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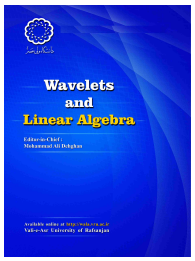

Quartic and pantic B-spline operational matrix of fractional integration

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Abstract

In this work, we proposed an effective method based on quartic and pantic B-spline scaling functions to solve partial differential equations of fractional order. Our method is based on dual functions of B-spline scaling functions. We derived the operational matrix of fractional integration of quartic and pantic B-spline scaling functions and used them to transform the mentioned equations to a system of algebraic equations. Some examples are presented to show the applicability and effectivity of the technique.

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1. Introduction

When we speak about integrals and derivatives of arbitrary order, we deal with fractional calculus. The first time Leibniz discuss the derivative of order $\alpha = \frac{1}{2}$ in a letter to L' Hopital in 1695 [8]. In recent years, the fractional calculus has been more interesting. They apply in many fields of science and engineering such as electrical networks, electromagnetic theory, and probability. There are some numerical method to solve integral equations of fractional order such as Least Squares Method [9], homotopy analysis [1] Sumudu decomposition method [4] and so on. The wavelet method is applicable for solving fractional equations; Haar wavelet [11], CAS wavelet [12], Chebyshev wavelets [7] and B-spline wavelet [6].

In this paper, we use B-spline scaling function of order 4 and 5. Explicit formula of B-spline and the regularity, symmetric and compact support of B-splines persuade us to use them for fractional equations.

This paper is organized as follows: In section 2 we describe some preliminaries on spline functions and fractional calculus. Fractional integral of quartic and pantic B-spline scaling function and the operational matrix in the fractional case are given in section 3. In section 4, we presented a technic for solving partial differential equations of fractional order by splines. We present some examples in section 5 to show the validity of the method.

2. Preliminaries

2.1. Fractional Integration and Derivatives

The primary objects of classical calculus are derivative and integral of functions. These two operations are inverse to each other in some sense. The Riemann-Liouville approach is based on the Cauchy formula for the n-fold integration:

$$I_a^n f(x) = \int_a^x \int_a^{t_{n-1}} \cdots \int_a^{t_1} f(t)\mathrm{d}t\mathrm{d}t_1 \cdots \mathrm{d}t_{n-1} = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)\mathrm{d}t, \tag{2.1}$$

Therefore it is a good basis for generalization. We generalize the Cauchy formula (2.1) in a way that the integer n is substituted by a positive real number α and the Gamma function is employed rather than the factorial [8]:

$$\begin{aligned} I_a^\alpha f(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\mathrm{d}t,\ \alpha > 0, \\ I^0 f(x) &= f(x). \end{aligned} \tag{2.2}$$

where $\Gamma(.)$ is the Gamma function, $x \in \mathbb{R}$, and $a > 0$. The fractional derivatives are described by using fractional integrals [5]. The Caputo fractional derivatives of order α is identified as:

$$\begin{aligned} {}^C D^\alpha f(x) &= I^{n-\alpha}(\frac{\mathrm{d}}{\mathrm{d}x})^n f(x) \\ &= \frac{1}{\Gamma(n-\alpha)}\int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}}\mathrm{d}t, n-1 < \alpha \le n, n \in \mathbb{N}. \end{aligned} \tag{2.3}$$

One of the useful and applicable relation between fractional integrals and derivatives in partial case is:

Lemma 2.1. *[10] If $l-1<\alpha\leq l$, and $n-1<\beta\leq n$ where $l,n\in\mathbb{N}$, then*

$$I_x^{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}u(x,t)=u(x,t)-\sum_{k=0}^{l-1}\frac{\partial^k}{\partial x^k}u(0^+,t)\frac{x^k}{k!},$$

and

$$I_t^{\beta}\frac{\partial^{\beta}}{\partial t^{\beta}}u(x,t)=u(x,t)-\sum_{k=0}^{n-1}\frac{\partial^k}{\partial t^k}u(x,0^+)\frac{t^k}{k!}.$$

2.2. B-spline functions

A spline is a function that is piecewise defined by polynomial functions, and possesses a high degree of smoothness at the knots, i.e. the places where the polynomial pieces connect. In this paper, In this paper we will use quartic and pantic splines to solve fractional partial differential equations and compare their results.

Definition 2.2. The cardinal B-splines $N_m(x)$ of order m are defined inductively by the following convolution product:

$$N_1(x)=\chi_{[0,1]}(x), \tag{2.4}$$
$$N_m(x)=N_1(x)*N_{m-1}(x),\ (m\geq 2). \tag{2.5}$$

B-splines of order m satisfy in two-scale relationship:

$$N_m(x)=\sum_{k=0}^{m}2^{1-m}\left(\begin{array}{c}m\\k\end{array}\right)N_m(2x-k). \tag{2.6}$$

Hence, these compactly supported functions generate an MRA with the dilation equations [2].

The B-spline of order m occupies m segments. Since for any j, the discretization step is $\frac{1}{2^j}$, thus for $j>0$, there are 2^j segments in $[0,1]$. Therefore, to have one or more inner scaling functions, we should have:

$$2^j>m. \tag{2.7}$$

Now for simplicty we assume $N_{m,j,k}(.)=N_m(2^j.-k)$, and $x_j=2^jx$. To take the scaling functions into the interval $[0,1]$ we consider

$$\varphi_{m,j,k}(x)=N_{m,j,k}(x)\chi_{[0,1]}(x), \tag{2.8}$$

where the quartic and pantic B-spline scaling functions are as follows:

$$N_{4,j,k}(x)=\frac{1}{6}\begin{cases}(x_j-k)^3, & k\leqslant x_j<k+1,\\ -3(x_j-k-1)^3+3(x_j-k-\frac{1}{2})^2+\frac{1}{4}, & k+1\leqslant x_j<k+2,\\ 3(x_j-k-2)^3-6(x_j-k-2)^2+4, & k+2\leqslant x_j<k+3,\\ (k+4-x_j)^3, & k+3\leqslant x_j<k+4,\\ 0, & \text{o.w.}\end{cases}$$

