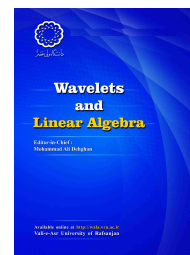


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### Triangularization over finite-dimensional division rings using the reduced trace

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#### ABSTRACT

In this paper we study triangularization of collections of matrices whose entries come from a finite-dimensional division ring. First, we give a generalization of Guralnick's theorem to the case of finite-dimensional division rings and then we show that in this case the reduced trace function is a suitable alternative for trace function by presenting two triangularization results. The first one is a generalization of a result due to Kaplansky and in the second one a triangularizability condition which is dependent on a single element is presented.

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#### 1. Introduction

Let  $\mathcal{V}$  be a finite dimensional right vector space over a division ring  $D$  and  $C$  be a collection of linear transformations on  $\mathcal{V}$ . A subspace  $\mathcal{M}$  of  $\mathcal{V}$  is said to be *invariant* under  $C$  if for all  $T \in C$ ,  $T\mathcal{M} \subseteq \mathcal{M}$ . The collection  $C$  is said to be *reducible* if  $C = \{0\}$  or there exists a subspace of  $\mathcal{V}$

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which is invariant under  $C$  and is different from  $\{0\}$  and  $\mathcal{V}$ . We say that  $C$  is *irreducible* if it is not reducible. The collection  $C$  is called *simultaneously triangularizable* or simply *triangularizable*, if there exists a maximal chain of subspaces of  $\mathcal{V}$  each of which is invariant under  $C$ . It is easy to see that if  $C$  is triangularizable then there exists a basis for  $\mathcal{V}$  such that the matrix representation of each operator in  $C$  with respect to this basis is an upper triangular matrix.

If  $C$  is a collection of square matrices over  $D$ , triangularizability is defined similarly, viewing  $C$  as a set of operators acting on  $D^n$  which is considered as a right vector space over  $D$ . Thus,  $C$  is triangularizable if and only if there exists an invertible  $n \times n$  matrix  $S$  with entries in  $D$  such that for all  $T \in C$ ,  $STS^{-1}$  is an upper triangular matrix. The division ring  $D$  is called *finite-dimensional* if the dimension of  $D$  as a vector space over its center  $F$ , is finite. A subfield  $L$  of  $D$  is called a *maximal subfield* of  $D$ , if there is no other subfield of  $D$  containing  $L$ . A field is called a *perfect* field if its finite extensions are simple extensions. A *semigroup* is a collection of matrices(operators) which is closed under multiplication.

Triangularizability of collections of matrices over fields has been of interest to several authors. In most cases, they consider certain collections such as semigroups, rings and algebras and look for conditions which guarantee the triangularizability of the collection. A survey of such results can be found in [10]. In case of matrices over a general division ring, the problem is harder and results are very restricted. Walter Sizer in his thesis [11] gathered a useful collection of triangularizability results over a general division ring but he considered similarities over extensions of the ground division ring. For more recent results one can refer to [6, 7, 12, 13]. For collections of matrices over quaternions stronger results are available [1, 7].

In this paper, we show that when  $D$  is finite-dimensional over its center  $F$ , some well-known results in case of fields can be extended to the case of division rings. In the next section, imposing a slight restriction on  $F$ , we show that for a semigroup  $\mathcal{S}$  of matrices over  $D$ , if  $AB - BA$  is nilpotent for all  $A, B \in \mathcal{S}$ , then  $\mathcal{S}$  is triangularizable; generalizing a result due to Guralnick [3]. In the third section, using the reduced trace function, we generalize a theorem of Kaplansky [4] saying that a semigroup of matrices with constant trace is triangularizable over the algebraic closure of the ground field, provided that the characteristics of the field is zero or greater than half the size of matrices. At the end, we consider the fact that if a triangularizable semigroup  $\mathcal{S}$  contains a nilpotent matrix  $A$ , then  $AX$  is also nilpotent and has reduced trace equal to zero for all  $X \in \mathcal{S}$ . We show that under some restrictions the converse holds. Now, we fix some notations.

