Wavelets and Linear Algebra 4(1) (2017) 1 - 9

Vali-e-Asr University
of Rafsanjan

Wavelets and Linear Algebra

http://wala.vru.ac.ir

Characterizations of amenable hypergroups

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Article Info

Article history:
Received 19 September 2016
Accepted 14 December 2016
Available online 01 July 2017
Communicated by Rajab Ali Kamyabi-Gol

Keywords:
Amenability, Banach algebras, Hypergroup algebras.

2000 MSC:
22A20, 43A60.

Abstract

Let K be a locally compact hypergroup with left Haar measure and let $L^1(K)$ be the complex Lebesgue space associated with it. Let $L^\infty(K)$ be the dual of $L^1(K)$. The purpose of this paper is to present some necessary and sufficient conditions for $L^\infty(K)^*$ to have a topologically left invariant mean. Some characterizations of amenable hypergroups are given.

1. Introduction

Hypergroups are locally compact spaces whose bounded Radon measures form an algebra which has similar properties to the convolution measures algebra of a locally compact group. Hypergroups arise as generalizations of the measure algebra of a locally compact group wherein

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http://doi.org/10.22072/wala.2017.23365

the product of two points is a probability measure rather than a single point. The abstract study of hypergroups began in the 1970s with Dunkl [3], Jewett [7], and Spector [15]. A detailed treatment can be found in the text of Bloom and Heyer [2]. Let K be a locally compact hypergroup with a fixed left Haar measure λ and modular function Δ defined by the identity

$$\Delta(y)\int\int f(t)d\delta_x * \delta_y(t)d\lambda(x) = \int f(x)d\lambda(x),$$

for continuous functions f vanishing off compact subsets of K. If f is a Borel function on K and $x \in K$, the left translation $_xf$ or L_xf is defined by

$$L_xf(y) = {}_xf(y) = \int f(t)d\delta_x * \delta_y(t) = f(x * y),$$

if the integral exists. We also consider $f^*(x) = f(\check{x})$ ($\check{x}$ is the involution of x). It is still unknown if an arbitrary hypergroup admits a left Haar measure. It particular, it remains unknown whether every amenable hypergroup admits a left Haar measure. But all the known examples such as commutative hypergroups and central hypergroups do. In this case, one can define the convolution algebra $L^1(K)$ with multiplication $f * g(x) = \int f(x * y)g(\check{y})d\lambda(y)$ for $f, g \in L^1(K)$.

Let $M(K)$ be the space of complex-valued, regular Borel measures on K. We denote by $M^1(K)$ the convex set formed by the probability measures on K. Recall that $L^1(K)$ is a Banach subalgebra and an ideal in $M(K)$ with a bounded approximate identity. It should be noted that these algebras include not only the group algebra $L^1(G)$ but also most of the semigroup algebras.

Let $L^\infty(K)$ be the Banach space of all essentially bounded Borel measurable functions on K with essential supremum norm. A linear functional $M \in L^\infty(K)^*$ is called a *mean* if $\langle M, f\rangle \geq 0$ whenever $f \geq 0$ and $\|M\| = 1$. We denote by $\mathcal{M}(L^\infty(K))$ the convex set of all means on $L^\infty(K)$. Each probability measure $\varphi \in L^1(K) \subseteq M(K)$ is a mean. Let $P^1(K) = \{\varphi \in L^1(K);\ \varphi \geq 0 \text{ and } \|\varphi\| = 1\}$. An application by Hahn-Banach theorem shows that $P^1(K)$ is weak* dense in the set of means on $L^\infty(K)$.

Let $C_b(K)$ denote the Banach space of all bounded continuous complex-valued functions on K. We say that a continuous function $f \in C_b(K)$ is left uniformly continuous if the map $x \mapsto L_xf$ is continuous in norm. We denote the collection of left uniformly continuous functions on K by $LUC(K)$. Let X be one of the spaces $LUC(K)$ or $L^\infty(K)$. A mean M on X is called *left invariant mean* if $\langle M, L_xf\rangle = \langle M, f\rangle$ for all $f \in X$ and $x \in K$. M is *topologically left invariant mean* if $\langle M, \varphi * f\rangle = \langle M, f\rangle$ for all $\varphi \in P^1(K)$ and $f \in X$. The convex set of left invariant [topologically left invariant] means on X is denoted by $LIM(X)$ [$TLIM(X)$]. A hypergroup K is called *amenable* if there is a LIM on $L^\infty(K)$. We mainly follow [2] in our notation and refer to [13] for basic functional analysis.

