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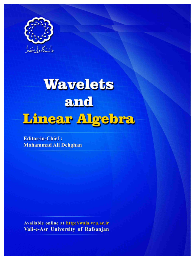

On the frames by multiplication and irregular frames of translates on LCA groups

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Abstract

Let X be a measure space and let E be a measurable subset of X with finite positive measure. In this paper, we investigate frame and Riesz basis properties of a family of functions multiplied by another measurable function in $L^2(E)$. Also, we study the equivalent conditions for a system of translates to be a Bessel family in $L^2(G)$ and to be a frame for P_E (the space of the band limited functions). Finally, we study the properties of frames of translates that preserved by convolution.

1. Introduction

Signal processing has been an active field in research for over 50 years, which provides tools and algorithms that are used in digital instruments. Signal processing and sampling theory are two

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significant areas that link two branch of computer science and mathematics. Frame theory, which is introduced in [7], is an important basis for sampling theory and signal processing. Frames of translates are a special class of frames that work with shifts of a function. These kinds of frames are essential in the theory of shift invariant spaces [4] that are used in sampling and wavelet theories [3, 5]. For $\varphi \in L^2(\mathbb{R})$, a system of translates generated by φ is defined as $\{T_k\varphi : k \in \mathbb{Z}\}$, where $T_k : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $T_kf(x) = f(x-k)$ is the shift operator. The properties of a system of translates have been studied by several authors including [6, 9]. In [9], the author discussed two closely related types of systems, namely the system of weighted exponential $\mathcal{E}(g) = \{e^{2\pi ikbx}g(x)\}_{k\in\mathbb{Z}}$ in $L^2(\mathbb{T})$, where $g \in L^2(\mathbb{T})$, $0 < b < 1$, and the system of translates $\{T_kg\}_{k\in\mathbb{Z}}$, (see also [6]).

For a locally compact abelian (LCA) group G, a uniform lattice L of G is a closed subgroup of G which is co-compact and discrete. In this setting, a regular system of translates generated by $\varphi \in L^2(G)$ is defined as $\mathcal{T}(\varphi) = \{T_k\varphi : k \in L\}$. An equivalent condition for $\mathcal{T}(\varphi)$ to be an orthonormal basis for $L^2(G)$ is given in [11]. In [12], we studied more properties of $\mathcal{T}(\varphi)$ in Hilbert space $L^2(G)$. In particular, we investigated some equivalent conditions for $\mathcal{T}(\varphi)$ to be a Bessel family, frame family and Riesz family in $L^2(G)$, in terms of the L^*-periodic function $\Phi_g(.) = \sum_{\gamma\in L^*}|\widehat{g}(.+\gamma)|^2$, in which $g \in L^2(G)$ and L^* is the annihilator of L in $\widehat{G}$.

In application, numerous problems need to be considered in the irregular case, for example when dealing with jittered samples in sound analysis. Irregular translates of a function lead to irregular frames of translates, which is an interesting generalization of regular frames of translates. For $\lambda \in \mathbb{R}^d$, the translation operator T_λ is defined as $T_\lambda f(x) = f(x-\lambda)$, in which $x \in \mathbb{R}^d$ and $f \in L^2(\mathbb{R}^d)$. For a countable index set $K \subseteq \mathbb{R}$ and $f \in L^2(\mathbb{R}^d)$, the properties of the system of translates $\{T_{\lambda_k}f\}_{k\in K}$ are investigated in [2], see also [1]. To be precise, the authors in [2] studied equivalent conditions for $\{T_{\lambda_k}f\}_{k\in K}$ to be a Bessel sequence for $L^2(\mathbb{R}^d)$, frame sequence in $L^2(\mathbb{R}^d)$ and to be a frame for $P_E = \{f \in L^2(\mathbb{R}^d);\ \mathrm{supp}\widehat{f} \subseteq E\}$, where E is a bounded subset of $\mathbb{R}^d$.