for $k = -3, \cdots, 2^j - 1$, and

$$N_{5,j,k}(x) = \frac{1}{24}\begin{cases} (x_j-k)^4, & k \leqslant x_j < k+1, \\ -4(x_j-k)^4 + 20(x_j-k)^3 - 30(x_j-k)^2 + 20(x_j-k) - 5, & k+1 \leqslant x_j < k+2, \\ 6(x_j-k)^4 - 60(x_j-k)^3 + 210(x_j-k)^2 - 300(x_j-k) + 155, & k+2 \leqslant x_j < k+3, \\ -4(x_j-k)^4 + 60(x_j-k)^3 - 330(x_j-k)^2 + 780(x_j-k) - 655, & k+3 \leqslant x_j < k+4, \\ (x_j-k-5)^4, & k+4 \leqslant x_j < k+5, \\ 0, & \text{o.w.} \end{cases}$$

for $k = -4, \cdots, 2^j - 1$.

2.3. *fractional integration of B-spline*

Now we present the fractional integration of B-spline scaling functions. For calculating the fractional integration of quartic B-spline for $k = 0, 1, \cdots, 2^j - 4$, by letting $x_j = 2^j x$ we can rewrite $\varphi_{4,j,k}(x)$ by using unit step function as the following:

$$\begin{aligned} \varphi_{4,j,k}(x) = {} & (x_j-k)^3 u(x-\frac{k}{2^j}) - 4(x_j-(k+1))^3 u(x-\frac{k+1}{2^j}) + 6(x_j-(k+2))^3 u(x-\frac{k+2}{2^j}) \\ & - 4(x_j-(k+3))^3 u(x-\frac{k+3}{2^j}) + (x_j-(k+4))^3 u(x-\frac{k+4}{2^j}). \end{aligned} \tag{2.9}$$

By taking Laplace transform of equation (2.9), we have:

$$\mathcal{L}\{\varphi_{4,j,k}(x)\} = \frac{2^{3j}}{s^4}(e^{-\frac{k}{2^j}s} - 4e^{-\frac{k+1}{2^j}s} + 6e^{-\frac{k+2}{2^j}s} - 4e^{-\frac{k+3}{2^j}s} + e^{-\frac{k+4}{2^j}s}),$$

thus, the Laplace transform of $I^\alpha \varphi_{4,j,k}(x)$ is:

$$\begin{aligned} \mathcal{L}\{I^\alpha \varphi_{4,j,k}(x)\} &= \frac{1}{\Gamma(\alpha)}\mathcal{L}\{x^{\alpha-1}\}\mathcal{L}\{\varphi_{4,j,k}(x)\} \\ &= \frac{2^{3j}}{s^{\alpha+4}}(e^{-\frac{k}{2^j}s} - 4e^{-\frac{k+1}{2^j}s} + 6e^{-\frac{k+2}{2^j}s} - 4e^{-\frac{k+3}{2^j}s} + e^{-\frac{k+4}{2^j}s}). \end{aligned}$$

Consequently, by taking Laplace inverse on both side of above equation, we have:

$$\begin{aligned} I^\alpha \varphi_{4,j,k}(x) = \frac{2^{3j}}{\Gamma(\alpha+4)}[& (x_j-k)^{\alpha+3} u(x-\frac{k}{2^j}) - 4(x_j-(k+1))^{\alpha+3} u(x-\frac{k+1}{2^j}) \\ & + 6(x_j-(k+2))^{\alpha+3} u(x-\frac{k+2}{2^j}) - 4(x_j-(k+3))^{\alpha+3} u(x-\frac{k+3}{2^j}) \\ & + (x_j-(k+4))^{\alpha+3} u(x-\frac{k+4}{2^j})], \end{aligned}$$

So by putting $b = \frac{2^{3j}}{\Gamma(\alpha+4)}$, for $k = 0, \cdots, 2^j - 4$, $j \geq 2$ the fractional integration of quartic B-spline is as follows:

$$I^\alpha \varphi_{4,j,k}(x) = b\begin{cases} 0, & x_j < k, \\ (x_j-k)^{\alpha+3}, & k \leq x_j < k+1, \\ (x_j-k)^{\alpha+3} - 4(x_j-(k+1))^{\alpha+3}, & k+1 \leq x_j < k+2, \\ (x_j-k)^{\alpha+3} - 4(x_j-(k+1))^{\alpha+3} + 6(x_j-(k+2))^{\alpha+3}, & k+2 \leq x_j < k+3, \\ (x_j-k)^{\alpha+3} - 4(x_j-(k+1))^{\alpha+3} + 6(x_j-(k+2))^{\alpha+3} & \\ -4(x_j-(k+3))^{\alpha+3}, & k+3 \leq x_j < k+4; \\ (x_j-k)^{\alpha+3} - 4(x_j-(k+1))^{\alpha+3} + 6(x_j-(k+2))^{\alpha+3} & \\ -4(x_j-(k+3))^{\alpha+3} + (x_j-(k+4))^{\alpha+3}, & k+4 \leq x_j; \end{cases}$$

and the boundary fractional integration functions by taking $\lambda = \frac{1}{2^{j\alpha}}$ are as follows:

