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Some inequalities related to 4-convex functions

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Abstract

In this paper, we consider the class of 4-convex functions and we obtain some inequalities related to 4-convex functions. Moreover, for $k \leq n$, we present a majorization $\prec_k$ on $\mathbb{R}_n$ and we give some equivalent conditions for $\prec_4$ on $\mathbb{R}_4$.

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1. Introduction

Let $x = (x_1, \ldots, x_n) \in \mathbb{R}_n$, where $x_i \geq 0$ and let p be a nonzero real number. The power mean of x is defined as $L_p(x) := \left(\frac{1}{n}\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}}$. We know that $\lim_{p\to+\infty} L_p(x) = max\{x_1, \ldots, x_n\}$, see [5]. The k^{th} order divided difference of $f : [a, b] \longrightarrow \mathbb{R}$ at distinct points $x_0, \ldots, x_n$ in $[a, b]$ is defined by $f[x_i] := f(x_i)$, and for $1 \leq k \leq n$,

$$f[x_0, \ldots, x_k] := \frac{f[x_1, \ldots, x_k] - f[x_0, \ldots, x_{k-1}]}{x_k - x_0}. \tag{1.1}$$

Also, we define $f[x, x] := \lim_{y\to x} f[x, y] = f'(x)$.

Convex function is appear in many fields of mathematics. In the last century mathematicians introduced and investigated many generalizations of convexity. The notion of nth order convexity or n-convexity was defined in terms of divided differences. The concept of n-convexity are motived by some basic questions in optimization and convex programming. In this paper, we use n-convexity to introduce a new concept of majorization.

It is perfectly reasonable, then, to consider new forms of majorization for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}_n$, wherein inequality $\sum_{i=1}^{n} f(x_i) \leq \sum_{i=1}^{n} f(y_i)$ is assumed to hold for the class of n-convex functions instead of convex ones. This is the theme of our paper.

Definition 1.1. Let $n \geq 0$. A function $f : [a, b] \longrightarrow \mathbb{R}$ is said to be n-convex on $[a, b]$ if $f[x_0, \ldots, x_n] \geq 0$, where $x_i \in [a, b]$, $i = 0, 1, \ldots, n$.

Let F is a real- valued function defined on the bounded closed interval $[a, b]$ and given the $(r + 1)$ points P_k, $0 \leq k \leq r$, with coordinates $(x_k, F(x_k))$, $0 \leq k \leq r$, respectively, there is a unique polynomial of degree at most r passing through these points given by

$$\pi_r(F; x; P_k) = \pi_r(x; P_k) = \sum_{k=0}^{r} F(x_k) \prod_{j=0, j\neq k}^{r} \frac{(x - x_j)}{(x_k - x_j)}.$$

Theorem 1.2. *[3, Theorem 5] Let*

$$P_k = (x_k, y_k),\ 1 \leq k \leq n,\ n \geq 2,\ a \leq x_1 < \ldots < x_n \leq b,$$

be any n distinct points on the graph of the function F. Then F is $n-$convex if and only if for all such sets of n distinct points, the graph lies alternately above and below the curve $y = \pi_{n-1}(F; x; P_k)$, lying below if $x_{n-1} \leq x \leq x_n$. Further $\pi_{n-1}(x; P_k) \leq F(x)$, $x_n \leq x \leq b$; and $\pi_{n-1}(x; P_k) \leq F(x)(\geq F(x))$ if $a \leq x < x_1$, n being even (odd).

Definition 1.3. Let $x, y \in \mathbb{R}_n$. Then x is said to be majorized by y, written $x \prec y$, if

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]},$$

for $k = 1, \ldots, n$ with equality at $k = n$, where $x_{[i]}$ and $y_{[i]}$ are the i^{th} largest component of the vectors x and y respectively.

The following theorem characterizes majorization in terms of convex (2-convex) functions on $\mathbb{R}$.

Theorem 1.4. *[5, Theorem 108] Let $x, y \in \mathbb{R}_n$. Then the following statements are equivalent:*

1. $x \prec y$
2. $\sum_{i=1}^{n} f(x_i) \leq \sum_{i=1}^{n} f(y_i)$, *for all convex functions* $f : \mathbb{R} \longrightarrow \mathbb{R}$.