Skantharajah [14] initiated the study of amenable hypergroups, extending the definition from groups. He discovered that despite the apparent analogy of this topic to groups, there are substantial differences, especially in connection with its relationship with the representation theory. Skantharajah [14] showed that if K is a hypergroup which admits a left Haar measure then the function spaces of $LUC(K)$, $C_b(K)$ and $L^\infty(K)$ all either admit a left invariant mean (if K is amenable), or all do not. Numerous authors continue to study various aspects of hypergroups including amenability properties [1, 6, 9, 10, 12, 16] and [17].

In this paper, among the other things, we investigate the structures of amenable hypergroups. A number of equivalent conditions characterizing amenable hypergroups are given.

2. Main results

Throughout this paper, unless explicitly stated otherwise, K will denote a locally compact hypergroup with a fixed left Haar measure λ. Recall that $L^1(K)^{**}$, the second conjugate space of $L^1(K)$, is a Banach algebra with the first Arens product. More specifically, let $F, G \in L^1(K)^{**}$, $f \in L^1(K)^*$ and $\varphi, \psi \in L^1(K)$; we define $f\varphi, Gf \in L^1(K)^*$, $FG \in L^1(K)^{**}$ by the equations

$$\langle f\varphi, \psi\rangle = \langle f, \varphi * \psi\rangle, \quad \langle Gf, \varphi\rangle = \langle G, f\varphi\rangle, \quad \langle GF, f\rangle = \langle G, Ff\rangle.$$

In [11] Medghalchi defined $L^1(K)^*L^1(K) = LUC(K)$ which is a Banach subspace of $L^1(K)^*$, and showed that $LUC(K)^*$ is a Banach algebra by an Arens-type product and that $L^1(K) \subseteq LUC(K)^*$. In the following Theorem, we give conditions on K and $L^\infty(K)$ that are sufficient to guarantee amenability of K.

Theorem 2.1. *A necessary and sufficient condition for the amenability of a locally compact hypergroup K is given by each of the following properties:*

(i) For every $f \in L^\infty(K)$, there exists a mean M_f on $L^\infty(G)$ such that $\langle M_f, f\varphi\rangle = \langle M_f, f\psi\rangle$ whenever $\varphi, \psi \in P^1(K)$

(ii) There exists a net $\{\varphi_\alpha\}$ in $P^1(K)$ such that, for every weakly compact subset $\mathcal{S}$ of $P^1(K)$, $\lim \|\varphi * \varphi_\alpha - \varphi_\alpha\|_1 = 0$ uniformly for every $\varphi \in \mathcal{S}$.

Proof. Let K be amenable, by [14, Theorem 3.2], we consider $M \in TLIM(L^\infty(G))$. Let $f \in L^\infty(K)$ and $\varphi \in P^1(K)$. For every $\psi \in L^1(K)$,

$$\begin{aligned}
\langle f\varphi, \psi\rangle &= \langle f, \varphi * \psi\rangle = \int f(x)\varphi * \psi(x)d\lambda(x) \\
&= \int f(x) \int \psi(\check{y} * x)\varphi(y)d\lambda(y)d\lambda(x) \\
&= \int\int \varphi(y) f(y * x)\psi(x)d\lambda(y)d\lambda(x) \\
&= \int \psi(x) \int \frac{(f_x\varphi)^*(y)}{\Delta(y)} d\lambda(y)d\lambda(x) \\
&= \int \psi(x) \int f(\check{y} * x)\varphi(\check{y})\Delta(\check{y})d\lambda(y)d\lambda(x).
\end{aligned}$$

We consider $\hat{\varphi}(y) = \varphi(\check{y})\Delta(\check{y})$. Obviously $\hat{\varphi} \in P^1(K)$. Therefore we have

$$\begin{aligned}
\langle f\varphi, \psi\rangle &= \int \psi(x) \int f(\check{y} * x)\hat{\varphi}(y)d\lambda(y)d\lambda(x) \\
&= \int \psi(x)\hat{\varphi} * f(x)d\lambda(x) = \langle \hat{\varphi} * f, \psi\rangle.
\end{aligned}$$

This shows that $f\varphi = \hat{\varphi} * f$. By definition of topological left invariant means, $\langle M, f\varphi\rangle = \langle M, \hat{\varphi} * f\rangle = \langle M, f\rangle$. Obviously $\langle M, f\varphi\rangle = \langle M, f\psi\rangle$ whenever $f \in L^\infty(K)$ and $\varphi, \psi \in P^1(K)$.