$$I^{\alpha}\varphi_{4,j,-3}(x) = \lambda \begin{cases} \frac{1}{6\Gamma(\alpha+1)}x_j^{\alpha} - \frac{1}{2\Gamma(\alpha+2)}x_j^{\alpha+1} + \frac{1}{\Gamma(\alpha+3)}x_j^{\alpha+2} - \frac{1}{\Gamma(\alpha+4)}x_j^{\alpha+3}, & 0 \le x_j < 1, \\ \frac{1}{6\Gamma(\alpha+1)}x_j^{\alpha} - \frac{1}{2\Gamma(\alpha+2)}x_j^{\alpha+1} + \frac{1}{\Gamma(\alpha+3)}x_j^{\alpha+2} + \frac{1}{\Gamma(\alpha+4)}[-x_j^{\alpha+3} \\ +(x_j-1)^{\alpha+3}], & 1 \le x_j; \end{cases}$$

and

$$I^{\alpha}\varphi_{4,j,-2}(x) = \lambda \begin{cases} \frac{2}{3\Gamma(\alpha+1)}x_j^{\alpha} - \frac{2}{\Gamma(\alpha+3)}x_j^{\alpha+2} + \frac{3}{\Gamma(\alpha+4)}x_j^{\alpha+3}, & 0 \le x_j < 1, \\ \frac{2}{3\Gamma(\alpha+1)}x_j^{\alpha} - \frac{2}{\Gamma(\alpha+3)}x_j^{\alpha+2} + \frac{1}{\Gamma(\alpha+4)}[3x_j^{\alpha+3} - 4(x_j-1)^{\alpha+3}], & 1 \le x_j < 2, \\ \frac{2}{3\Gamma(\alpha+1)}x_j^{\alpha} - \frac{2}{\Gamma(\alpha+3)}x_j^{\alpha+2} + \frac{1}{\Gamma(\alpha+4)}[3x_j^{\alpha+3} - 4(x_j-1)^{\alpha+3} \\ +(x_j-2)^{\alpha+3}], & 2 \le x_j; \end{cases}$$

and

$$I^{\alpha}\varphi_{4,j,-1}(x) = \lambda \begin{cases} \frac{1}{6\Gamma(\alpha+1)}x_j^{\alpha} + \frac{1}{2\Gamma(\alpha+2)}x_j^{\alpha+1} + \frac{1}{\Gamma(\alpha+3)}x_j^{\alpha+2} - \frac{1}{\Gamma(\alpha+4)}3x_j^{\alpha+3}, & 0 \le x_j < 1, \\ \frac{1}{6\Gamma(\alpha+1)}x_j^{\alpha} + \frac{1}{2\Gamma(\alpha+2)}x_j^{\alpha+1} + \frac{1}{\Gamma(\alpha+3)}x_j^{\alpha+2} + \frac{1}{\Gamma(\alpha+4)}[-3x_j^{\alpha+3} \\ +6(x_j-1)^{\alpha+3}], & 1 \le x_j < 2, \\ \frac{1}{6\Gamma(\alpha+1)}x_j^{\alpha} + \frac{1}{2\Gamma(\alpha+2)}x_j^{\alpha+1} + \frac{1}{\Gamma(\alpha+3)}x_j^{\alpha+2} + \frac{1}{\Gamma(\alpha+4)}[-3x_j^{\alpha+3} \\ +6(x_j-1)^{\alpha+3} - 4(x_j-2)^{\alpha+3}], & 2 \le x_j < 3, \\ \frac{1}{6\Gamma(\alpha+1)}x_j^{\alpha} + \frac{1}{2\Gamma(\alpha+2)}x_j^{\alpha+1} + \frac{1}{\Gamma(\alpha+3)}x_j^{\alpha+2} + \frac{1}{\Gamma(\alpha+4)}[-3x_j^{\alpha+3} \\ +6(x_j-1)^{\alpha+3} - 4(x_j-2)^{\alpha+3} + (x_j-3)^{\alpha+3}], & 3 \le x_j; \end{cases}$$

and for $k = 2^j - 3$ we have:

$$I^{\alpha}\varphi_{4,j,k}(x) = b \begin{cases} 0, & x_j < k, \\ (x_j-k)^{\alpha+3}, & k \le x_j < k+1, \\ (x_j-k)^{\alpha+3} - 4(x_j-(k+1))^{\alpha+3}, & k+1 \le x_j < k+2, \\ (x_j-k)^{\alpha+3} - 4(x_j-(k+1))^{\alpha+3} + 6(x_j-(k+2))^{\alpha+3}, & k+2 \le x_j < k+3, \end{cases}$$

and for $k = 2^j - 2$ is obtained:

$$I^{\alpha}\varphi_{4,j,k}(x) = b \begin{cases} 0, & x_j < k, \\ (x_j-k)^{\alpha+3}, & k \le x_j < k+1, \\ (x_j-k)^{\alpha+3} - 4(x_j-(k+1))^{\alpha+3}, & k+1 \le x_j < k+2, \end{cases}$$

and for $k = 2^j - 1$ we have:

$$I^{\alpha}\varphi_{4,j,k}(x) = b \begin{cases} 0, & x_j < k, \\ (x_j-k)^{\alpha+3}, & k \le x_j < k+1, \end{cases}$$

The fractional integration of pantic B-spline scaling functions for $j \ge 3$ and taking $\gamma = \frac{2^{4j}}{\Gamma(\alpha+5)}$ and $\lambda = \frac{1}{2^{j\alpha}}$ is obtained:

$$I^{\alpha}\varphi_{5,j,-4}(x) = \lambda \begin{cases} \frac{1}{24\Gamma(\alpha+1)}x_j^{\alpha} - \frac{1}{6\Gamma(\alpha+2)}x_j^{\alpha+1} + \frac{1}{2\Gamma(\alpha+3)}x_j^{\alpha+2} - \frac{1}{\Gamma(\alpha+4)}x_j^{\alpha+3} \\ +\frac{1}{\Gamma(\alpha+5)}x_j^{\alpha+4}, & 0 \le x_j < 1, \\ \frac{1}{24\Gamma(\alpha+1)}x_j^{\alpha} - \frac{1}{6\Gamma(\alpha+2)}x_j^{\alpha+1} + \frac{1}{2\Gamma(\alpha+3)}x_j^{\alpha+2} - \frac{1}{\Gamma(\alpha+4)}x_j^{\alpha+3} \\ +\frac{1}{\Gamma(\alpha+5)}[x_j^{\alpha+4} - (x_j-1)^{\alpha+4}], & 1 \le x_j; \end{cases}$$