Furthermore, if f is strictly convex, then the equality can occur, only when two vectors x and y are permutations of each other.

In [1, 2, 6], the authors presented some consequences of inequalities describing the behavior of 3-convex functions.

Theorem 1.5. *[2, Theorem 2] Suppose that $x_1, x_2, x_3, y_1, y_2, y_3$ are real numbers. Then the inequality $\sum_{i=1}^{3} f(x_i) \leq \sum_{i=1}^{3} f(y_i)$ is valid for all 3-convex functions $f : \mathbb{R} \to \mathbb{R}$ if and only if*

$$\begin{aligned} x_1 + x_2 + x_3 &= y_1 + y_2 + y_3, \\ x_1^2 + x_2^2 + x_3^2 &= y_1^2 + y_2^2 + y_3^2, \\ max\{x_1, x_2, x_3\} &\leq max\{y_1, y_2, y_3\}. \end{aligned}$$

In this note, we state an extension of these results for 4-convex functions. Let $x_1, \ldots, x_n$ be variables. For $k \geq 1$, the k^{th} power sum is denoted by

$$p_k(x_1, \ldots, x_n) := \sum_{i=1}^{n} x_i^k = x_1^k + \cdots + x_n^k. \tag{1.2}$$

Let $1 \leq k \leq n$. The k^{th} elementary symmetric polynomial (that is, the sum of all distinct products of k distinct variables) is denoted by

$$e_k(x_1, \ldots, x_n) := \sum_{1 \leq l_1 < \cdots < l_k \leq n} x_{l_1} \cdots x_{l_k}, \quad \& \quad e_0(x_1, \ldots, x_n) = 1. \tag{1.3}$$

Newton's identities, can be used to recursively express elementary symmetric polynomials in terms of power sums (for more information see [7]).

$$ke_k(x_1, \ldots, x_n) = \sum_{i=1}^{k} (-1)^{i-1} e_{k-i}(x_1, \ldots, x_n) p_i(x_1, \ldots, x_n). \tag{1.4}$$

Let $f(x) := x^n + \sum_{k=1}^{n} a_{n-k} x^{n-k} = \prod_{k=1}^{n} (x - \alpha_k)$. By Vieta's formulas [4], for $1 \leq k \leq n$,

$$a_{n-k} = (-1)^k e_k(\alpha_1, \ldots, \alpha_n). \tag{1.5}$$

It is clear that $a_0 = e_n(\alpha_1, \ldots, \alpha_n) = \prod_{i=1}^{n} \alpha_i$ and $a_{n-1} = e_1(\alpha_1, \ldots, \alpha_n) = \sum_{i=1}^{n} \alpha_i$.

2. 4-Convex functions

In this section, we will state two key lemmas to find some simpler conditions for inequalities on 4-convex functions.

Lemma 2.1. *Let the polynomials* $f(x) = x^n + \sum_{k=1}^{n} a_{n-k}x^{n-k} = \prod_{k=1}^{n}(x - \alpha_k)$ *and* $g(x) = x^n + \sum_{k=1}^{n} b_{n-k}x^{n-k} = \prod_{k=1}^{n}(x - \beta_k)$ *be given. If* $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ *and there exists* $1 \le m \le n$ *such that* $p_j(\alpha) = p_j(\beta)$, *for all* $1 \le j \le m - 1$, *then* $f - g$ *is a polynomial of degree less than or equal* $n - m$. *In particular, if m=n, then* $f - g$ *is a constant polynomial.*

Proof. We will show that $e_j(\alpha) = e_j(\beta),\ \ 1 \le j \le m - 1$. By (1.4), for $x = (x_1, \ldots, x_n) \in \mathbb{R}_n$,

$$ke_k(x) = \sum_{i=1}^{k}(-1)^{i-1}e_{k-i}(x)p_i(x),\ 1 \le k \le n. \tag{2.1}$$