We aim to study the properties of the system $\{T_\lambda f\}_{\lambda\in\Lambda}$, in which Λ is a countable subset of an LCA group G. Due to the condition on Λ, which is a countable subset (not necessarily a uniform lattice), we call $\{T_\lambda f\}_{\lambda\in\Lambda}$ an irregular system of translates. To achieve our goal, we study the general properties of the system $\Psi = \{\psi_k\}_{k\in K} \subseteq L^2(X)$, where X is a measure space and K is a countable index set, multiplied by a measurable function φ. As an application, in the setting of LCA groups, we characterize the equivalent conditions for a system of irregular frames of translates to be a Bessel family in $L^2(G)$ and to be a frame for the space of band limited functions. We also study the properties of a system of irregular frames of translates that are preserved by convolution.

This paper is organized as follows. Section 2 is devoted to state and fix some preliminaries and notation related to frames theory and LCA groups. In section 3, we present conditions for a sequence in $L^2(E)$ to be a frame family, Bessel family and Riesz basis. In section 4, we apply the results of section 3 to investigate frames of irregular translates in LCA groups.

2. Preliminaries and Notations

Let $\mathcal{H}$ be a Hilbert space. A countable subset $\Omega \subseteq \mathcal{H}$, is called a *frame* for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{g\in\Omega} |\langle f, g\rangle|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}. \tag{2.1}$$

If Ω satisfies only the right hand inequality in (2.1), it is called a *Bessel family*. If $A = B$, then Ω is called a *tight frame*. The set Ω is called a *frame family* if it is a frame for the space of closure span of Ω.

The countable set $\Omega \subseteq \mathcal{H}$ is called a *Riesz family* if there exist two constants $A, B > 0$, called *Riesz bounds*, such that

$$A\sum_{\eta\in\Omega} |c_\eta|^2 \leq \|\sum_{\eta\in\Omega} c_\eta \eta\|^2 \leq B\sum_{\eta\in\Omega} |c_\eta|^2,$$

for any finitely supported $(c_\eta)_{\eta\in\Omega} \subseteq \mathbb{C}$ (i.e. all but finitely many (c_η) is zero). If a Riesz family Ω is complete in $\mathcal{H}$, i.e. $\overline{\text{span}(\Omega)} = \mathcal{H}$, then Ω is called a *Riesz basis*.

It is well-known that any Riesz basis is a frame (see [6]), but the converse is not necessarily accurate.

Assume that G is a second countable LCA group. It is well-known that such a group possesses a Haar measure μ_G that is unique up to a multiplication by positive constants. Let $\widehat{G}$ denote the dual group of G equipped with the compact convergence topology. The elements of $\widehat{G}$, which we usually denote by ξ, are characters on G, but one can also regard elements of G as characters on $\widehat{G}$. More precisely, $\widehat{\widehat{G}} \cong G$ (Pontrjagin duality theorem). To see more detail we refer to [8, Chapter 4].

The Fourier transform$\widehat{\ }$: $L^1(G) \to C_0(\widehat{G})$, $f \to \widehat{f}$, which is defined by $\widehat{f}(\xi) = \int_G f(x)\overline{\xi}(x)dx$, can be extended to a unitary isomorphism from $L^2(G)$ to $L^2(\widehat{G})$ known as the Plancherel transform. For more detail on LCA groups see [8, 10].

3. Frames by multiplication

In this section we will investigate the properties of the family $\{\psi_k.\varphi\}_{k\in K}$ for the family $\{\psi_k\}_{k\in K}$ and the function φ. Through this section, we assume that (X, μ) is a measure space and $E \subseteq X$ has finite positive measure, and that K is a countable index set.

Lemma 3.1. *With the notation as above, let $\{\psi_k\}_{k\in K}$ be complete in $L^2(E)$ and $\varphi \in L^2(X)$ such that $\{\varphi\psi_k\}_{k\in K}$ is in $L^2(E)$ and $\mu(\{t \in E \ : \ \varphi(t) = 0\}) = 0$. Assume that there exists a positive constant B such that $|\varphi(t)| \leq B$ for almost every $t \in E$. Then $\{\varphi\psi_k\}_{k\in K}$ is complete in $L^2(E)$.*

Proof. Let $\{\psi_k\}_{k\in K}$ be complete in $L^2(E)$. For $f \in L^2(E)$ assume that $\langle f, \varphi\psi_k\rangle = 0$, for every $k \in K$. Then $\langle f\overline{\varphi}, \psi_k\rangle = 0$, for every $k \in K$, and hence $f\overline{\varphi} = 0$ a.e. Using the hypothesis we get $f = 0$ a.e. on E. □

Using Lemma 3.1, immediately the following result follows.