and

$$I^{\alpha}\varphi_{5,j,-3}(x) = \lambda \begin{cases} \frac{11}{24\Gamma(\alpha+1)}x_j^{\alpha} - \frac{1}{2\Gamma(\alpha+2)}x_j^{\alpha+1} - \frac{1}{2\Gamma(\alpha+3)}x_j^{\alpha+2} + \frac{3}{\Gamma(\alpha+4)}x_j^{\alpha+3} \\ -\frac{4}{\Gamma(\alpha+5)}x_j^{\alpha+4}, & 0 \le x_j < 1, \\ \frac{11}{24\Gamma(\alpha+1)}x_j^{\alpha} - \frac{1}{2\Gamma(\alpha+2)}x_j^{\alpha+1} - \frac{1}{2\Gamma(\alpha+3)}x_j^{\alpha+2} + \frac{3}{\Gamma(\alpha+4)}x_j^{\alpha+3} \\ -\frac{1}{\Gamma(\alpha+5)}[4x_j^{\alpha+4} - 5(x_j-1)^{\alpha+4}], & 1 \le x_j < 2, \\ \frac{11}{24\Gamma(\alpha+1)}x_j^{\alpha} - \frac{1}{2\Gamma(\alpha+2)}x_j^{\alpha+1} - \frac{1}{2\Gamma(\alpha+3)}x_j^{\alpha+2} + \frac{3}{\Gamma(\alpha+4)}x_j^{\alpha+3} \\ -\frac{1}{\Gamma(\alpha+5)}[4x_j^{\alpha+4} - 5(x_j-1)^{\alpha+4} + (x_j-2)^{\alpha+4}], & 2 \le x_j; \end{cases}$$

and

$$I^{\alpha}\varphi_{5,j,-2}(x)=\lambda\begin{cases}\frac{11}{24\Gamma(\alpha+1)}x_j^{\alpha}+\frac{1}{2\Gamma(\alpha+2)}x_j^{\alpha+1}-\frac{1}{2\Gamma(\alpha+3)}x_j^{\alpha+2}-\frac{3}{\Gamma(\alpha+4)}x_j^{\alpha+3}\\ +\frac{6}{\Gamma(\alpha+5)}x_j^{\alpha+4}, & 0\le x_j<1,\\ \frac{11}{24\Gamma(\alpha+1)}x_j^{\alpha}+\frac{1}{2\Gamma(\alpha+2)}x_j^{\alpha+1}-\frac{1}{2\Gamma(\alpha+3)}x_j^{\alpha+2}-\frac{3}{\Gamma(\alpha+4)}x_j^{\alpha+3}\\ +\frac{1}{\Gamma(\alpha+5)}[6x_j^{\alpha+4}-10(x_j-1)^{\alpha+4}], & 1\le x_j<2,\\ \frac{11}{24\Gamma(\alpha+1)}x_j^{\alpha}+\frac{1}{2\Gamma(\alpha+2)}x_j^{\alpha+1}-\frac{1}{2\Gamma(\alpha+3)}x_j^{\alpha+2}-\frac{3}{\Gamma(\alpha+4)}x_j^{\alpha+3}\\ +\frac{1}{\Gamma(\alpha+5)}[6x_j^{\alpha+4}-10(x_j-1)^{\alpha+4}+5(x_j-2)^{\alpha+4}], & 2\le x_j<3,\\ \frac{11}{24\Gamma(\alpha+1)}x_j^{\alpha}+\frac{1}{2\Gamma(\alpha+2)}x_j^{\alpha+1}-\frac{1}{2\Gamma(\alpha+3)}x_j^{\alpha+2}-\frac{3}{\Gamma(\alpha+4)}x_j^{\alpha+3}\\ +\frac{1}{\Gamma(\alpha+5)}[6x_j^{\alpha+4}-10(x_j-1)^{\alpha+4}+5(x_j-2)^{\alpha+4}-(x_j-3)^{\alpha+4}], & 3\le x_j;\end{cases}$$

and

$$I^{\alpha}\varphi_{5,j,-1}(x)=\lambda\begin{cases}\frac{1}{24\Gamma(\alpha+1)}x_j^{\alpha}+\frac{1}{6\Gamma(\alpha+2)}x_j^{\alpha+1}+\frac{1}{2\Gamma(\alpha+3)}x_j^{\alpha+2}+\frac{1}{\Gamma(\alpha+4)}x_j^{\alpha+3}\\ -\frac{4}{\Gamma(\alpha+5)}x_j^{\alpha+4}, & 0\le x_j<1,\\ \frac{1}{24\Gamma(\alpha+1)}x_j^{\alpha}+\frac{1}{6\Gamma(\alpha+2)}x_j^{\alpha+1}+\frac{1}{2\Gamma(\alpha+3)}x_j^{\alpha+2}+\frac{1}{\Gamma(\alpha+4)}x_j^{\alpha+3}\\ +\frac{1}{\Gamma(\alpha+5)}[-4x_j^{\alpha+4}+10(x_j-1)^{\alpha+4}], & 1\le x_j<2,\\ \frac{1}{24\Gamma(\alpha+1)}x_j^{\alpha}+\frac{1}{6\Gamma(\alpha+2)}x_j^{\alpha+1}+\frac{1}{2\Gamma(\alpha+3)}x_j^{\alpha+2}+\frac{1}{\Gamma(\alpha+4)}x_j^{\alpha+3}\\ +\frac{1}{\Gamma(\alpha+5)}[-4x_j^{\alpha+4}+10(x_j-1)^{\alpha+4}-10(x_j-2)^{\alpha+4}], & 2\le x_j<3,\\ \frac{1}{24\Gamma(\alpha+1)}x_j^{\alpha}+\frac{1}{6\Gamma(\alpha+2)}x_j^{\alpha+1}+\frac{1}{2\Gamma(\alpha+3)}x_j^{\alpha+2}+\frac{1}{\Gamma(\alpha+4)}x_j^{\alpha+3}\\ +\frac{1}{\Gamma(\alpha+5)}[-4x_j^{\alpha+4}+10(x_j-1)^{\alpha+4}-10(x_j-2)^{\alpha+4}\\ +5(x_j-3)^{\alpha+4}], & 3\le x_j<4,\\ \frac{1}{24\Gamma(\alpha+1)}x_j^{\alpha}+\frac{1}{6\Gamma(\alpha+2)}x_j^{\alpha+1}+\frac{1}{2\Gamma(\alpha+3)}x_j^{\alpha+2}+\frac{1}{\Gamma(\alpha+4)}x_j^{\alpha+3}\\ +\frac{1}{\Gamma(\alpha+5)}[-4x_j^{\alpha+4}+10(x_j-1)^{\alpha+4}-10(x_j-2)^{\alpha+4}\\ +5(x_j-3)^{\alpha+4}-(x_j-4)^{\alpha+4}], & 4\le x_j;\end{cases}$$