$(i) \Rightarrow (ii)$ If $f \in L^\infty(K)$, we denote by $\mathcal{M}_f(L^\infty(K))$ the set of all means M on $L^\infty(K)$ such that $\langle M, f\varphi\rangle = \langle M, f\psi\rangle$ whenever $\varphi, \psi \in P^1(K)$. Let $\mathcal{F}$ denote the collection of all finite subsets F of $L^\infty(K)$, and for every $F \in \mathcal{F}$, let

$$\mathcal{M}_F = \{m \in \mathcal{M}(L^\infty(K));\ \langle M, f\varphi\rangle = \langle M, f\psi\rangle \text{ for all } f \in F \text{ and } \varphi, \psi \in P^1(K)\}.$$

We show that $\mathcal{M}_F \neq \emptyset$ for each $F \in \mathcal{F}$. We achieve this by induction on the number of elements in F. When $n = 1$, this is clear. Let $F = \{f_1, ..., f_{n+1}\}$ be a finite subset of $L^\infty(K)$. Let $M_1 \in \bigcap_{i=1}^n \mathcal{M}_{f_i}(L^\infty(K))$. By hypothesis, there exists $M_2 \in \mathcal{M}_{M_1 f_{n+1}}(L^\infty(K))$. Let $M = M_2 M_1$. For $1 \le i \le n$ and $\varphi, \psi \in P^1(K)$, we have

$$\langle M_2 M_1, f_i\varphi\rangle = \langle M_2, M_1 f_i\varphi\rangle = \langle M_2, M_1 f_i\psi\rangle = \langle M_2 M_1, f_i\psi\rangle.$$

On the other hand,

$$\langle M_2 M_1, f_{n+1}\varphi\rangle = \langle M_2, M_1 f_{n+1}\varphi\rangle = \langle M_2, M_1 f_{n+1}\psi\rangle = \langle M_2 M_1, f_{n+1}\psi\rangle.$$

This shows that $\mathcal{M}_F \neq \emptyset$. Choose $N \in \bigcap\{\mathcal{M}_F; F \in \mathcal{F}\}$. As $P^1(K)$ is weak*-dense in $\mathcal{M}(L^\infty(K))$, there exists a net $\{\varphi_\alpha\}$ in $P^1(K)$ such that $\varphi_\alpha \to N$ in the weak*-topology. If $f \in L^\infty(K)$ and $\varphi \in P^1(K)$, we have

$$\begin{aligned} \langle NN, f\varphi\rangle &= \langle N, Nf\varphi\rangle = \lim_\alpha \langle \varphi_\alpha, Nf\varphi\rangle = \lim_\alpha \langle N, f\varphi * \varphi_\alpha\rangle \\ &= \lim_\alpha \langle N, f\varphi_\alpha\rangle = \lim_\alpha \langle \varphi_\alpha, Nf\rangle = \langle NN, f\rangle. \end{aligned}$$

Obviously $\langle NN, f\rangle \ge 0$ whenever $f \in L^\infty(K)^+$, also

$$\langle NN, 1\rangle = \langle N, N1\rangle = \langle N, 1\rangle = 1.$$

Therefore we have $NN \in TLIM(L^\infty(K)$. By [17, Remark 1.4], there exists a net $\{\varphi_\alpha\}$ in $P^1(K)$ such that, for every compact subset C of K, $\lim_\alpha \|\delta_x * \varphi_\alpha - \varphi_\alpha\|_1 = 0$ uniformly for every $x \in C$. Let C be a compact subset of K and $\epsilon > 0$ be given. Let $\varphi \in P^1(K) \subseteq M(K)$ with $\{x \in K; \varphi(x) \neq 0\} \subseteq C$. There exists α_0 such that for $\alpha \ge \alpha_0$, $\|\delta_x * \varphi_\alpha - \varphi_\alpha\|_1 < \epsilon$ whenever $x \in C$. Since $x \mapsto \delta_x * \varphi_\alpha$ is continuous from K to $L^1(K)$, by [13] we have

$$\begin{aligned} \left|\langle f, \varphi * \varphi_\alpha - \varphi_\alpha\rangle\right| &= \left|\int \langle f, \delta_x * \varphi_\alpha - \varphi_\alpha\rangle d\varphi(x)\right| \\ &\le \|f\|\, \|\delta_x * \varphi_\alpha - \varphi_\alpha\|_1 < \epsilon\|f\|, \end{aligned}$$

whenever $f \in L^\infty(K)$ and $\alpha \ge \alpha_0$. This shows that for every compact subset C of K, $\lim_\alpha \|\varphi * \varphi_\alpha - \varphi_\alpha\|_1 = 0$ uniformly for every $\varphi \in P^1(K)$ with $\varphi(C^c) = 0$. Now, let $\mathcal{S}$ be a weakly compact subset of $P^1(K)$ and $\epsilon > 0$ be given. By [4, Theorem 4.21.2], there exists a compact subset C in K such that $\int_{K\setminus C} \varphi(x) d\lambda(x) < \frac{\epsilon}{4}$ for all $\varphi \in \mathcal{S}$. There exists $\alpha_0 \in I$ such that, for every $\alpha \in I$ with $\alpha \ge \alpha_0$