By taking $k = 1$ and $x = \alpha$ in (2.1), $e_1(\alpha) = p_1(\alpha)$. Since $p_1(\alpha) = p_1(\beta)$, we obtain that $e_1(\alpha) = e_1(\beta)$. Also, by taking $k = 2$ and $x = \alpha$ in (2.1), we have

$$2e_2(\alpha) = e_1(\alpha)p_1(\alpha) - p_2(\alpha).$$

Since $p_j(\alpha) = p_j(\beta)$, $j = 1, 2$ and $e_1(\alpha) = e_1(\beta)$, we obtain that $e_2(\alpha) = e_2(\beta)$. By (1.4), we know that e_k can be written recursively in terms of power sums p_k. Now by continuing this method $e_i(\alpha) = e_i(\beta)$, $1 \le i \le m-1$. Then by (1.5), $a_{n-k} = (-1)^k e_k(\alpha) = (-1)^k e_k(\beta) = b_{n-k}$, $k = 1, \ldots, m-1$. Therefore $f - g = (a_{n-m} - b_{n-m})x^{n-m} + \cdots + (a_1 - b_1)x + (a_0 - b_0)$ is a polynomial of degree less than or equal $n - m$. In particular, if m=n, then $f - g = a_0 - b_0$ is a constant polynomial and the proof is complete. □

Lemma 2.2. *Let* $\alpha_k, \beta_k,\ k = 1, \ldots, n$ *be real numbers such that* $\alpha_1 \ge \cdots \ge \alpha_n$ *and* $\beta_1 \ge \cdots \ge \beta_n$ *and* $\sum_{k=1}^{n} \alpha_k^j = \sum_{k=1}^{n} \beta_k^j$ *for all* $1 \le j \le n - 1$. *Then the following assertions hold.*

1. *If* $\alpha_p = \beta_q$, *for some* $1 \le p, q \le n$, *then* $\alpha_i = \beta_i$ *for all* $i = 1, \ldots, n$.

2. *If* $\alpha_1 < \beta_1$, *then* $(-1)^{n-1}\alpha_n < (-1)^{n-1}\beta_n$.

Proof. We consider two functions $f, g : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = \prod_{k=1}^{n}(x - \alpha_k)$ and $g(x) = \prod_{k=1}^{n}(x - \beta_k)$. Then, there exist $a_i, b_i \in \mathbb{R},\ i = 0, \ldots, n - 1$ such that $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ and $g(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_0$. By (1.5), $e_k(\alpha) = (-1)^k a_{n-k}$ and $e_k(\beta) = (-1)^k b_{n-k}$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$. By Lemma 2.1, we know that $f(x) - g(x) = a_0 - b_0$, for any $x \in \mathbb{R}$. Define $\gamma := a_0 - b_0$.

1. Let $\alpha_p = \beta_q$ for some $1 \le p, q \le n$. Then $f(\beta_q) = f(\alpha_p) = 0$ and $g(\alpha_p) = g(\beta_q) = 0$. Therefore, $\gamma = f(\alpha_p) - g(\alpha_p) = 0$ and hence $0 = \gamma = f(x) - g(x)$ for any $x \in \mathbb{R}$. Then $\alpha_i = \beta_i$ for all $i = 2, \ldots, n$.

2. Let $\alpha_1 < \beta_1$. We know that α_1 is the largest root of the monic polynomial $f(x)$. Then $f(x) \geq 0$ for any $x \geq \alpha_1$. Since $\beta_1 > \alpha_1$, we obtain that $f(\beta_1) > 0 = g(\beta_1)$ and hence $\gamma = f(\beta_1) - g(\beta_1) > 0$. Now, we consider two cases:
   Case 1: suppose that n is even. We know that β_n is the smallest root of the monic polynomial $g(x)$. Then $g(x) \geq 0$ for any $x \leq \beta_n$. Since $f(x) > g(x)$ for all $x \in \mathbb{R}$, we obtain that $f(x) > 0$ for all $x \leq \beta_n$. Therefore, $\beta_n < \alpha_n$.
   Case 2: suppose that n is odd. We know that α_n is the smallest root of the monic polynomial $f(x)$. By the same method as above, $g(x) < f(x) \leq 0$ for any $x \leq \alpha_n$. Therefore, $\beta_n > \alpha_n$. □

In [2], G. Bennett presented a p-free inequality. Now, by using Lemma 2.2, in the following theorem, we extend [2, Theorem 1].