and for $k=0,\cdots,2^j-5$ we have:

$$I^{\alpha}\varphi_{5,j,k}(x)=\gamma\begin{cases}0, & x_j<k,\\ (x_j-k)^{\alpha+4}, & k\le x_j<k+1,\\ (x_j-k)^{\alpha+4}-5(x_j-(k+1))^{\alpha+4}, & k+1\le x_j<k+2,\\ (x_j-k)^{\alpha+4}-5(x_j-(k+1))^{\alpha+4}+10(x_j-(k+2))^{\alpha+4}, & k+2\le x_j<k+3,\\ (x_j-k)^{\alpha+4}-5(x_j-(k+1))^{\alpha+4}+10(x_j-(k+2))^{\alpha+4}\\ -10(x_j-(k+3))^{\alpha+4}, & k+3\le x_j<k+4,\\ (x_j-k)^{\alpha+4}-5(x_j-(k+1))^{\alpha+4}+10(x_j-(k+2))^{\alpha+4}\\ -10(x_j-(k+3))^{\alpha+4}+5(x_j-(k+4))^{\alpha+4}, & k+4\le x_j<k+5,\\ (x_j-k)^{\alpha+4}-5(x_j-(k+1))^{\alpha+4}+10(x_j-(k+2))^{\alpha+4}-\\ 10(x_j-(k+3))^{\alpha+4}+5(x_j-(k+4))^{\alpha+4}-(x_j-(k+5))^{\alpha+4}, & k+5\le x_j;\end{cases}$$

and for $k=2^j-4$ we have:

$$I^{\alpha}\varphi_{5,j,k}(x)=\gamma\begin{cases}0, & x_j<k,\\ (x_j-k)^{\alpha+4}, & k\le x_j<k+1,\\ (x_j-k)^{\alpha+4}-5(x_j-(k+1))^{\alpha+4}, & k+1\le x_j<k+2,\\ (x_j-k)^{\alpha+4}-5(x_j-(k+1))^{\alpha+4}+10(x_j-(k+2))^{\alpha+4}, & k+2\le x_j<k+3,\\ (x_j-k)^{\alpha+4}-5(x_j-(k+1))^{\alpha+4}+10(x_j-(k+2))^{\alpha+4}\\ -10(x_j-(k+3))^{\alpha+4}, & k+3\le x_j<k+4;\end{cases}$$

and for $k=2^j-3$ we have:

$$I^{\alpha}\varphi_{5,j,k}(x)=\gamma\begin{cases}0, & x_j<k,\\ (x_j-k)^{\alpha+4}, & k\le x_j<k+1,\\ (x_j-k)^{\alpha+4}-5(x_j-(k+1))^{\alpha+4}, & k+1\le x_j<k+2,\\ (x_j-k)^{\alpha+4}-5(x_j-(k+1))^{\alpha+4}+10(x_j-(k+2))^{\alpha+4}, & k+2\le x_j<k+3;\end{cases}$$

and for $k = 2^j - 2$ we have:

$$I^\alpha \varphi_{5,j,k}(x) = \gamma \begin{cases} 0, & x_j < k, \\ (x_j - k)^{\alpha+4}, & k \le x_j < k+1, \\ (x_j - k)^{\alpha+4} - 5(x_j - (k+1))^{\alpha+4}, & k+1 \le x_j < k+2; \end{cases}$$

and for $k = 2^j - 1$ we have:

$$I^\alpha \varphi_{5,j,k}(x) = \gamma \begin{cases} 0, & x_j < k, \\ (x_j - k)^{\alpha+4}, & k \le x_j < k+1; \end{cases}$$

3. Function Approximation

We can expanded any function f defined on $[0, 1]$ by spline scaling functions as the following:

$$f(x) \approx \sum_{k=r}^{2^j-1} c_k \varphi_{m,j,k}(x) = C^T \Phi_m(x). \tag{3.1}$$