For a division ring  $D$  with center  $F$  and a natural number  $n$ , the algebra of all square matrices of order  $n$  with entries in  $D$  will be denoted by  $M_n(D)$ . We denote the characteristics of  $D$  by  $\text{Char}(D)$ . The dimension of  $D$  over  $F$  is denoted by  $\dim_F D$ . When  $D$  is finite-dimensional, it is well-known that  $\dim_F D$  is the square of a natural number. We define the *degree* of  $D$  to be the square root of  $\dim_F D$  and denote it by  $\text{deg}(D)$ . For a collection  $C$  of matrices, the term  $\text{Alg}_F(C)$  will denote the  $F$ -algebra generated by  $C$ . It is easy to see that if  $C$  is a semigroup,  $\text{Alg}_F(C)$  is equal to the linear span of  $C$  over  $F$ . The *trace* of a matrix  $A$  in  $M_n(D)$  is the summation of the diagonal entries of  $A$  and is denoted by  $\text{tr}(A)$ . The term  $I_n$  represents the identity matrix of size  $n$  and the algebraic closure of a field  $L$  is denoted by  $\bar{L}$ .

## 2. Some generalizations to finite-dimensional division rings

Finite-dimensional division rings are very important in the theory of division rings since they have many applications and are easier to work with. One of the important properties of this class of division rings is the existence of a representation for them in the algebra of matrices over a field. Since we apply this representation several times throughout this paper, we give a brief review of it here. For a detailed study one can refer to [2].

Let  $D$  be a finite-dimensional division ring with center  $F$  and let  $L$  be any maximal subfield of  $D$ . Then,  $L$  contains  $F$  and  $\dim_F L = \deg(D)$ . That is,

$$\dim_F D = (\dim_F L)^2.$$

If  $\deg(D) = m$ , then we have an  $F$ -algebra isomorphism

$$\varphi : M_n(D) \otimes_F L \longrightarrow M_{nm}(L). \quad (2.1)$$

The isomorphism  $\varphi$  sends  $\lambda I_n \otimes 1$  to  $\lambda I_{nm}$  for all  $\lambda \in F$ . If we identify the subset  $M_n(D) \otimes 1$  of the left hand side by  $M_n(D)$ , then 2.1 shows that  $M_n(D)$  can be embedded in  $M_{nm}(L)$ . In particular, when  $n = 1$ , 2.1 takes the form

$$\varphi : D \otimes_F L \longrightarrow M_m(L). \quad (2.2)$$

The main result of this section is the generalization of a theorem due to Guralnick which asserts that a semigroup  $\mathcal{S}$  of triangularizable matrices over a field is triangularizable, if and only if  $AB - BA$  is nilpotent for all  $A, B \in \mathcal{S}$  [3]. Another generalization can be found in [12]. We assume that either  $F$  is a perfect field or  $\text{Char}(F) = p > nm/2$ . This is a slight restriction since when  $\text{Char}(F) = 0$ ,  $F$  is a perfect field. To prove the main result of this section, we first consider algebras of matrices.

**Lemma 2.1.** *Let  $n, m \in \mathbb{N}$  and let  $D$  be a finite-dimensional division ring with center  $F$  and  $\deg(D) = m$  such that either  $F$  is perfect or  $\text{Char}(F) = p > nm/2$ . Let  $\mathcal{A}$  be an  $F$ -algebra of triangularizable matrices in  $M_n(D)$  such that  $AB - BA$  is nilpotent for all  $A, B \in \mathcal{A}$ . Then,  $\mathcal{A}$  is triangularizable.*

*Proof.* Clearly, all properties of  $\mathcal{A}$  stated in the statement of the theorem are inherited by quotients. Thus, using the Triangularization Lemma [11], we need only to prove that  $\mathcal{A}$  is reducible, if  $n > 1$ . By contradiction, if  $\mathcal{A}$  is irreducible it contains the identity matrix and it follows from Density Theorem [5, P. 192] that  $\mathcal{A} \simeq M_s(E)$  for some  $s \in \mathbb{N}$  and some division  $F$ -algebra  $E$ . Now, the condition that  $AB - BA$  is nilpotent for all  $A, B \in \mathcal{A}$  immediately implies that  $s = 1$  and  $E$  is a field. Thus,  $\mathcal{A}$  is a finite field extension of  $F = FI_n$ . We show that  $\mathcal{A}$  is a simple extension of  $F$ . Let  $K$  be the separable closure of  $F$  in  $\mathcal{A}$ . If  $\mathcal{A} = K$ , we are done. If not,  $F$  is not perfect,  $\text{Char}(F) = p > nm/2$  and  $\mathcal{A}$  is purely inseparable over  $K$ . So,  $p$  divides  $\dim_K \mathcal{A}$  which implies that it divides  $\dim_F \mathcal{A}$  as well. Now,  $\mathcal{A}$  is contained in a maximal subfield of  $M_n(D)$  and hence  $\dim_F \mathcal{A} \leq nm < 2p$  [2, P. 45, Theorem 3]. So,  $\dim_F \mathcal{A} = p$  and  $\mathcal{A}$  is a simple extension of  $F$ . Thus,  $\mathcal{A} = F[A]$  for some  $A \in \mathcal{A}$ . Since  $A$  is triangularizable,  $\mathcal{A}$  is reducible; a contradiction.  $\square$

Now, we can prove the theorem for semigroups of matrices.