Proposition 3.2. *Let $\{\psi_k\}_{k\in K}$ be a frame of $L^2(E)$. If $\varphi \in L^2(X)$ such that $\{\varphi\psi_k\}$ is in $L^2(E)$, $\mu(\{t \in E : \varphi(t) = 0\}) = 0$, and there exists a positive constant B such that $|\varphi(t)| \leq B$, for almost every $t \in E$, then $\{\varphi\psi_k\}$ is complete in $L^2(E)$.*

The following proposition expresses the necessary and sufficient condition for $\{\varphi\psi_k\}_{k\in K}$ to be a frame for $L^2(E)$.

Proposition 3.3. *With the above notation, let φ be a measurable function on X and let $\{\psi_k\}_{k\in K}$ be a frame of $L^2(E)$. Then $\{\varphi\psi_k\}_{k\in K}$ is a frame of $L^2(E)$ if and only if there exist constants A and B such that*

$$0 < A \leq B < \infty, \quad \text{and} \quad A \leq |\varphi(t)| \leq B \text{ for a.e. } t \in E. \tag{3.1}$$

Proof. Let $\{\psi_k\}_{k\in K}$ and $\{\varphi\psi_k\}_{k\in K}$ be two frames for $L^2(E)$. Assume that for every $A > 0$, there exists a set $U \subseteq E$ of positive measure such that $|\varphi(t)| < A$, for every $t \in U$. Let $E_n = \{t \in E : |\varphi(t)| < \frac{1}{n}\}$, for $n \in \mathbb{N}$, in which $\mu(E_n) > 0$. Define

$$f_n(t) = \begin{cases} \dfrac{1}{\sqrt{\mu(E_n)}} & \text{for } t \in E_n, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

It is clear that $\|f_n\|_2 = 1$ for any $n \in \mathbb{N}$, hence $f_n \in L^2(E)$.

Suppose that α is a lower frame bound for $\{\varphi\psi_k\}_{k\in K}$ and M is an upper frame bound for $\{\psi_k\}_{k\in K}$, then

$$\begin{aligned} \alpha &\leq \sum_{k\in K} |\langle f_n, \varphi\psi_k\rangle|^2 \\ &= \sum_{k\in K} |\langle f_n\overline{\varphi}, \psi_k\rangle|^2 \leq M\|f_n\varphi\|_2^2 \\ &= M\int_{E_n} |f_n\varphi|^2 d\mu = \frac{1}{\mu(E_n)} M\int_{E_n} |\varphi|^2 d\mu \leq \frac{M}{n^2} \to 0, \end{aligned}$$

which is a contradiction.

Now, assume that for every $B > 0$ there exists a set $V \subseteq E$ of positive measure such that $|\varphi(t)| > B$, for any $t \in V$. Let $E_s = \{t \in E : |\varphi(t)| > s\}$, for $s \in \mathbb{N}$, where $\mu(E_s) > 0$. Define $f_s(t)$ similar to (3.2). Let m be the lower bound for $\{\psi_k\}_{k\in K}$. Then

$$\begin{aligned} \sum_{k\in K} |\langle f_s, \varphi\psi_k\rangle|^2 &= \sum_{k\in K} \langle f_s\overline{\varphi}, \psi_k\rangle|^2 \geq m\|f_s\varphi\|_2^2 \\ &= m\int_{E_s} |f_s\varphi|^2 d\mu = \frac{1}{\mu(E_s)} m\int_{E_s} |\varphi|^2 d\mu \geq ms^2 \to +\infty, \end{aligned}$$

which is a contradiction, hence there exists a constant B satisfies (3.1).