Theorem 2.3. *Suppose that $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4$ are positive numbers. Then the following inequalities hold:*

$$\begin{aligned} x_1^p + x_2^p + x_3^p + x_4^p &\leq y_1^p + y_2^p + y_3^p + y_4^p, \quad p \in (-\infty, 0] \cup [1, 2] \cup [3, \infty) \\ x_1^p + x_2^p + x_3^p + x_4^p &\geq y_1^p + y_2^p + y_3^p + y_4^p, \quad p \in [0, 1] \cup [2, 3], \end{aligned} \tag{2.2}$$

if and only if the following conditions are satisfied:

$$x_1 + x_2 + x_3 + x_4 = y_1 + y_2 + y_3 + y_4, \tag{2.3}$$
$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2, \tag{2.4}$$
$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = y_1^3 + y_2^3 + y_3^3 + y_4^3, \tag{2.5}$$
$$max\{x_1, x_2, x_3, x_4\} \leq max\{y_1, y_2, y_3, y_4\}. \tag{2.6}$$

Proof. If inequalities in (2.2) hold, then (2.3), (2.4) and (2.5) follow by taking $p = 1, 2, 3$ in (2.2). Now, we rephrase (2.2) in terms of L_p-means, $p \geq 3$.

$$\left(\frac{x_1^p + x_2^p + x_3^p + x_4^p}{4}\right)^{\frac{1}{p}} \leq \left(\frac{y_1^p + y_2^p + y_3^p + y_4^p}{4}\right)^{\frac{1}{p}}. \tag{2.7}$$

Then (2.6) follows by making $p \to +\infty$ in (2.7).
Conversely, we assume that the sets $\{x_1, x_2, x_3, x_4\}$ and $\{y_1, y_2, y_3, y_4\}$ are disjoint. If they have a point in common then by Lemma 2.2 they coincide and the result holds. It will be convenient to assume that the sets $\{x_1, x_2, x_3, x_4\}$ and $\{y_1, y_2, y_3, y_4\}$ are arranged in decreasing order

$$x_1 \geq x_2 \geq x_3 \geq x_4 \quad and \quad y_1 \geq y_2 \geq y_3 \geq y_4.$$

We will show that

$$y_1 > x_1 \geq x_2 > y_2 \geq y_3 > x_3 \geq x_4 > y_4. \tag{2.8}$$

We consider two functions $h, l : \mathbb{R} \longrightarrow \mathbb{R}$, defined by

$$h(x) = \prod_{i=1}^{4}(x - x_i) \quad , \quad l(x) = \prod_{i=1}^{4}(x - y_i).$$

The first and last strict inequalities in (2.8) are followed by (2.6) and Lemma 2.2. If the third inequality fails to hold, then $x_2 \leq y_2$. Now, by (2.6) we have $x_1 + x_2 \leq y_1 + y_2$. Since $x_4 \geq y_4$, by (2.3) we have $x_1 + x_2 + x_3 \leq y_1 + y_2 + y_3$. Then equation (2.3) and Definition 1.3 implies that $(x_1, x_2, x_3, x_4) \prec (y_1, y_2, y_3, y_4)$. By Theorem 1.4 we have

$$f(x_1) + f(x_2) + f(x_3) + f(x_4) < f(y_1) + f(y_2) + f(y_3) + f(y_4),$$

for all strictly convex functions $f : [y_4, y_1] \longrightarrow \mathbb{R}$. By considering the strictly convex function $f(x) = x^2$, we obtain a contradiction by (2.4). Therefore $x_2 > y_2$. If the fifth inequality fails to hold, then $y_3 \leq x_3$. Since $x_4 \geq y_4$, we have $y_3 + y_4 \leq x_3 + x_4$. We deduce from (2.3) that $x_1 + x_2 \leq y_1 + y_2$. The same argument as above implies that $y_3 > x_3$ and hence $y_1 > x_1 \geq x_2 > y_2 \geq y_3 > x_3 \geq x_4 > y_4$. Now, we will show that