**Theorem 2.2.** *Let  $n, m \in \mathbb{N}$  and let  $D$  be a finite-dimensional division ring with center  $F$  and  $\deg(D) = m$  such that either  $F$  is perfect or  $\text{Char}(F) = p > nm/2$ . Let  $\mathcal{S}$  be a semigroup of triangularizable matrices in  $M_n(D)$  such that  $AB - BA$  is nilpotent for all  $A, B \in \mathcal{S}$ . Then,  $\mathcal{S}$  is triangularizable.*

*Proof.* Let  $L, \varphi$ , and  $m$  be as in 2.1. Thus,  $AB - BA$  is nilpotent for any pair of matrices  $A$  and  $B$  in  $\varphi(\mathcal{S})$ . By Guralnick's theorem [3],  $\varphi(\mathcal{S})$  is triangularizable over  $\bar{L}$ . So,  $\text{Alg}_F(\varphi(\mathcal{S}))$  is also triangularizable over  $\bar{L}$ . Again  $AB - BA$  is nilpotent for any pair of matrices  $A$  and  $B$  in  $\text{Alg}_F(\varphi(\mathcal{S}))$  which means that  $AB - BA$  is nilpotent for any pair of matrices  $A$  and  $B$  in  $\text{Alg}_F(\mathcal{S})$  in  $M_n(D)$ . Now, by Lemma 2.1,  $\text{Alg}_F(\mathcal{S})$  is triangularizable. This completes the proof.  $\square$

**Corollary 2.3.** *Let  $n, m \in \mathbb{N}$  and let  $D$  be a finite-dimensional division ring with center  $F$  and  $\deg(D) = m$  such that either  $F$  is perfect or  $\text{Char}(F) = p > nm/2$ . Then, any commutative semigroup of triangularizable matrices in  $M_n(D)$  is triangularizable.*

### 3. Reduced trace; a suitable alternative

The trace function plays an important role in the theory of triangularization over fields. In case of division rings, trace is not very useful since it is not preserved under similarities. In the theory of finite-dimensional division rings the reduced trace function is well-known as an alternative for the trace function. In this section we use the reduced trace as a tool in the theory of triangularization over division rings. First, we give a brief review of reduced trace and its properties adopted from [2].

Let  $L, \varphi$ , and  $m$  be as in 2.1. For  $A \in M_n(D)$ , the reduced trace of  $A$  is defined as

$$\text{RTr}_{M_n(D)/F}(A) = \text{tr}(\varphi(A \otimes 1)). \quad (3.1)$$

Surprisingly, for all  $A \in M_n(D)$ ,  $\text{RTr}_{M_n(D)/F}(A)$  belongs to  $F$  and does not depend on the chosen maximal subfield  $L$  and the isomorphism  $\varphi$ . It is immediate from the definition that the reduced trace has the following properties.

- (i)  $\text{RTr}_{M_n(D)/F}$  is  $F$ -linear.
- (ii)  $\text{RTr}_{M_n(D)/F}(aI_n) = nma$  if  $a \in F$ .
- (iii)  $\text{RTr}_{M_n(D)/F}(AB) = \text{RTr}_{M_n(D)/F}(BA)$  for all  $A, B \in M_n(D)$ .

It is worth mentioning that, in case  $n = 1$ ,  $\text{RTr}_{D/F} : D \rightarrow F$  is of special importance. Moreover, when  $D = F$ , the reduced trace is just the trace function.

The following theorem, reveals the relation between the reduced trace and the usual trace function. Note that, unlike the trace function, reduced trace is preserved under similarity.

**Theorem A.** *Let  $D$  be a finite-dimensional division ring with center  $F$  and  $n \in \mathbb{N}$ . Then,*

$$\text{RTr}_{M_n(D)/F}(A) = \text{RTr}_{D/F}(\text{tr}(A))$$

for all  $A \in M_n(D)$ .

Let  $F$  be a field with  $\text{Char}(F) = 0$  or  $\text{Char}(F) > n/2$  and let  $\mathcal{S}$  be a semigroup in  $M_n(F)$ . A well-known theorem due to Kaplansky states that if trace is constant on  $\mathcal{S}$ , then  $\mathcal{S}$  is triangularizable over  $\bar{F}$  [4]. Here, we generalize this result.