For the other implication, suppose that there exist positive constants $A, B > 0$ such that $A \leq |\varphi(t)| \leq B$ a.e. for $t \in E$ and $\{\psi_k\}_{k\in K}$ is a frame for $L^2(E)$ with an upper frame bound M and a lower frame bound m. As $\varphi \in L^\infty(E)$, for $f \in L^2(E)$ we get $f\varphi \in L^2(E)$,

$$\sum_{k\in K} |\langle f, \varphi\psi_k\rangle|^2 = \sum_{k\in K} |\langle f\overline{\varphi}, \psi_k\rangle|^2.$$

So

$$m\|f\overline{\varphi}\|^2 \le \sum_{k\in K} |\langle f, \varphi\psi_k\rangle|^2 \le M\|f\overline{\varphi}\|^2, \text{ for any } f \in L^2(E).$$

On the other hand, we have

$$\|f\overline{\varphi}\|^2 \ge A^2\|f\|^2 \text{ and } \|f\overline{\varphi}\|^2 \le B^2\|f\|^2,$$

hence

$$mA^2\|f\|^2 \le \sum_{k\in K} |\langle f, \varphi\psi_k\rangle|^2 \le MB^2\|f\|^2, \text{ for any } f \in L^2(E),$$

which completes the proof. □

In the special case, we conclude the following proposition.

Proposition 3.4. *Let φ be a measurable function on X and let $\{\psi_k\}_{k\in K}$ be a tight frame of $L^2(E)$. Then $\{\varphi\psi_k\}_{k\in K}$ is a tight frame of $L^2(E)$ if and only if there exists a positive constant A such that*

$$|\varphi(t)| = A \ \text{ for a.e. } \ t \in E.$$

For Riesz basis property we achieve the following result.

Proposition 3.5. *Let φ be a measurable function on X and let $\{\psi_k\}_{k\in K}$ be a Riesz basis for $L^2(E)$. Then $\{\varphi\psi_k\}_{k\in K}$ is a Riesz basis for $L^2(E)$ if and only if there exist constants A and B such that*

$$0 < A \le B < \infty \ \text{ and } \ A \le |\varphi(t)| \le B \text{ for a.e. } t \in E. \tag{3.3}$$

Proof. Let $\{\varphi\psi_k\}_{k\in K}$ be a Riesz basis for $L^2(E)$. Then it is a frame for $L^2(E)$, so by Proposition 3.3 there exist constants A and B such that (3.3) holds. Conversely, assume that there exist constants A and B such that (3.3) holds. Then by Proposition 3.3, $\{\varphi\psi_k\}_{k\in K}$ is a frame for $L^2(E)$. Therefore, for every $f \in L^2(E)$,

$$f = \sum_{k\in K} c_k\varphi\psi_k, \tag{3.4}$$

in which $\{c_k\}_{k\in K} \in \ell^2(K)$. Since $|\varphi(t)| \ge A$, for a.e. $t \in E$, we have

$$\frac{f}{\varphi} = \sum_{k\in K} c_k\psi_k \in L^2(E).$$

Using the fact that $\{\psi_k\}_{k\in K}$ is a Riesz basis for $L^2(E)$, we conclude that the coefficients $\{c_k\}_{k\in K}$ are unique. This completes the proof. □

Also, in the case of Bessel family, we have the following proposition. The proof is straightforward and so we remove it.

Proposition 3.6. *Let φ be a measurable function on X, and let $\{\psi_k\}_{k\in K}$ be a frame for $L^2(E)$. Then $\{\varphi\psi_k\}_{k\in K}$ is a Bessel family for $L^2(E)$ if and only if there exists a constant $B > 0$ such that*

$$|\varphi(t)| \le B \text{ for a.e. } t \in E.$$

Proposition 3.7. *Let φ be a measurable function on X. If there exist constants A and B such that*

$$0 < A \leq B < \infty, \text{ and } A \leq |\varphi(t)| \leq B \text{ for a.e. } t \in E,$$

and $\{\varphi\psi_k\}_{k\in K}$ is a frame for $L^2(E)$, then $\{\psi_k\}_{k\in K}$ is a frame for $L^2(E)$.