and every $\varphi \in P^1(K)$ with $\varphi(C^c) = 0$, $\|\varphi * \varphi_\alpha - \varphi_\alpha\|_1 < \frac{\epsilon}{4}$. Hence for all $\alpha \in I$ with $\alpha \geq \alpha_0$ and $\varphi \in \mathcal{S}$,

$$\begin{aligned}\|\varphi * \varphi_\alpha - \varphi_\alpha\|_1 &= \|\varphi\chi_C * \varphi_\alpha + \varphi\chi_{K\setminus C} * \varphi_\alpha - \|\varphi\chi_C\|_1\varphi_\alpha - \|\varphi\chi_{K\setminus C}\|_1\varphi_\alpha\|_1 \\ &\leq \|\varphi\chi_C * \varphi_\alpha - \|\varphi\chi_C\|_1\varphi_\alpha\|_1 + \|\varphi\chi_{K\setminus C} * \varphi_\alpha - \|\varphi\chi_{K\setminus C}\|_1\varphi_\alpha\|_1 \\ &\leq \|\varphi\chi_C\|_1\left\|\frac{\varphi\chi_C * \varphi_\alpha}{\|\varphi\chi_C\|_1} - \varphi_\alpha\right\|_1 + \frac{\epsilon}{2} < \epsilon.\end{aligned}$$

By [14, Theorem 4.1], (ii) characterizes amenability of K. This completes our proof. □

Corollary 2.2. *A locally compact hypergroup K is amenable if and only if the following condition holds for every $f \in L^\infty(K)$:*

$$\inf\left\{\sup\{|\langle f, \varphi * \mu - \psi * \mu\rangle|;\ \varphi, \psi \in P^1(K)\},\ \mu \in M^1(K)\right\} = 0.$$

Proof. Assume K to be amenable. Let $M \in TLIM(L^\infty(K))$. It is known that, there exists a net $\{\varphi_\alpha\}$ in $P^1(K)$ such that $\varphi_\alpha \to M$ in the weak*-topology. If $f \in L^\infty(K)$ and $\varphi, \psi \in P^1(K)$, we have

$$\begin{aligned}\lim_\alpha\langle f, \varphi * \varphi_\alpha - \psi * \varphi_\alpha\rangle &= \lim_\alpha\langle \varphi_\alpha, f\varphi - f\psi\rangle \\ &= \langle M, f\varphi - f\psi\rangle = 0.\end{aligned}$$

We conclude that

$$\inf\left\{\sup\{|\langle f, \varphi * \mu - \psi * \mu\rangle|;\ \varphi, \psi \in P^1(K)\},\ \mu \in M^1(K)\right\} = 0.$$

To prove the converse, we assume that the condition holds. Let $f \in L^\infty(K)$ and $\epsilon > 0$ be given. By hypothesis, there exists $\mu_\epsilon \in M^1(K)$ such that $|\langle f, \varphi * \mu_\epsilon - \psi * \mu_\epsilon\rangle| < \frac{\epsilon}{2}$ whenever $\varphi, \psi \in P^1(K)$. Consider a fixed element $\varphi_0 \in P^1(K)$. Since $L^1(K)$ is an ideal in $M(K)$, we have $\varphi_0 * \mu_\epsilon \in P^1(K)$. As $\mathcal{M}(L^\infty(K))$ is weak*-compact, without loss of generality we can assume that the net $\{\varphi_0 * \mu_\epsilon\}$ converging to a mean M_f in the weak*-topology of $L^\infty(K)$. Given $\epsilon > 0$ and $\varphi, \psi \in P^1(K)$, there exists $\epsilon_0 \leq \epsilon$ such that $|\langle M_f - \phi_0 * \mu_{\epsilon_0}, f\varphi - f\psi\rangle| < \frac{\epsilon}{2}$. Therefore we have

$$\begin{aligned}\left|\langle M_f, f\varphi - f\psi\rangle\right| &\leq \left|\langle M_f - \phi_0 * \mu_{\epsilon_0}, f\varphi - f\psi\rangle\right| + \left|\langle \phi_0 * \mu_{\epsilon_0}, f\varphi - f\psi\rangle\right| \\ &\leq \frac{\epsilon}{2} + \left|\langle f, \varphi * \phi_0 * \mu_{\epsilon_0} - \psi * \phi_0 * \mu_{\epsilon_0}\rangle\right| < \epsilon.\end{aligned}$$