$$\int_{y_4}^{x_4} \varphi(x)\,dx + \int_{y_2}^{x_2} \varphi(x)\,dx \leq \int_{x_3}^{y_3} \varphi(x)\,dx + \int_{x_1}^{y_1} \varphi(x)\,dx, \tag{2.9}$$

for all 3-convex functions $\varphi : [y_4, y_1] \longrightarrow \mathbb{R}$. We consider a quadratic function g that agree with φ at x_4, y_3 and x_2. By Theorem 1.2, we know that $\varphi(x) \leq g(x)$ for $x \in [y_4, x_4]$ or $x \in [y_2, x_2]$ and $\varphi(x) \geq g(x)$ for $x \in [x_3, y_3]$ or $x \in [x_1, y_1]$. By (2.3), (2.4) and (2.5), the inequality (2.9) is an equality for g. Therefore,

$$\begin{aligned}\int_{y_4}^{x_4} \varphi(x)\,dx + \int_{y_2}^{x_2} \varphi(x)\,dx &\leq \int_{y_4}^{x_4} g(x)\,dx + \int_{y_2}^{x_2} g(x)\,dx \\ = \int_{x_3}^{y_3} g(x)\,dx + \int_{x_1}^{y_1} g(x)\,dx &\leq \int_{x_3}^{y_3} \varphi(x)\,dx + \int_{x_1}^{y_1} \varphi(x)\,dx.\end{aligned}$$

Now, applying (2.9) to the following 3-convex functions, the result holds.

$$\varphi(x) = \begin{cases} px^{p-1} & p \leq 0 \;\; or \;\; 1 \leq p \leq 2 \;\; or \;\; p \geq 3, \\ -px^{p-1} & 0 \leq p \leq 1 \;\; or \;\; 2 \leq p \leq 3. \end{cases}$$

□

Example 2.4. Let $x_1 = x_2 = 2,\ x_3 = x_4 = 7$ and $y_1 = 1,\ y_2 = 4,\ y_3 = 5,\ y_4 = 8$. Since

$$x_1^i + x_2^i + x_3^i + x_4^i = y_1^i + y_2^i + y_3^i + y_4^i,$$

for $i = 1, 2, 3$ and

$$\max\{x_1, x_2, x_3, x_4\} \leq \max\{y_1, y_2, y_3, y_4\},$$

for $p \leq 0$ or $1 \leq p \leq 2$ or $p \geq 3$, we have

$$2(2^p) + 2(7^p) \leq 1^p + 4^p + 5^p + 8^p.$$

The inequality reverses direction if $0 \leq p \leq 1$ or $2 \leq p \leq 3$.

In the following, we define *k-majorization* $\prec_k$ on $\mathbb{R}_n$, $k \leq n$.

Definition 2.5. Let $k \leq n$ be positive integers. The vector $x = (x_1, x_2, \dots, x_n)$ is said to be k-majorized by $y = (y_1, y_2, \dots, y_n)$, denoted by $x \prec_k y$, if $\Sigma_{i=1}^n f(x_i) \leq \Sigma_{i=1}^n f(y_i)$ for all k-convex functions $f : \mathbb{R} \to \mathbb{R}$.

In the following theorem, we extend [2, Theorem 2] for 4-convex functions.