**Theorem 3.1.** *Let  $n, m \in \mathbb{N}$ , and let  $D$  be a finite-dimensional division ring with center  $F$  and  $\deg(D) = m$ . Assume that  $\text{Char}(D) = 0$  or  $\text{Char}(D) > nm/2$ . Let  $\mathcal{S}$  be a semigroup of triangularizable matrices in  $M_n(D)$  with constant reduced trace. Then,  $\mathcal{S}$  is triangularizable.*

*Proof.* Let  $L, \varphi$ , and  $m$  be defined as in 2.1. According to the assumption, the semigroup  $\varphi(\mathcal{S})$  in  $M_{nm}(L)$  has constant trace. Thus, it is triangularizable over  $\bar{L}$  in view of Kaplansky's theorem. So,  $AB - BA$  is nilpotent for all  $A$  and  $B$  in  $\varphi(\mathcal{S})$ ; which in turn implies that  $AB - BA$  is nilpotent for all  $A$  and  $B$  in  $\mathcal{S}$ . Now the proof is complete using Theorem 2.2.  $\square$

Using Theorem A, we have the following corollary of the above theorem.

**Corollary 3.2.** *Let  $n, m \in \mathbb{N}$ , and let  $D$  be a finite-dimensional division ring with  $\deg(D) = m$ . Assume that  $\text{Char}(D) = 0$  or  $\text{Char}(D) > nm/2$ . Let  $\mathcal{S}$  be a semigroup of triangularizable matrices in  $M_n(D)$  with constant trace. Then,  $\mathcal{S}$  is triangularizable.*

Let  $F$  be a field and  $\mathcal{A}$  be an  $F$ -algebra of triangularizable matrices in  $M_n(F)$ . It is well-known that  $\mathcal{A}$  is triangularizable if and only if for  $A, B \in \mathcal{A}$ ,  $AB$  is nilpotent, whenever  $A$  or  $B$  is nilpotent [10]. When  $\mathcal{A}$  is triangularizable and  $A$  is a fixed nilpotent matrix in  $\mathcal{A}$ , again  $AB$  is nilpotent for all  $B \in \mathcal{A}$ . The converse is not true unless  $A^{n-1} \neq 0$ . In fact, we have the following generalization to semigroups of matrices over division rings.

**Theorem 3.3.** *Let  $n, m \in \mathbb{N}$ , and let  $D$  be a finite-dimensional division ring with center  $F$  where  $\text{Char}(D) = 0$  or  $\text{Char}(D) > nm/2$ . Assume that  $\mathcal{S}$  is a semigroup of triangularizable matrices in  $M_n(D)$  containing a nilpotent matrix  $A$  such that  $A^{n-1} \neq 0$ . Then,  $\mathcal{S}$  is triangularizable if and only if  $\text{RTr}_{M_n(D)/F}(AX) = 0$  for all  $X \in \mathcal{S}$ .*

*Proof.* Again let  $L, \varphi$ , and  $m$  be as in 2.1. If  $\mathcal{S}$  is triangularizable, the result follows from Theorem A. For the converse, note that similar to the case of fields  $A$  is similar to an elementary Jordan block [7] and it is easy to see that it has a unique triangularizing chain obtained by the standard basis of  $D^n$ . So, it suffices to prove that for any  $B \in \mathcal{S}$ , the pair  $\{A, B\}$  is triangularizable. Let  $\mathcal{T}$  be the semigroup generated by  $A$  and  $B$ . It is straight to see that the trace function is permutable on  $\varphi(\mathcal{T})$ . By Radjavi's trace theorem [9],  $\varphi(\mathcal{T})$  is triangularizable over  $\bar{L}$  and by Theorem 2.2  $\mathcal{T}$  is triangularizable.  $\square$

*Remark 3.4.* The original proof of Theorem 3.3 is rather longer. The presented proof is adopted from a proof provided by the referee in case of fields(Corollary 3.5).

**Corollary 3.5.** *Let  $n \in \mathbb{N}$ , and let  $F$  be a field with  $\text{Char}(F) = 0$  or  $\text{Char}(F) > n/2$ . Assume that  $\mathcal{S}$  is a semigroup of triangularizable matrices in  $M_n(F)$  containing a nilpotent matrix  $A$  such that  $A^{n-1} \neq 0$ . Then,  $\mathcal{S}$  is triangularizable if and only if  $\text{tr}(AX) = 0$  for all  $X \in \mathcal{S}$ .*

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