As $\epsilon > 0$ is arbitrary, we conclude that $\langle M_f, f\varphi\rangle = \langle M_f, f\psi\rangle$ whenever $\varphi, \psi \in P^1(K)$. By Theorem 2.1, K is amenable. □

For each member f of $L^\infty(K)$, define $\lambda_f : L^1(K) \to L^\infty(K)$ by $\lambda_f(\varphi) = f\varphi$. Then $L^\infty(K)$ can be embedded into $\mathcal{B}(L^1(K), L^\infty(K))$ by a linear map Λ so that $\Lambda(f) = \lambda_f$. Since $\mathcal{B}(L^1(K), L^\infty(K))$ carries naturally the strong and the weak operator topology, Λ allows us to consider their induced topologies on $L^\infty(K)$, which we denote by τ_c and τ_w, respectively. By [14, Lemma 3.1], $TLIM(L^\infty(K)) \subseteq LIM(L^\infty(K))$. Granirer in [5] has shown that for a non-discrete abelian locally compact group G, there is an $M \in LIM(L^\infty(G)) \setminus TLIM(L^\infty(G))$. In the following Proposition, we compare the sets of means admitting the different types of invariance properties and consider relations existing between these sets.

Proposition 2.3. *Let K be an amenable hypergroup. The following properties hold:*

(i) If $M \in P^1(K) \cap LIM(L^\infty(K))$, then also $M \in TLIM(L^\infty(K))$.

(ii) If $M \in LIM(L^\infty(K))$ is weak*-continuous, then $M \in TLIM(L^\infty(K))$.

(iii) If $M \in LIM(L^\infty(K))$ is τ_c-continuous, then $M \in TLIM(L^\infty(K))$.

Proof. Suppose that $f \in L^\infty(K))$ and $\varphi \in P^1(K)$. By density we may suppose that $\varphi \in P^1(K) \subseteq M(K)$ has compact support. By hypothesis $M \in P^1(K)$, and so $x \mapsto \langle \delta_x * M, f\rangle$ is continuous. By [13, Theorem 3.27], the integral $\int \langle \delta_x * M, f\rangle \varphi(x) d\lambda(x)$ exists in the sense [13, Definition 3.26]. By hypothesis $M \in LIM(L^\infty(K))$, and so

$$\langle M, f\varphi\rangle = \int \langle \delta_x * M, f\rangle \varphi(x) d\lambda(x) = \langle M, f\rangle.$$

We conclude that $M \in TLIM(L^\infty(K))$.

(*ii*) We can show that $M \in Z_t(L^1(K)^{**})$ (recall that the topological center of $L^1(K)^{**}$ consists of all the functionals $F \in L^1(K)^{**}$ such that left multiplication by F is weak*-weak* continuous on $L^1(K)^{**}$. The topological center of $L^1(K)^{**}$ is denoted by $Z_t(L^1(K)^{**})$). Indeed, if $\{F_\alpha\}$ is a net in $L^1(K)^{**}$ and $F_\alpha \to F$ in the weak*-weak* topology, then

$$\langle MF_\alpha, f\rangle = \langle M, F_\alpha f\rangle \to \langle M, Ff\rangle$$

whenever $f \in L^\infty(K)$. Hence $MF_\alpha \to MF$ in the weak*-topology. Therefore $M \in Z_t(L^1(K)^{**}) = L^1(K)$, see [8]. By hypothesis, $M \in P^1(K) \cap LIM(L^\infty(K))$ and so $M \in TLIM(L^\infty(K))$.

(*iii*) Let $f \in L^\infty(K)$ and $\varphi \in P^1(K)$. We claim that

$$f\varphi \in \overline{\Big\{ \sum_{i=1}^n \alpha_i {}_{x_i}f; x_i \in K, \alpha_i \geq 0, n \in \mathbb{N}, \sum_{i=1}^n \alpha_i = 1 \Big\}},$$

