \cdots + x_{1n})D.$$

Then $T(\Delta_m) = \{D\}$ and therefore, $P_{m,n} \neq \emptyset$. □

Theorem 2.8. *A linear operator $T : M_m \to M_n$ is a preserver of doubly stochastic matrices, if and only if, for every permutation $P \in M_m$, $TP \in \Delta_n$.*

Proof. Let $T \in P_{m,n}$. Since every permutation $P \in M_m$ is doubly stochastic matrix, so $TP \in \Delta_n$. Conversely suppose $D \in \Delta_m$. By Birkhoff's theorem there exist non-negative real numbers $\lambda_1, \ldots, \lambda_k$ and permutations $P_1, \ldots, P_k \in M_m$ such that

$$D = \lambda_1 P_1 + \cdots + \lambda_k P_k.$$

Now $TP_1, \ldots, TP_k \in \Delta_n$ and Δ_n is a convex set. Hence,
$TD = \lambda_1 TP_1 + \cdots + \lambda_k TP_k \in \Delta_n$. □

Remark 2.9. Since for every permutation $P : \mathbb{R}^m \to \mathbb{R}^m$, we have $P = I_\theta$, where $\theta : \{1, \ldots, m\} \to \{1, \ldots, m\}$ is a $1-1$ and on-to function and I is identity matrix and for any matrix $M = [a_{ij}]$, $M_\theta = [a_{i\theta(j)}]$, so according to Theorem 2.8, the linear operator $T : M_m \to M_m$ is a preserver of doubly stochastic matrices if and only if for every $1-1$ and on-to function $\theta : \{1, \cdots, m\} \to \{1, \ldots, m\}$; $T(I_\theta) \in \Delta_m$.

3. Strong Linear preservers of doubly stochastic matrices

In this section we give some examples of strong preservers of doubly stochastic matrices. We also determine the condition of existence and the structure of strong preserver of doubly stochastic matrices.

Theorem 3.1. *If $T : M_m \to M_n$ is a strong preserver of doubly stochastic matrices, then $m \geq n$.*

Proof. By Definition 2.1, we have $T(\Delta_m) = \Delta_n$. Then by Theorem 1.5, and 2.8, $(n-1)^2 + 1 = \dim\Omega(n, \mathbb{R}) = \dim T(\Omega(m.\mathbb{R})) \leq \dim\Omega(m, \mathbb{R}) = (m-1)^2 + 1$ therefore, $m \geq n$. □

Remark 3.2. Example 2.6, shows that $SP_{n,n} \neq \emptyset$ (actually $SP_{n,n}$ has simple members like identity and Transpose operators). But we saw the linear operator is defined in Example 2.2, preserves (not strongly preserves) the Birkhoff's polytope. Actually by Theorem 3.1, $SP_{2,4} = \emptyset$ and also if $m < n$, then $SP_{m,n} = \emptyset$.

Now we give an example of operator $T : M_3 \to M_2$, where $T \in SP_{3,2}$.

Example 3.3. The following matrices
$P_1 = Id = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}, P_2 = \begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\1 & 0 & 0\end{bmatrix}, P_3 = \begin{bmatrix}0 & 0 & 1\\1 & 0 & 0\\0 & 1 & 0\end{bmatrix}, P_4 = \begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix},$
$P_5 = \begin{bmatrix}0 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{bmatrix}$ and $P_6 = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix};$
are all permutation matrices in M_3, also $\{P_1, \dots, P_5\}$ is linear independent. But

$$P_1 + P_2 + P_3 = P_4 + P_5 + P_6. \tag{3.1}$$

That means $\{P_1, \dots, P_5, P_6\}$ is linear dependent.
Now, we define $T : M_3 \to M_2$ as follows

$$T(P_1) = T(P_3) = T(P_4) = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, T(P_2) = T(P_5) = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}.$$

By (3.1) it will be obtained

$$T(P_6) = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}.$$

We show $T \in SP_{3,2}$.
By Birkhoff's theorem if $D \in \Delta_3$, then there exist $\alpha_i \geq 0\,(1 \leq i \leq 6)$ such that $\sum_{i=1}^{6} \alpha_i = 1$ and $D = \sum_{i=1}^{6} \alpha_i P_i$. Therefore,

$$T(D) = \sum_{i=1}^{6} \alpha_i T(P_i) = (\alpha_1 + \alpha_3 + \alpha_4 + \alpha_6)\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + (\alpha_2 + \alpha_5)\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} =$$

$$\begin{bmatrix}\alpha_1 + \alpha_3 + \alpha_4 + \alpha_6 & \alpha_2 + \alpha_5\\ \alpha_2 + \alpha_5 & \alpha_1 + \alpha_3 + \alpha_4 + \alpha_6\end{bmatrix} \in \Delta_2.$$

So $T \in P_{3,2}$ and therefore, $T(\Delta_3) \subseteq \Delta_2$.
In addition, since $T(P_1) = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, T(P_5) = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$, we have

$$\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\} \subseteq T(\Delta_3) \subseteq \Delta_2.$$

We know $T(\Delta_2)$ is a convex set, then

$$\Delta_2 = \text{co}\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\} \subseteq T(\Delta_3) \subseteq \Delta_2.$$

Now as a result $T(\Delta_3) = \Delta_2$, that means $T \in SP_{3,2}$.