Theorem 2.6. *Let $x = (x_1, x_2, x_3, x_4),\ y = (y_1, y_2, y_3, y_4) \in \mathbb{R}_4$. Then $x \prec_4 y$ if and only if hypotheses (2.3)-(2.6) hold.*

Proof. Let $x \prec_4 y$. Then by choosing $f_j(x) := \pm x^j$ for $j = 1, 2, 3$, we obtain that $x_1^j + x_2^j + x_3^j + x_4^j \leq y_1^j + y_2^j + y_3^j + y_4^j$ and $x_1^j + x_2^j + x_3^j + x_4^j \geq y_1^j + y_2^j + y_3^j + y_4^j,\ j = 1, 2, 3$. Therefore, (2.3)-(2.5) hold. It is enough to show that (2.6) holds. Let $m := max\{y_1, y_2, y_3, y_4\}$. We consider non negative 4-convex function

$$f(x) := \begin{cases} (x-m)^3 & x > m, \\ 0 & x \leq m. \end{cases}$$

Since $x \prec_4 y$ and f is a nonnegative 4-convex function, we obtain that

$$\begin{aligned} &0 \leq f(x_1) + f(x_2) + f(x_3) + f(x_4) \leq f(y_1) + f(y_2) + f(y_3) + f(y_4) \\ &= (y_1 - m)^3 + (y_2 - m)^3 + (y_3 - m)^3 + (y_4 - m)^3 \leq 0. \end{aligned}$$

Therefore, $f(x_1) + f(x_2) + f(x_3) + f(x_4) = 0$ and we obtain that $f(x_1) = f(x_2) = f(x_3) = f(x_4) = 0$. The definition of $f(x)$ implies that $x_i \leq m,\ i = 1, 2, 3, 4$. Then $max\{x_1, x_2, x_3, x_4\} \leq m = max\{y_1, y_2, y_3, y_4\}$ and (2.6) holds. Conversely, By Lemma 2.2, without loss of generality, we assume that the sets $\{x_1, x_2, x_3, x_4\}$ and $\{y_1, y_2, y_3, y_4\}$ are disjoint. It will be convenient to assume that the sets $\{x_1, x_2, x_3, x_4\}$ and $\{y_1, y_2, y_3, y_4\}$ are arranged in decreasing order $x_1 \geq x_2 \geq x_3 \geq x_4$ and $y_1 \geq y_2 \geq y_3 \geq y_4$. Then by the same method as in the proof of Theorem 2.3, we obtain that $y_1 > x_1 \geq x_2 > y_2 \geq y_3 > x_3 \geq x_4 > y_4$. Now, let f be an arbitrary 4-convex function. we consider four cases:

Case 1: Let $y_2 \neq y_3$. Since f is 4−convex function, by using (1.1) several times

$$\begin{aligned} 0 \leq f[x_i, y_1, y_2, y_3, y_4] &= \frac{f[y_1, y_2, y_3, y_4] - f[x_i, y_1, y_2, y_3]}{y_4 - x_i} \\ &= \frac{1}{y_4 - x_i}\left(\frac{f[y_2, y_3, y_4] - f[y_1, y_2, y_3]}{y_4 - y_1} - \frac{f[y_1, y_2, y_3,] - f[x_i, y_1, y_2]}{y_3 - x_i}\right) \\ &= \frac{1}{(y_4 - x_i)(y_4 - y_1)(y_4 - y_2)}\left(\frac{f(y_4) - f(y_3)}{y_4 - y_3} - \frac{f(y_3) - f(y_2)}{y_3 - y_2}\right) \\ &- \frac{1}{(y_4 - x_i)(y_4 - y_1)(y_3 - y_1)}\left(\frac{f(y_3) - f(y_2)}{y_3 - y_2} - \frac{f(y_2) - f(y_1)}{y_2 - y_1}\right) \\ &- \frac{1}{(y_4 - x_i)(y_3 - x_i)(y_3 - y_1)}\left(\frac{f(y_3) - f(y_2)}{y_3 - y_2} - \frac{f(y_2) - f(y_1)}{y_2 - y_1}\right) \\ &+ \frac{1}{(y_4 - x_i)(y_3 - x_i)(y_2 - x_i)}\left(\frac{f(y_2) - f(y_1)}{y_2 - y_1} - \frac{f(y_1) - f(x_i)}{y_1 - x_i}\right). \end{aligned}$$