where closure is taken in τ_c-topology. Indeed, let $\epsilon > 0$ be given. Choose $\delta > 0$ such that $\delta(2\|f\| + 1) = \epsilon$. There exists a compact subset C in K such that $\int_{K\setminus C} \hat{\varphi}(x) d\lambda(x) < \delta$. Let $y \in K$. As the mapping $x \mapsto {}_xf$ is τ_c-continuous, there exists a symmetric open neighborhood V_y of y in K such that $\|{}_xf - {}_yf\| < \delta$ whenever $x \in V_y$. We may determine a subset $\{x_1, ..., x_n\}$ in K such that $C \subseteq \bigcup_{i=1}^n V_{x_i}$ and $\|{}_xf - {}_{x_i}f\| < \delta$ whenever $x \in V_{x_i} \bigcap K$; $i = 1, ..., n$. Now, we consider $x_0 = e$, $C_0 = K \setminus C$, $C_1 = V_{x_1} \cap C$, $C_i = V_{x_i} \cap (C \setminus \bigcup_{k=1}^{i-1} V_{x_k})$ and $\alpha_i = \int_{C_i} \hat{\varphi}(x) d\lambda(x)$ for $i = 0, 1, ..., n$. Obviously, $\sum_{i=0}^n \alpha_i = 1$. For $\psi \in L^1(K)$,

$$\begin{aligned}
\Big|\langle f\varphi - \sum_{i=0}^n \alpha_i {}_{x_i}f, \psi\rangle\Big| &= \Big| \int_K f(\check{y} * x)\hat{\varphi}(y) - \sum_{i=0}^n \alpha_i {}_{x_i}f(x)\psi(x) d\lambda(x)\Big| \\
&\leq \sum_{i=0}^n \int_{C_i} \Big(f(\check{y} * x) - {}_{x_i}f(x)\Big)\hat{\varphi}(y)\psi(x) d\lambda(y) d\lambda(x) \\
&\leq \frac{\epsilon}{2}\|\psi\|_1 + \sum_{i=1}^n \int_{C_i} \|{}_yf - {}_{x_i}f\| \, \|\psi\|_1 \hat{\varphi}(y) d\lambda(y) < \epsilon\|\psi\|_1.
\end{aligned}$$

The statement follows from τ_c-continuity of M. □

Theorem 2.4. *If K is a locally compact hypergroup, the following properties are equivalent:*

(i) K is amenable.

(ii) For every $\psi \in L^1(K)$;

$$D_\psi = \inf\left\{\|\psi * \varphi\|_1;\ \varphi \in P^1(K)\right\} \leq \left|\int \psi(x)d\lambda(x)\right|.$$

Proof. Suppose that K is amenable. We assume first that $\int \psi(x)d\lambda(x) = 0$. Assume that $\inf\{\|\psi * \varphi\|_1;\ \varphi \in P^1(K)\} > 0$. Let $n \in \mathbb{N}$. Separating $\{\phi \in L^1(K);\ \|\phi\|_1 < \frac{n}{n+1}D_\psi$ and $\{\psi * \varphi;\ \varphi \in P^1(K)\}$, we can find $f \in L^\infty(K)$ and $\alpha \in \mathbb{R}$ such that

$$Re\langle f, \phi\rangle \leq \alpha < Re\langle f, \psi * \varphi\rangle \qquad (1)$$

for all $\varphi \in P^1(K)$ and $\phi \in L^1(K)$ with $\|\phi\|_1 < \frac{n}{n+1}D_\psi$. Without loss of generality, we can assume that $\|f\| = 1$. Thus $\sup\{Re\langle f, \phi\rangle;\ \|\phi\|_1 < \frac{n}{n+1}D_\psi\} = \frac{n}{n+1}D_\psi$, and so we can take $\alpha = \frac{n}{n+1}D_\psi$ in (1). For every $n \in \mathbb{N}$, let

$$A_n = \left\{g \in L^\infty(K); \|g\| = 1, Re\langle g, \phi\rangle \geq \frac{n}{n+1}D_\psi \text{ for all } \phi \in \{\psi * \varphi;\ \varphi \in P^1(K)\}\right\}.$$

The sets A_n are obviously relatively weak*-compact. Since each A_n is nonempty and since A_n is relatively weak*-compact, it follows that the weak*-closures of the sets A_n have on point g in common. It follows that $Re\langle g, \psi * \varphi\rangle \geq D_\psi$ for all $\varphi \in P^1(K)$. Now let M be a topologically left invariant mean on $L^\infty(K)$. Let $\{\varphi_\alpha\}$ be a net in $P^1(K)$ such that $\varphi_\alpha \to M$ in the weak*-topology. We get that

$$\begin{aligned} |\langle M, g\psi\rangle| &= \lim_\alpha |\langle \varphi_\alpha, g\psi\rangle| = \lim_\alpha |\langle g, \psi * \varphi_\alpha\rangle| \\ &\geq \lim_\alpha Re\langle g, \psi * \varphi_\alpha\rangle \geq D_\psi. \end{aligned}$$

Write $\psi = (\psi_1^+ - \psi_1^-) + i(\psi_2^+ - \psi_2^-)$ where ψ_1, ψ_2 are respectively the real and imaginary parts of ψ, and for $i = 1, 2$, ψ_i^+ and ψ_i^- are respectively the positive and negative parts of ψ_i. It is easy to see that

$$\langle M, g\psi\rangle = \langle M, g\psi_1^+\rangle - \langle M, g\psi_1^-\rangle + i\langle M, g\psi_2^+\rangle - i\langle M, g\psi_2^-\rangle = \int \psi(x)d\lambda(x) = 0.$$

Thus $D_\psi = 0$ giving a contradiction.