Since $suppN_m \subseteq [0, m)$, to have inner scaling functions we should have $supp\varphi_{m,j,r} \subseteq [0, m)$. So we can transmit only $m - 1$ times. In other words we have $r = -m + 1$. For example we have for quartic and pantic splines $r = -3$ and $r = -4$, respectively. The vectors C and $\Phi_m(x)$ in (3.1) are $(2^j - r)$-vectors given by:

$$C = [c_r, c_{r+1}, \cdots, c_{2^j-1}]^T, \tag{3.2}$$

$$\Phi_m(x) = [\varphi_{m,j,r}(x), \varphi_{m,j,r+1}(x), \cdots, \varphi_{m,j,2^j-1}(x)]^T, \tag{3.3}$$

where

$$c_k = \int_0^1 f(x)\tilde{\varphi}_{m,j,k}(x)\mathrm{d}x,$$

the functions $\tilde{\varphi}_{m,j,k}(x)$ are dual functions of $\varphi_{m,j,k}$ for $k = r, r+1, \cdots, 2^j - 1$ and $j \in \mathbb{Z}$. Let $\tilde{\Phi}_m$ be the vector of dual functions of Φ_m as:

$$\tilde{\Phi}_m(x) = [\tilde{\varphi}_{m,j,r}(x), \tilde{\varphi}_{m,j,r+1}(x), \cdots, \tilde{\varphi}_{m,j,2^j-1}(x)]^T,$$

so we have by duality principle:

$$\int_0^1 \tilde{\Phi}_m(x)\Phi_m(x)^T \mathrm{d}x = I, \tag{3.4}$$

where I is the identity matrix.

3.1. operational matrices

Now we want to calculate the operational matrix of fractional integration of B-spline scaling functions.

Theorem 3.1. *If the matrix $P_m = [p_{i,r}]$ is defined as:*

$$P_m = \int_0^1 \Phi_m(x)\Phi_m^T(x)\mathrm{d}x, \tag{3.5}$$

then we get:

a)

$$P_4 = \frac{1}{2^j}\begin{bmatrix} \frac{1}{252} & \frac{43}{1680} & \frac{1}{84} & \frac{1}{5040} & 0 & 0 & 0 & \cdots & 0 \\ \frac{43}{1680} & \frac{151}{630} & \frac{59}{280} & \frac{1}{42} & \frac{1}{5040} & 0 & 0 & \cdots & 0 \\ \frac{1}{84} & \frac{59}{280} & \frac{599}{1260} & \frac{397}{1680} & \frac{1}{42} & \frac{1}{5040} & 0 & \cdots & 0 \\ \frac{1}{5040} & \frac{1}{42} & \frac{397}{1680} & \frac{151}{315} & \frac{397}{1680} & \frac{1}{42} & \frac{1}{5040} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{5040} & \frac{1}{42} & \frac{397}{1680} & \frac{151}{315} & \frac{397}{1680} & \frac{1}{42} & \frac{1}{5040} \\ 0 & \cdots & 0 & \frac{1}{5040} & \frac{1}{42} & \frac{397}{1680} & \frac{599}{1260} & \frac{59}{280} & \frac{1}{84} \\ 0 & \cdots & 0 & 0 & \frac{1}{5040} & \frac{1}{42} & \frac{59}{280} & \frac{151}{630} & \frac{43}{1680} \\ 0 & \cdots & 0 & 0 & 0 & \frac{1}{5040} & \frac{1}{84} & \frac{43}{1680} & \frac{1}{252} \end{bmatrix},$$

where for $j \geq 2$, P_4 is a $(2^j+3)\times(2^j+3)$ symmetric matrix.

b) $P_5 = \frac{1}{2^{j-3}}$

$$\begin{bmatrix} \frac{1}{41472} & \frac{16}{41435} & \frac{39}{71612} & \frac{10}{115659} & \frac{1}{2903040} & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{16}{41435} & \frac{143}{18144} & \frac{249}{16385} & \frac{46}{10251} & \frac{19}{109876} & \frac{1}{2903040} & 0 & 0 & 0 & \cdots & 0 \\ \frac{39}{71612} & \frac{249}{16385} & \frac{94}{2047} & \frac{40}{1333} & \frac{85}{16892} & \frac{19}{109876} & \frac{1}{2903040} & 0 & 0 & \cdots & 0 \\ \frac{10}{115659} & \frac{46}{10251} & \frac{40}{1333} & \frac{79}{1469} & \frac{437}{14378} & \frac{85}{16892} & \frac{19}{109876} & \frac{1}{2903040} & 0 & \cdots & 0 \\ \frac{1}{2903040} & \frac{19}{109876} & \frac{85}{16892} & \frac{437}{14378} & \frac{179}{3327} & \frac{437}{14378} & \frac{85}{16892} & \frac{19}{109876} & \frac{1}{2903040} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{2903040} & \frac{19}{109876} & \frac{85}{16892} & \frac{437}{14378} & \frac{179}{3327} & \frac{437}{14378} & \frac{85}{16892} & \frac{19}{109876} & \frac{1}{2903040} \\ 0 & \cdots & 0 & \frac{1}{2903040} & \frac{19}{109876} & \frac{85}{16892} & \frac{437}{14378} & \frac{79}{1469} & \frac{40}{1333} & \frac{46}{10251} & \frac{10}{115659} \\ 0 & \cdots & 0 & 0 & \frac{1}{2903040} & \frac{19}{109876} & \frac{85}{16892} & \frac{40}{1333} & \frac{94}{2047} & \frac{249}{16385} & \frac{39}{71612} \\ 0 & \cdots & 0 & 0 & 0 & \frac{1}{2903040} & \frac{19}{109876} & \frac{46}{10251} & \frac{249}{16385} & \frac{143}{18144} & \frac{16}{41435} \\ 0 & \cdots & 0 & 0 & 0 & 0 & \frac{1}{2903040} & \frac{10}{115659} & \frac{39}{71612} & \frac{16}{41435} & \frac{1}{41472} \end{bmatrix},$$

where for $j \geq 3$, P_5 is a $(2^j+4)\times(2^j+4)$ symmetric matrix.

c) The vector of dual functions is as:

$$\tilde{\Phi}_m = (P_m)^{-1}\Phi_m.$$

Proof. The theorem can be easily proved by using the following formula:

$$p_{i,k} = \int_0^1 \varphi_{m,j,i}(x)\varphi_{m,j,k}(x)\mathrm{d}x,\ j \geqslant m; i,k = r,\cdots,2^j-1, r = -3,-4.$$

Definition 3.2. Suppose that $\Phi_m(x)$ is the vector of B-spline scaling functions and $I^\alpha\Phi_m(x)$ is expanded by B-spline functions as follows:

$$(I^\alpha\Phi_m)(x) \cong F_m^\alpha\Phi_m(x).$$

The matrix F_m^α is called B-spline operational matrix of fractional integration.