Proof. Assume that $\{\varphi\psi_k\}_{k\in K}$ is a frame for $L^2(E)$ with the lower and upper frame bounds α and β, respectively. Then for every $f \in L^2(E)$ we have

$$\sum_{k\in K} |\langle f, \psi_k\rangle|^2 = \sum_{k\in K} |\langle f, \frac{1}{\varphi}\varphi\psi_k\rangle|^2 = \sum_{k\in K} |\langle f\frac{1}{\overline{\varphi}}, \varphi\psi_k\rangle|^2$$
$$\leq \beta\|f\frac{1}{\overline{\varphi}}\|^2 \leq \frac{\beta}{A^2}\|f\|^2.$$

Similarly, we get

$$\sum_{k\in K} |\langle f, \psi_k\rangle|^2 \geq \frac{\alpha}{B^2}\|f\|^2.$$

□

4. Application for frames of translates on LCA groups

In this section we will use the results of the previous section and will achieve some results for the frames of translates. Let G be an LCA group with a countable subset Λ. A system of translates generated by $\varphi \in L^2(G)$ is defined as $\{T_\lambda\varphi : \lambda \in \Lambda\}$, we call it irregular system of translates. For $\lambda \in \Lambda$, we denote by X_λ the associated character on $\widehat{G}$ which is defined as $X_\lambda(\chi) = \chi(\lambda)$. It is well-known that for $\varphi \in L^2(G)$, we have $\widehat{T_\lambda\varphi}(\xi) = X_\lambda(\xi)\widehat{\varphi}(\xi)$. Assume that $E \subseteq \widehat{G}$ is a measurable set with finite positive measure, the space of the band limited functions (Paley Wiener space) is defined as

$$P_E := \{f \in L^2(G) : \text{supp}\widehat{f} \subseteq E\}.$$

The following theorem investigates the necessary and sufficient condition for a system $\{T_\lambda h\}_{\lambda\in\Lambda}$, where $h \in P_E$, to be a Bessel family in $L^2(G)$ and to be a frame for P_E.

Theorem 4.1. *Let Λ be a subset of G such that $\{X_\lambda\}_{\lambda\in\Lambda}$ be a frame for $L^2(E)$, in which $E \subseteq \widehat{G}$ is a measurable set with finite positive measure, and let $h \in P_E$. Then*

(a) $\{T_\lambda h\}_{\lambda\in\Lambda}$ is a Bessel family in $L^2(G)$ if and only if there exists $B > 0$ such that $|\widehat{h}(\xi)| \leq B$ for a.e. $\xi \in \widehat{G}$.

(b) $\{T_\lambda h\}_{\lambda\in\Lambda}$ is a frame for P_E if and only if there exist $0 < A \leq B$ such that $A \leq |\widehat{h}(\xi)| \leq B$ for a.e. $\xi \in E$.

Proof. (a) Assume that $\{T_\lambda h\}_{\lambda\in\Lambda}$ is a Bessel family for $L^2(G)$ and $f \in L^2(G)$. Then there exists $\beta > 0$ such that

$$\sum_{\lambda\in\Lambda} |\langle f, T_\lambda h\rangle_{L^2(G)}|^2 \leq \beta\|f\|_{L^2(G)}, \text{ for every } f \in L^2(G).$$

Using the Plancherel Theorem and the assumption supp$\widehat{h} \subseteq E$, we get

$$\sum_{\lambda\in\Lambda} |\langle \widehat{f}, X_\lambda \widehat{h}\rangle_{L^2(E)}|^2 \le \beta \|\widehat{f}\|^2_{L^2(\widehat{G})}, \quad \text{for every } f \in L^2(G).$$

For $g \in L^2(E)$, there exists $f \in P_E$ such that $g = \hat{f}|_E$ a.e. Thus,

$$\sum_{\lambda\in\Lambda} |\langle g, X_\lambda \widehat{h}\rangle_{L^2(E)}|^2 \le \beta \|g\|_{L^2(E)}, \quad \text{for every } g \in L^2(E).$$