Easy computations show that

$$\begin{aligned}0 \leq & \frac{f(x_i)}{(y_4 - x_i)(y_3 - x_i)(y_2 - x_i)(y_1 - x_i)} + \frac{f(y_4)}{(y_4 - x_i)(y_4 - y_1)(y_4 - y_2)(y_4 - y_3)} \\ & - \frac{f(y_3)}{(y_4 - y_3)(y_3 - y_2)(y_3 - y_1)(y_3 - x_i)} + \frac{f(y_2)}{(y_4 - y_2)(y_3 - y_2)(y_2 - y_1)(y_2 - x_i)} \\ & - \frac{f(y_1)}{(y_4 - y_1)(y_3 - y_1)(y_2 - y_1)(y_1 - x_i)}.\end{aligned}$$

Therefore $f(x_i) \leq \sum_{k=1}^{4} f(y_k) \prod_{j=1, j\neq k}^{4} \frac{(y_j - x_i)}{(y_j - y_k)}$, $i = 1, 2, 3, 4$ and hence

$$\sum_{i=1}^{4} f(x_i) \leq \sum_{i=1}^{4} \sum_{k=1}^{4} f(y_k) \prod_{j=1, j\neq k}^{4} \frac{(y_j - x_i)}{(y_j - y_k)} = \sum_{k=1}^{4} f(y_k) \left(\sum_{i=1}^{4} \prod_{j=1, j\neq k}^{4} \frac{(y_j - x_i)}{(y_j - y_k)} \right).$$

By using (2.3)-(2.5), we obtain that $\sum_{i=1}^{4} \prod_{j=1, j\neq k}^{4} \frac{(y_j - x_i)}{(y_j - y_k)} = 1$. Then $\sum_{i=1}^{4} f(x_i) \leq \sum_{i=1}^{4} f(y_k)$ for all 4−convex functions, and hence $x \prec_4 y$.

Case 2: Let $y_2 = y_3$, $x_1 \neq x_2$, and $x_3 \neq x_4$. Since f is 4−convex function, the divided difference $f[x_1, x_2, x_3, x_4, y_i] \geq 0$. Then by the same method as in Case 1, we obtain that $f(y_i) \geq \sum_{k=1}^{4} f(x_k) \prod_{j=1, j\neq k}^{4} \frac{(y_i - x_j)}{(x_k - x_j)}$. Again, by using the same method as above, $\sum_{i=1}^{4} f(y_i) \geq \sum_{i=1}^{4} f(x_i)$ for all 4−convex functions f, and hence $x \prec_4 y$.

Case 3: Let $y_2 = y_3$, $x_1 = x_2$, and $x_3 = x_4$. Since f is 4−convex function, the divided difference $f[x_1, x_1, x_3, x_3, x] \geq 0$. Note that $f[l, l] := f'(l)$. By the same method as above, we have

$$\begin{aligned} 0 \leq f[x_1, x_1, x_3, x_3, y_i] &= \frac{f[x_1, x_3, x_3, y_i] - f[x_1, x_1, x_3, x_3]}{y_i - x_1} \\ &= \frac{1}{(y_i - x_1)^2 (x - x_3)} \left(\frac{f(x) - f(x_3)}{x - x_3} - f'(x_3) \right) \\ &- \frac{1}{(y_i - x_1)^2 (x_3 - x_1)} \left(f'(x_3) - \frac{f(x_3) - f(x_1)}{x_3 - x_1} \right) \\ &- \frac{1}{(y_i - x_1)(x_3 - x_1)^2} \left(f'(x_3) - \frac{f(x_3) - f(x_1)}{x_3 - x_1} \right) \\ &+ \frac{1}{(y_i - x_1)(x_3 - x_1)^2} \left(\frac{f(x_3) - f(x_1)}{x_3 - x_1} - f'(x_1) \right) \end{aligned}$$