We use the next lemma to determine the structure of strong linear preserver of doubly stochastic matrices.

Lemma 3.4. *If $T : M_m \to M_n$ is a linear operator, then $T(\Delta_m) = \text{co}(T(\pi_m))$.*

Proof. It is clear that a permutation matrix is a doubly stochastic matrix. Therefore $\pi_m \subseteq \Delta_m$ and so $T(\pi_m) \subseteq T(\Delta_m)$. On the other hand T is linear and Δ_m is convex, then $T(\Delta_m)$ is convex and as a result

$$\mathrm{co}(T(\pi_m)) \subseteq T(\Delta_m). \tag{3.2}$$

Now if $D \in \Delta_m$ by using Birkhoff's theorem there exist $\alpha_i \geq 0 (1 \leq i \leq k)$, $\sum_{i=1}^{k} \alpha_i = 1$ and $P_1, P_2, \ldots, P_k \in \pi_m$ such that $D = \sum_{i=1}^{k} \alpha_i P_i$.
Therefore

$$T(D) = T(\sum_{i=1}^{k} \alpha_i P_i) = \sum_{i=1}^{k} \alpha_i T(P_i) \in \mathrm{co}(T(\pi_m)).$$

Hence

$$T(\Delta_m) \subseteq \mathrm{co}(T(\pi_m)). \tag{3.3}$$

From (3.2) and (3.3) $T(\Delta_m) = \mathrm{co}(T(\pi_m))$. □

The next lemma shows that if the convex hull of a finite subset of M_n equals to Δ_n, it necessary contains all permutation matrices.

Lemma 3.5. *For matrices $A_1, \ldots, A_k \in M_n$ if $\mathrm{co}\{A_1, \ldots, A_k\} = \Delta_n$, then*

(a) $k \geq n!$,

(b) $\pi_n \subseteq \{A_1, \ldots, A_k\}$.

Proof. Since π has $n!$ members, then (a) is the direct result of (b) so we just prove (b).
First, note that the relation $\mathrm{co}\{A_1, \ldots, A_k\} = \Delta_n$ expresses that every $A_i, (1 \leq i \leq k)$ is doubly stochastic matrix.
Now, suppose $A_1 = [a^1_{ij}], \ldots, A_k = [a^k_{ij}]$. Also consider $P = [p_{ij}] : \mathbb{R}^n \to \mathbb{R}^n$ as a permutation, we show $P \in \{A_1, \ldots, A_k\}$.
It is clear that each $P_{ij} = 0$ or 1, on the other hand $P \in \Delta_n = \mathrm{co}\{A_1, \ldots, A_k\}$. Then there exist real numbers $0 \leq \lambda_1, \ldots, \lambda_k \leq 1$ such that $P = \lambda_1 A_1 + \cdots + \lambda_k A_k$.
Let $\{i : \lambda_i > 0\} = \{i_1, \ldots, i_l\}$. Without loss of generality, we may assume that $i_1 = 1, \ldots, i_l = l$, and so $\lambda_i = 0$, for any $l + 1 \leq i \leq k$. Using the recent relation we have

$$P_{ij} = \lambda_1 a^1_{ij} + \cdots + \lambda_l a^l_{ij} \leq \lambda_1 + \cdots + \lambda_l = 1, \ (i, j = 1, \ldots, n). \tag{3.4}$$

If for i and j, $1 \leq i, j \leq n$, $P_{ij} = 1$, then because $0 \leq a^1_{ij}, \ldots, a^l_{ij} \leq 1$, it follows that

$$a^1_{ij} = \ldots = a^l_{ij} = 1 = P_{ij}. \tag{3.5}$$

Also if $P_{ij} = 0$, by (3.4)

$$a^1_{ij} = \cdots = a^l_{ij} = 0 = P_{ij}. \tag{3.6}$$

From (3.5) and (3.6) it turns out $P = A_1 = \cdots = A_l$. Therefore $P \in \{A_1, \ldots, A_k\}$. □

Theorem 3.6. *Suppose $m \geq n$ and $T : M_m \to M_n$ is an arbitrary linear operator. In this case $T \in SP_{m,n}$ if and only if $T \in P_{m,n}$ and there exist permutations $P_1, \cdots, P_{n!} \in M_m$ such that $\{TP_1, \ldots, TP_{n!}\} = \pi_n$.*