By Proposition 3.6, there exists $B > 0$ such that $|\widehat{h}(\xi)| \le B$ for a.e. $\xi \in E$. Since supp$\widehat{h} \subseteq E$, we have $|\widehat{h}(\xi)| \le B$ for a.e. $\xi \in \widehat{G}$. Conversely, suppose that there exists $B > 0$ such that $|\widehat{h}(\xi)| \le B$ for a.e. $\xi \in \widehat{G}$. Using Proposition 3.6, we get

$$\begin{aligned}\sum_{\lambda\in\Lambda} |\langle f, T_\lambda h\rangle_{L^2(G)}|^2 &= \sum_{\lambda\in\Lambda} |\langle \widehat{f}, X_\lambda \widehat{h}\rangle_{L^2(\widehat{G})}|^2 \\ &= \sum_{\lambda\in\Lambda} |\langle \widehat{f}, X_\lambda \widehat{h}\rangle_{L^2(E)}|^2 \\ &\le \beta \|\widehat{f}\|_{L^2(E)} \le \beta \|\widehat{f}\|_{L^2(\widehat{G})} = \beta \|f\|_{L^2(G)},\end{aligned}$$

for every $f \in L^2(G)$. That is $\{T_\lambda h\}_{\lambda\in\Lambda}$ is a Bessel family in $L^2(G)$.
(b) If $\{T_\lambda h\}_{\lambda\in\Lambda}$ is a frame for P_E, there exist $0 < A \le B < \infty$ such that

$$A\|f\|_{P_E} \le \sum_{\lambda\in\Lambda} |\langle f, T_\lambda h\rangle_{P_E}|^2 \le B\|f\|_{P_E},$$

equivalently,

$$A\|\widehat{f}\|_{L^2(E)} \le \sum_{\lambda\in\Lambda} |\langle \widehat{f}, X_\lambda \widehat{h}\rangle_{L^2(E)}|^2 \le B\|\widehat{f}\|_{L^2(E)}. \tag{4.1}$$

Based on Proposition 3.3, (4.1) is equivalent to existing constants A and B such that

$$0 < A \le B < +\infty, \quad A \le |\widehat{h}(\xi)| \le B, \quad \text{for a.e. } \xi \in E.$$

□

The forthcoming result is a consequence of Theorem 4.1.

Corollary 4.2. *For $h \in P_E$, let $\widehat{h}$ be continuous. Then there does not exist $\Lambda \subseteq G$ such that $\{h(. - \lambda)\}_{\lambda\in\Lambda}$ is a frame of P_E.*

In the next proposition, we investigate the properties of frames of translates that are preserved by convolution.

Proposition 4.3. *For $\Lambda \subseteq G$, let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a frame for $L^2(E)$, and $f, g \in P_E$. Then*

- *(a) If $\{T_\lambda f\}_{\lambda\in\Lambda}$ and $\{T_\lambda g\}_{\lambda\in\Lambda}$ are Bessel families of $L^2(G)$, then $\{T_\lambda(f * g)\}_{\lambda\in\Lambda}$ is a Bessel family of $L^2(G)$.*

(b) *If $\{T_\lambda f\}_{\lambda\in\Lambda}$ and $\{T_\lambda g\}_{\lambda\in\Lambda}$ are Bessel families for P_E, then $\{T_\lambda(f * g)\}_{\lambda\in\Lambda}$ is a Bessel family for P_E.*

(c) *If $\{T_\lambda f\}_{\lambda\in\Lambda}$ and $\{T_\lambda(f * g)\}_{\lambda\in\Lambda}$ are two frames for P_E, then $\{T_\lambda g\}_{\lambda\in\Lambda}$ is a frame for P_E.*

(d) *If $\{T_\lambda(f * g)\}_{\lambda\in\Lambda}$ is a Bessel family in $L^2(G)$, and there exists $C > 0$ such that $|\widehat{f}(\xi)| \geq C$ a.e., then $\{T_\lambda g\}_{\lambda\in\Lambda}$ is a Bessel family in $L^2G)$.*