Now suppose $\psi \in L^1(K)$ is general, and let $\varphi_0 \in P^1(K)$, $c = \int \psi(x)d\lambda(x)$. Applying the preceding result to $\psi - c\varphi_0$, we have $\inf\{\|\psi * \varphi - c\varphi_0 * \varphi\|_1;\ \varphi \in P^1(K)\} = 0$. For every $\varphi \in P^1(K)$,

$$\begin{aligned} \|\psi * \varphi - c\varphi_0 * \varphi\|_1 &\geq \|\psi * \varphi\|_1 - |c|\|\varphi_0 * \varphi\|_1 \\ &= \|\psi * \varphi\|_1 - |c|. \end{aligned}$$

It follows that $\inf\{\|\psi * \varphi\|_1;\ \varphi \in P^1(K)\} \leq |c|$.

Conversely, assume that the condition holds for all $\psi \in L^1(K)$. We claim that for every finite subset F of $P^1(K)$ and $\epsilon > 0$, there exists $\varphi_{F,\epsilon} \in P^1(K)$ such that $\|\varphi_{F,\epsilon} * \varphi - \varphi_{F,\epsilon}\|_1 < \epsilon$ for all $\varphi \in F$. Let $\phi_0 \in P^1(K)$ be fixed. Let $F = \{\varphi_1, ..., \varphi_n\}$ and $\epsilon > 0$. Obviously $\int \varphi_1 * \phi_0 - \phi_0(x)d\lambda(x) = 0$. By hypothesis, there exists $\phi_1 \in P^1(K)$ such that $\|(\varphi_1 * \phi_0 - \phi_0) * \phi_1\|_1 < \epsilon$. Since $\int \varphi_2 * \phi_0 * \phi_1 - \phi_0 * \phi_1(x)d\lambda(x) = 0$, we can find $\phi_2 \in P^1(K)$ such that

$$\|(\varphi_2 * \phi_0 * \phi_1 - \phi_0 * \phi_1) * \phi_2\|_1 < \epsilon.$$

Proceeding in this way we can find $\phi \in P^1(K)$ such that $\|\varphi_i * \phi - \phi\|_1 < \epsilon$ whenever $i \in \{1, ..., n\}$. This shows that K is amenable. □

Let E be a linear space. A representation π of $P^1(K)$ is a homomorphism $\varphi \mapsto \pi_\varphi$ into the algebra of operators on E. That is, for each $\varphi \in P^1(K)$, π_φ is an operator on E, and for $\varphi, \psi \in P^1(K)$, we have $\pi_{\varphi*\psi} = \pi_\varphi \pi_\psi$. π is called weakly continuous provided, for each $x \in E$ and $x^* \in E^*$, the function $\varphi \mapsto \langle x^*, \pi_\varphi(x)\rangle$ is continuous on $P^1(K)$. In the next Theorem FP denotes the set of fixed points of $\{\pi_\varphi;\ \varphi \in P^1(K)\}$, i.e.,

$$FP = \Big\{x \in E;\ \pi_\varphi(x) = x \text{ for all } \varphi \in P^1(K)\Big\}.$$

Theorem 2.5. *For a locally compact hypergroup K, the following are equivalent:*

(i) K is amenable.

(ii) Whenever π is a weakly continuous representation of $P^1(K)$ by weakly continuous operators on E such that $\{\pi_\varphi(x);\ \varphi \in P^1(K)\}$ is relatively weakly compact for every $x \in E$, then

$$\Big\{\pi_\varphi(x);\ \varphi \in P^1(K)\Big\} \cap FP \neq \emptyset.$$

Proof. Assume that $L^\infty(K)$ has a topological left invariant mean. By [14, Theorem 4.1], there is a net $\varphi_\alpha \in P^1(K)$ such that $\varphi * \varphi_\alpha - \varphi_\alpha \to 0$ in the weak topology. Consider the net $\pi_{\varphi_\alpha}(x)$ where $x \in E$ is arbitrary but fixed. By compactness of $\overline{\{\pi_\varphi(x); \varphi \in P^1(K)\}}$, without loss of generality, we can assume that $\pi_{\varphi_\alpha}(x) \to x_0$ in the weak topology. We claim that x_0 is the required fixed point. For it $\varphi \in P^1(K)$, we have