Proof. Let $T \in SP_{m,n}$. It is clear that $T \in P_{m,n}$. On the other hand by Lemma 3.4 we have $\text{co}(T(\pi_m)) = T(\Delta_m) = \Delta_n$.
Since π_m is a finite set (and has m! members), then $T(\pi_m)$ is a finite set. Lemma 3.5 concludes that $\pi_n \subseteq T(\pi_m)$ and thus there exist $P_1, \ldots, P_{n!}$ in π_m such that $\{TP_1, \ldots, TP_{n!}\} = \pi_n$.
Conversely because $T \in P_{m,n}$, so

$$T(\Delta_m) \subseteq \Delta_n. \tag{3.7}$$

On the other hand by assumptions we have

$$\pi_n = \{T(P_1), \ldots, T(P_{n!})\} \subseteq T(\pi_m) \subseteq T(\Delta_m). \tag{3.8}$$

It follows from (3.7) and (3.8), $T \in SP_{m,n}$. □

From the above theorem we get immediately the following corollary:

Corollary 3.7. *The operator $T : M_m \to M_n \in SP_{m,n}$ if and only if the following conditions are met:*

(a) *If $P \in M_m$ is a permutation matrix, then $T(P) \in \Delta_n$,*

(b) *There exist permutation matrices $P_1, \ldots, P_{n!} \in M_m$ such that $T(P_1), \ldots, T(P_{n!})$ are all permutation matrices of M_n.*

Example 3.8. Suppose that $T : M_4 \to M_2$ is defined as follows:

$$TX = \frac{1}{2}\begin{bmatrix} x_{11} + x_{12} + x_{21} + x_{22} & x_{13} + x_{14} + x_{23} + x_{24} \\ x_{31} + x_{32} + x_{41} + x_{42} & x_{33} + x_{34} + x_{43} + x_{44} \end{bmatrix}, X = [x_{ij}], 1 \leq i, j \leq 4.$$

If $X \in \Delta_4$, then by a simple calculation we see that $TX \in \Delta_2$, and so $T \in P_{4,2}$.
On the other hand because $T(Id_4) = Id_2$ and $T\Big(\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\Big) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,
according to Theorem 3.6, $T \in SP_{4,2}$.

Remark 3.9. If $T \in SP_{m,n}$, then $\text{rank}T \geq (n-1)^2 + 1$. By Theorem 2.8, T is strong preserver of vector space of generalized doubly stochastic matrices, i.e. $T\Omega_m = \Omega_n$. Thus $\text{rank}T = \dim\text{Im}T \geq \dim T(\Omega_m) = \dim\Omega_n = (n-1)^2 + 1$.

Example 3.10. Suppose $D_1, \ldots, D_6$ are doubly stochastic matrices in Δ_5, also let $\sum_{i=1}^{6} \lambda_i = 1$, $\lambda_1, \ldots, \lambda_6 \geq 0$, and let $T : M_6 \to M_5$ has the following formula:

$$TX = \lambda_1(\sum_{j=1}^{6} x_{1j})D_1 + \cdots + \lambda_6(\sum_{j=1}^{6} x_{6j})D_6.$$

Now if $X \in \Delta_6$, then

$$TX = \lambda_1 D_1 + \cdots + \lambda_6 D_6 \in \mathrm{co}(\Delta_5) = \Delta_5.$$

Therefore, $T \in P_{6,5}$. If $T \in SP_{6,5}$ by previous remark, we have $26 = 5^2 + 1 \leq \mathrm{rank}T = \dim\langle D_1, \ldots, D_6\rangle \leq 6$, that it is a contradiction. Therefore, $T \notin SP_{6,5}$.

Theorem 2.7, states that for each $m, n \in \mathbb{N}$, we have $P_{m,n} \neq \emptyset$. According to the Remark 3.2, we also have $SP_{m,n} = \emptyset$, for $m < n$. Now, the question arises is whether $SP_{m,n} \neq \emptyset$, for every $m, n \in \mathbb{N}$ with $m \geq n$? From corollary 3.7 we get the affirmative answer to the problem if for any $m \geq n$ there exists a linear operator $T : M_m \to M_n$ that satisfies conditions (a) and (b) of Corollary 3.7.
Actually because the identity linear operator is in $SP_{n,n}$, then $SP_{n,n} \neq \emptyset$, also according to Examples 3.3, and 3.8, we have $SP_{3,2} \neq \emptyset$ and $SP_{4,2} \neq \emptyset$.
It can be said that, $SP_{m,n} \neq \emptyset$ for every $m > n$, such that $m, n \in \mathbb{N}$ is equivalent to $SP_{n+1,n} \neq \emptyset$ for every $n \in \mathbb{N}$. Because if $SP_{n+1,n} \neq \emptyset$ for every $n \in \mathbb{N}$, then there exist linear operators $T_1 \in SP_{m,m-1}, T_2 \in SP_{m-1,m-2}, \ldots, T_{m-n} \in SP_{n,n-1}$. Therefor the combination of them $T_{m-n} \ldots T_2 T_1$ is located in $SP_{m,n}$.

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