Proof. (a) Assume that $\{T_\lambda f\}_{\lambda\in\Lambda}$ and $\{T_\lambda g\}_{\lambda\in\Lambda}$ are Bessel families of $L^2(G)$. By Theorem 4.1, there exist positive constants B_1, B_2 such that $|\widehat{f}(\xi)| \leq B_1$ a.e., and $|\widehat{g}(\xi)| \leq B_2$ a.e. It is well known that $\widehat{f * g} = \widehat{f}.\widehat{g}$, so $|\widehat{f * g}(\xi)| \leq B_1 B_2$ a.e. Since $f * g \in P_E$ we get $\{T_\lambda(f * g)\}_{\lambda\in\Lambda}$ is a Bessel family of $L^2(G)$.
The proof of part (b) is similar.
(c) Let $\{T_\lambda f\}_{\lambda\in\Lambda}$ and $\{T_\lambda(f * g)\}_{\lambda\in\Lambda}$ be two frames for P_E, then by Theorem 4.1 there exist $B_1 \geq A_1 > 0$ and $B_2 \geq A_2 > 0$ such that

$$A_1 \leq |\widehat{f}(\xi)| \leq B_1, \quad \text{for a.e. } \xi \in E,$$
$$A_2 \leq |\widehat{fg}(\xi)| \leq B_2, \quad \text{for a.e. } \xi \in E.$$

Therefore,

$$\frac{A_2}{B_1} \leq |\widehat{g}(\xi)| \leq \frac{B_2}{A_1}, \quad \text{for a.e. } \xi \in E.$$

Hence, we get the result.
Analogously, part (d) can be proven. □

We close this section with the following proposition which states the necessary and sufficient condition for the union of frame families of translates to be a frame.

Proposition 4.4. *Suppose $h_j \in P_{E_j}$ for all $j \in J$, in which $\{E_j\}_{j\in J}$ is a family of finite positive measure subsets of $\widehat{G}$. Assume that $\{X_\lambda 1_{E_j}\}_{\lambda\in\Lambda}$ is a frame for $L^2(E_j)$ with frame bounds m_j and M_j for every $j \in J$. If $m = \inf m_j > 0$ and $M = \sup M_j < +\infty$, then $\{T_\lambda h_j\}_{\lambda\in\Lambda, j\in J}$ is a frame for $P_{\cup_{j\in J} E_j}$ if and only if there exist constants $B \geq A > 0$ such that*

$$A \leq \sum_{j\in J} |\widehat{h}_j(\xi)|^2 \leq B, \quad a.e.\ in\ \cup_{j\in J} E_j. \tag{4.2}$$

Proof. Let $\{T_\lambda h_j\}_{\lambda\in\Lambda, j\in J}$ be a frame for $P_{\cup_{j\in J} E_j}$. Assume that for every $A > 0$ there exists a set $U \subseteq \cup_{j\in J} E_j$ of positive measure such that $\sum_{j\in J} |\widehat{h}_j(\xi)|^2 < A$ for every $\xi \in U$. For $n \in \mathbb{N}$, consider $E_n = \{\xi \in \cup_{j\in J} E_j \ : \ \sum_{j\in J} |\widehat{h}_j(\xi)|^2 < 1/n\}$, where $\mu(E_n) > 0$. Define

$$f_n(t) = \begin{cases} \dfrac{1}{\sqrt{\mu(E_n)}} & \text{for } t \in E_n \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

Then $f_n \in L^2(\cup_{j\in J} E_j)$ for every $n \in \mathbb{N}$. Let α be a lower frame bound of $\{T_\lambda h_j\}_{\lambda\in\Lambda, j\in J}$ and M_j is a upper frame bound of $\{X_\lambda 1_{E_j}\}_{\lambda\in\Lambda}$. Then