Therefore, for $i = 1, 2, 3, 4$,

$$\begin{aligned} f(y_i) \geq & \frac{(y_i - x_1)(y_i - x_3)^2}{(x_3 - x_1)^2} f'(x_1) + \frac{(y_i - x_1)^2 (y_i - x_3)}{(x_3 - x_1)^2} f'(x_3) \\ & + \frac{(y_i - x_3)^2 (x_3 + 2x - 3x_1)}{(x_3 - x_1)^3} f(x_1) \\ & - \frac{(y_i - x_3)^2 (x_3 - x_1) - (x_3 - x_1)^3 + 2(y_i - x_1)(y_i - x_3)^2}{(x_3 - x_1)^3} f(x_3). \end{aligned}$$

By using (2.3)-(2.5), the coefficients of $f'(x_1), f'(x_3), f(x_1), f(x_3)$ are equal 1. Thus, $\sum_{i=1}^{4} f(y_i) \geq \sum_{i=1}^{4} f(x_i)$ for all 4-convex functions f, and hence $x \prec_4 y$.

Case 4: Let $y_2 = y_3$ and ($x_1 \neq x_2, x_3 = x_4$ or $x_1 = x_2, x_3 \neq x_4$). We consider the divided differences $f[x_1, x_2, x_3, x_3, x]$ or $f[x_1, x_1, x_3, x_4, x]$ respectively. By the same method as in Case 3, $\sum_{i=1}^{4} f(y_i) \geq \sum_{i=1}^{4} f(x_i)$ for all 4-convex functions f, and hence $x \prec_4 y$. □

The following example gives us a pair of vectors x, y where x is 4-majorized but not majorized by y.

Example 2.7. Let $x = (2, 2, 7, 7)$ and $y = (1, 4, 5, 8)$. By Definition 1.3, it is clear that the majorization fails but Example 2.4 and Theorem 2.6 imply that $(2, 2, 7, 7) \prec_4 (1, 4, 5, 8)$.

In the following remark, the equivalent conditions for $x \prec_k y$ in $\mathbb{R}_k$, $k = 2, 3, 4$ are summarized.

Remark 2.8.

1. Let $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}_2$. Then by Theorem 1.4, $x \prec_2 y$ if and only if the following hold:

$$x_1 + x_2 = y_1 + y_2,$$
$$max\{x_1, x_2\} \leq max\{y_1, y_2\}.$$

2. Let $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}_3$. Then by Theorem 1.5, we obtain that $x \prec_3 y$ if and only if the following hold:

$$x_1 + x_2 + x_3 = y_1 + y_2 + y_3,$$
$$x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2,$$
$$max\{x_1, x_2, x_3\} \leq max\{y_1, y_2, y_3\}.$$

3. Let $x = (x_1, x_2, x_3, x_4), y = (y_1, y_2, y_3, y_4) \in \mathbb{R}_4$. Then by Theorem 2.6 $x \prec_4 y$ if and only if the following hold:

$$x_1 + x_2 + x_3 + x_4 = y_1 + y_2 + y_3 + y_4,$$
$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2,$$
$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = y_1^3 + y_2^3 + y_3^3 + y_4^3,$$
$$max\{x_1, x_2, x_3, x_4\} \leq max\{y_1, y_2, y_3, y_4\}.$$

In the above remark, we state equivalent conditions for $x \prec_k y$ in $\mathbb{R}_k$, $k = 2, 3, 4$.

Remark 2.9. Let $x = (x_1, x_2, \ldots, x_k)$, $y = (y_1, y_2, \ldots, y_k) \in \mathbb{R}_k$, $k = 2, 3, 4$. Then by Remark 2.8, $x \prec_k y$ and $y \prec_k x$ hold if and only if x and y are permutation of each other.

It would be nice to characterize $x \prec_k y$ in $\mathbb{R}_k$ for $k \geq 5$.

conjecture 2.10. *Let $x = (x_1, x_2, \ldots, x_k), y = (y_1, y_2, \ldots, y_k) \in \mathbb{R}_k$, $k \geq 5$. Then $x \prec_k y$ if and only if the following hold:*

$$x_1^i + x_2^i + \cdots + x_k^i = y_1^i + y_2^i + \cdots + y_k^i, \quad i = 1, 2, \ldots, k-1,$$
$$max\{x_1, x_2, \ldots, x_k\} \leq max\{y_1, y_2, \ldots, y_k\}.$$

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