$$\begin{aligned} \pi_\varphi(x_0) &= \pi_\varphi\Big(\lim_\alpha \pi_{\varphi_\alpha}(x)\Big) = \lim_\alpha \pi_\varphi\Big(\pi_{\varphi_\alpha}(x)\Big) = \lim_\alpha \pi_\varphi\Big(\pi_{\varphi_\alpha}(x)\Big) \\ &= \lim_\alpha \pi_{\varphi*\varphi_\alpha}(x) = \lim_\alpha \pi_{\varphi_\alpha}(x) = x_0. \end{aligned}$$

Conversely, let $E = LUC(K)^*$ with weak*-topology. For each $\varphi \in P^1(K)$, define a map $\pi_\varphi : LUC(K)^* \to LUC(K)^*$ by $\pi_\varphi(F) = \varphi F$. Recall that weak*-topology of $LUC(K)^*$ is a locally convex vector topology on $LUC(K)^*$ and that every linear functional on $LUC(K)^*$ that is weak*-continuous has the form $F \mapsto \langle F, f\rangle$ for some $f \in LUC(K)$. Therefore $(LUC(K)^*)^* = LUC(K)$. We claim that π_φ is weak-weak continuous. For if $F_\alpha \to F$ in the weak topology, then for any $f \in LUC(K)$, $\langle F_\alpha, f\rangle \to \langle F, f\rangle$. Obviously, $\langle \varphi F_\alpha, f\rangle \to \langle \varphi F, f\rangle$. Hence $\pi_\varphi(F_\alpha) \to \pi_\varphi(F)$ weakly in $LUC(K)^*$. It is easy to see that $\pi_{\varphi*\psi} = \pi_\varphi\pi_\psi$ for all $\varphi, \psi \in P^1(K)$. By the Banach-Alaoglu theorem $\{\pi_\varphi(F); \varphi \in P^1(K)\}$ is relatively weakly compact in $LUC(K)^*$ for every $F \in LUC(K)^*$. It follows from the compactness of the product space $\prod\{\overline{\pi_\varphi(F)};\ F \in LUC(K)^*\}$ that $\overline{\{\pi_\varphi;\ \varphi \in P^1(K)\}}$ is a compact subset of $\mathcal{L}(LUC(K)^*)$ in the weak operator topology. By hypothesis,

$$C(F) = \Big\{T \in \overline{\{\pi_\varphi;\ \varphi \in P^1(K)\}};\ T(F) \in \mathcal{F} \neq \emptyset\Big\};$$

and $C(F)$ is a compact subset of $\overline{\{\pi_\varphi; \varphi \in P^1(K)\}}$. We showing that the family $\{C(F); F \in LUC(K)^*\}$ has the finite intersection property. We prove that if, for $F_1, ..., F_{n+1} \in LUC(K)^*$, $\bigcap\{C(F_i);\ 1 \leq i \leq n\} \neq \emptyset$, then also $\bigcap\{C(F_i);\ 1 \leq i \leq n+1\} \neq \emptyset$. If $T_1 \in \bigcap\{C(F_i);\ 1 \leq i \leq n\}$ and if $T_2 \in C(T_1(F_{n+1}))$, then for every $i = 1, ..., n$ and every $T \in \overline{\{\pi_\varphi; \varphi \in P^1(K)\}}$,

$$T(T_2T_1(F_i)) = TT_2(T_1(F_i)) = T_1(F_i) = T_2T_1(F_i).$$

Moreover $T(T_2T_1(F_{n+1})) = T_2T_1(F_{n+1})$. Hence $T_2T_1 \in \bigcap\{C(F_i);\ 1 \leq i \leq n+1\}$. Since $\overline{\{\pi_\varphi;\ \varphi \in P^1(K)\}}$ is compact, it follows that $\bigcap\{C(F);\ F \in LUC(K)^*\} \neq \emptyset$. If P is any member of this intersection, then there is an operator $P \in \mathcal{L}(LUC(K)^*)$ and a net $\{T_\alpha\}$ in $\{\pi_\varphi;\ \varphi \in P^1(K)\}$ such that $T_\alpha(F) \to P(F)$ for all $F \in LUC(K)^*$ and $\pi_\varphi P = P$ for all $\varphi \in P^1(K)$. If M is any mean on $LUC(K)$, it is easy to see that $MP \in TLIM(LUC(K)^*)$. □

Acknowledgments

I would like to thank the referee for his/her careful reading of my paper and many valuable suggestions.

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