$$\begin{aligned}
\alpha \leq \sum_{\lambda\in\Lambda, j\in J} |\langle T_\lambda h_j, f_n\rangle|^2 &= \sum_{\lambda\in\Lambda, j\in J} |\langle X_\lambda 1_{E_j}\widehat{h}_j, \widehat{f_n}\rangle|^2 \\
&= \sum_{\lambda\in\Lambda, j\in J} |\langle X_\lambda 1_{E_j}, \overline{\widehat{h}_j}\widehat{f_n}\rangle|^2 \\
&\leq \sum_{j\in J} M_j \|\widehat{f_n}\widehat{h}_j\|^2 \\
&\leq M \sum_{j\in J} \int_{E_n} |\widehat{f_n}\widehat{h}_j|^2(\xi) d\xi \\
&\leq M \int_{E_n} \sum_{j\in J} |\widehat{h}_j(\xi)|^2 |\widehat{f_n}(\xi)|^2 d\xi \leq \frac{M}{n} \to 0, \text{ as } n \to \infty
\end{aligned}$$

which is a contradiction.

To prove the existence of the upper frame bound, assume that for every $B > 0$, there exists a set $V \subseteq \cup_{j\in J} E_j$ of positive measure such that $\sum_{j\in J} |\widehat{h}_j(\xi)|^2 > B$, for a.e. $\xi \in V$. For $\ell \in \mathbb{N}$, let $E_\ell = \{\xi \in \cup_{j\in J} E_j \ : \ \sum_{j\in J} |\widehat{h}_j(\xi)|^2 > \ell\}$, then $\mu(E_\ell) > 0$ for every $\ell \in \mathbb{N}$. Define $f_\ell(\xi)$ similar to (4.3). Let m_j be the lower frame bound of $\{T_\lambda h_j\}_{\lambda\in\Lambda, j\in J}$. Then

$$\begin{aligned}
\sum_{\lambda\in\Lambda, j\in J} |\langle T_\lambda h_j, f_\ell\rangle|^2 &= \sum_{\lambda\in\Lambda, j\in J} |\langle X_\lambda 1_{E_j}\widehat{h}_j, \widehat{f_\ell}\rangle|^2 \\
&= \sum_{\lambda\in\Lambda, j\in J} |\langle X_\lambda 1_{E_j}, \overline{\widehat{h}_j}\widehat{f_\ell}\rangle|^2 \\
&\geq \sum_{j\in J} m_j \|\widehat{f_\ell}\widehat{h}_j\|^2 \\
&\geq m \sum_{j\in J} \int_{E_\ell} |\widehat{f_\ell}\widehat{h}_j|^2(\xi) d\xi \geq m\ell^2.
\end{aligned}$$

Since $\ell \in \mathbb{N}$ is arbitrary, it is again a contradiction, so there exists a constant $B > 0$ satisfied (4.2).

For the other implication, assume that there exist positive constants $B \geq A > 0$ such that

$$A \leq \sum_{j\in J} |\widehat{h}_j(\xi)|^2 \leq B, \quad a.e. \ in \ \cup_{j\in J} E_j.$$

Suppose that $\{X_\lambda 1_{E_j}\}_{\lambda\in\Lambda}$ is a frame for $L^2(E_j)$, for each $j \in J$. For $f \in P_{\cup_{j\in J}E_j}$, we have

$$\begin{aligned}
\sum_{\lambda\in\Lambda, j\in J} |\langle T_\lambda h_j, f\rangle_{L^2(G)}|^2 &= \sum_{\lambda\in\Lambda, j\in J} |\langle X_\lambda 1_{E_j}\widehat{h}_j, \widehat{f}\rangle_{L^2(\widehat{G})}|^2 \\
&= \sum_{\lambda\in\Lambda, j\in J} |\langle X_\lambda 1_{E_j}\widehat{h}_j, \widehat{f}\rangle_{L^2(E_j)}|^2 \\
&= \sum_{j\in J}\sum_{\lambda\in\Lambda} |\langle X_\lambda 1_{E_j}, \overline{\widehat{h}_j}\widehat{f}\rangle_{L^2(E_j)}|^2 \\
&\le \sum_{j\in J} M_j \|\overline{\widehat{h}_j}\widehat{f}\|^2 \\
&\le M \sum_{j\in J} \int_{\widehat{G}} |\overline{\widehat{h}_j}\widehat{f}|^2(\xi)d\xi \le MB\|f\|^2.
\end{aligned}$$

Similarly, the other inequality can be proven. □

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