We calculate F_m^α as follows:

$$\begin{aligned} F_m^\alpha &= \int_0^1 I^\alpha\Phi_m(t)\tilde{\Phi}_m^T(t)\mathrm{d}t = \int_0^1 I^\alpha\Phi_m(t)(P_m^{-1}\Phi_m)^T(t)\mathrm{d}t \\ &= \left(\int_0^1 I^\alpha\Phi_m(t)\Phi_m^T(t)\mathrm{d}(t)\right)P_m^{-1} = E_mP_m^{-1}, \end{aligned} \tag{3.6}$$

where $E_m = [e_{i,k}]$,

$$e_{i,k} = \int_0^1 I^{\alpha}\varphi_{m,j,k}(t)\varphi_{m,j,i}(t)\mathrm{d}t.$$

The following theorem determines the matrix E_m for $m = 4, 5$.

Theorem 3.3. *Suppose E_m is the matrix in the equation (3.6), then:*

a) The $(2^j + 3) \times (2^j + 3)$ matrix E_4 for quartic B-spline is obtained as bellow:

$$E_4 = \frac{1}{2^{j(\alpha+1)}\Gamma(\alpha+8)} \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \cdots & \gamma_{2^j} & \gamma_{2^j+1} & \gamma_{2^j+2} & \gamma_{2^j+3} \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 & \cdots & \eta_{2^j} & \eta_{2^j+1} & \eta_{2^j+2} & \gamma_{2^j+2} \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 & \cdots & \delta_{2^j} & \delta_{2^j+1} & \eta_{2^j+1} & \gamma_{2^j+1} \\ 1 & \xi_1 & \xi_2 & \xi_3 & \cdots & \xi_{2^j} & \delta_{2^j} & \eta_{2^j} & \gamma_{2^j} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \xi_2 & \xi_3 & \delta_4 & \eta_4 & \gamma_4 \\ 0 & 0 & 0 & \cdots & \xi_1 & \xi_2 & \delta_3 & \eta_3 & \gamma_3 \\ 0 & 0 & 0 & \cdots & 1 & \xi_1 & \delta_2 & \eta_2 & \gamma_2 \\ 0 & 0 & 0 & \cdots & 0 & 1 & \delta_1 & \eta_1 & \gamma_1 \end{bmatrix},$$

b) For pantic B-spline, E_5 is an $(2^j + 4) \times (2^j + 4)$ matrix as bellow:

$$E_5 = \frac{1}{2^{j(\alpha+1)}\Gamma(\alpha+10)} \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \cdots & \gamma_{2^j} & \gamma_{2^j+1} & \gamma_{2^j+2} & \gamma_{2^j+3} \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 & \cdots & \eta_{2^j} & \eta_{2^j+1} & \eta_{2^j+2} & \gamma_{2^j+2} \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 & \cdots & \delta_{2^j} & \delta_{2^j+1} & \eta_{2^j+1} & \gamma_{2^j+1} \\ \nu_1 & \nu_2 & \nu_3 & \nu_4 & \nu_5 & \cdots & \nu_{2^j} & \delta_{2^j} & \eta_{2^j} & \gamma_{2^j} \\ 1 & \xi_1 & \xi_2 & \xi_3 & \xi_4 & \cdots & \xi_{2^j-1} & \delta_{2^j-1} & \eta_{2^j-1} & \gamma_{2^j-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \xi_2 & \xi_3 & \nu_4 & \delta_4 & \eta_4 & \gamma_4 \\ 0 & 0 & 0 & \cdots & \xi_1 & \xi_2 & \nu_3 & \delta_3 & \eta_3 & \gamma_3 \\ 0 & 0 & 0 & \cdots & 1 & \xi_1 & \nu_2 & \delta_2 & \eta_2 & \gamma_2 \\ 0 & 0 & 0 & \cdots & 0 & 1 & \nu_1 & \delta_1 & \eta_1 & \gamma_1 \end{bmatrix},$$

As an example for $j = 3$ and $\alpha = 1$ we have:

$$E_4 = \begin{bmatrix} \frac{1}{73728} & \frac{41}{168203} & \frac{44}{74649} & \frac{47}{72235} & \frac{1}{1536} & \frac{1}{1536} & \frac{1}{1536} & \frac{1}{1536} & \frac{23}{36864} & \frac{1}{3072} & \frac{1}{36864} \\ \frac{21}{256825} & \frac{1}{512} & \frac{83}{13530} & \frac{69}{8942} & \frac{157}{20097} & \frac{1}{128} & \frac{1}{128} & \frac{1}{128} & \frac{23}{3072} & \frac{1}{256} & \frac{1}{3072} \\ \frac{6}{173965} & \frac{28}{20703} & \frac{145}{20209} & \frac{71}{5373} & \frac{257}{17274} & \frac{51}{3406} & \frac{23}{1536} & \frac{23}{1536} & \frac{239}{16655} & \frac{23}{3072} & \frac{23}{36864} \\ \frac{1}{2580480} & \frac{6}{62431} & \frac{80}{45461} & \frac{1}{128} & \frac{57}{4111} & \frac{154}{9917} & \frac{630}{40321} & \frac{1}{64} & \frac{23}{1536} & \frac{1}{128} & \frac{1}{1536} \\ 0 & \frac{1}{2580480} & \frac{6}{62431} & \frac{80}{45461} & \frac{1}{128} & \frac{57}{4111} & \frac{154}{9917} & \frac{630}{40321} & \frac{23}{1536} & \frac{1}{128} & \frac{1}{1536} \\ 0 & 0 & \frac{1}{2580480} & \frac{6}{62431} & \frac{80}{45461} & \frac{1}{128} & \frac{57}{4111} & \frac{154}{9917} & \frac{51}{3406} & \frac{1}{128} & \frac{1}{1536} \\ 0 & 0 & 0 & \frac{1}{2580480} & \frac{6}{62431} & \frac{80}{45461} & \frac{1}{128} & \frac{57}{4111} & \frac{257}{17274} & \frac{157}{20097} & \frac{1}{1536} \\ 0 & 0 & 0 & 0 & \frac{1}{2580480} & \frac{6}{62431} & \frac{80}{45461} & \frac{1}{128} & \frac{71}{5373} & \frac{69}{8942} & \frac{47}{72235} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2580480} & \frac{6}{62431} & \frac{80}{45461} & \frac{145}{20209} & \frac{83}{13530} & \frac{44}{74649} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2580480} & \frac{6}{62431} & \frac{28}{20703} & \frac{1}{512} & \frac{41}{168203} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2580480} & \frac{6}{173965} & \frac{21}{256825} & \frac{1}{73728} \end{bmatrix},$$

and also we have:

$$E_5 = \begin{bmatrix} \frac{1}{1843200} & \frac{5}{238296} & \frac{9}{100321} & \frac{5}{39221} & \frac{4}{30721} & \frac{1}{7680} & \frac{1}{7680} & \frac{1}{7680} & \frac{13}{100679} & \frac{31}{307200} & \frac{3}{102400} & \frac{1}{921600} \\ \frac{1}{120271} & \frac{38}{96079} & \frac{46}{22605} & \frac{78}{23519} & \frac{109}{31043} & \frac{317}{90169} & \frac{9}{2560} & \frac{9}{2560} & \frac{121}{34707} & \frac{40}{14681} & \frac{81}{102400} & \frac{3}{102400} \\ \frac{1}{89290} & \frac{87}{126149} & \frac{476}{101441} & \frac{393}{39508} & \frac{74}{6219} & \frac{77}{6361} & \frac{31}{2560} & \frac{31}{2560} & \frac{193}{16072} & \frac{961}{102400} & \frac{40}{14681} & \frac{31}{307200} \\ \frac{1}{609562} & \frac{17}{100082} & \frac{40}{19407} & \frac{131}{17051} & \frac{20}{1501} & \frac{169}{11057} & \frac{250}{16139} & \frac{119}{7680} & \frac{262}{17051} & \frac{193}{16072} & \frac{121}{34707} & \frac{13}{100679} \\ \frac{1}{232243200} & \frac{3}{687110} & \frac{17}{80815} & \frac{33}{15205} & \frac{1}{128} & \frac{207}{15385} & \frac{79}{5125} & \frac{56}{3585} & \frac{119}{7680} & \frac{31}{2560} & \frac{9}{2560} & \frac{1}{7680} \\ 0 & \frac{1}{232243200} & \frac{3}{687110} & \frac{17}{80815} & \frac{33}{15205} & \frac{1}{128} & \frac{207}{15385} & \frac{79}{5125} & \frac{250}{16139} & \frac{31}{2560} & \frac{9}{2560} & \frac{1}{7680} \\ 0 & 0 & \frac{1}{232243200} & \frac{3}{687110} & \frac{17}{80815} & \frac{33}{15205} & \frac{1}{128} & \frac{207}{15385} & \frac{169}{11057} & \frac{77}{6361} & \frac{317}{90169} & \frac{1}{7680} \\ 0 & 0 & 0 & \frac{1}{232243200} & \frac{3}{687110} & \frac{17}{80815} & \frac{33}{15205} & \frac{1}{128} & \frac{20}{1501} & \frac{74}{6219} & \frac{109}{31043} & \frac{4}{30721} \\ 0 & 0 & 0 & 0 & \frac{1}{232243200} & \frac{3}{687110} & \frac{17}{80815} & \frac{33}{15205} & \frac{131}{17051} & \frac{393}{39508} & \frac{78}{23519} & \frac{5}{39221} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{232243200} & \frac{3}{687110} & \frac{17}{80815} & \frac{40}{19407} & \frac{476}{101441} & \frac{46}{22605} & \frac{9}{100321} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{232243200} & \frac{3}{687110} & \frac{17}{100082} & \frac{87}{126149} & \frac{38}{96079} & \frac{5}{238296} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{232243200} & \frac{1}{609562} & \frac{1}{89290} & \frac{1}{120271} & \frac{1}{1843200} \end{bmatrix}.$$

4. Applying B-spline Operational Matrices

In this section, we want to apply the mentioned operational matrices for solving partial differential equations. Consider the following equation:

$$a\frac{\partial^\alpha u}{\partial x^\alpha} + b\frac{\partial^\beta u}{\partial t^\beta} = f(x,t), \tag{4.1}$$

with the boundary conditions

$$u(0,t) = g_0(t),$$
$$u(x,0) = g_1(x),$$

where a and b are constant, $u(x,t) \in L^2([0,1]\times[0,1])$, $0 < \alpha \leq 1$ and $0 < \beta \leq 1$. We approximate $u(x,t)$ with B-spline functions for a fixed j,

$$u(x,t) \approx \sum_{i,k} \varphi_{m,j,i}(x) C_{i,k} \varphi_{m,j,k}(t) = \Phi_m^T(x) C \Phi_m(t), \tag{4.2}$$

where C is a $(2^j - r)\times(2^j - r)$ unknown matrix where $r = -m+1$. Now we apply first I_x^α and then I_t^β on both sides of the equation (4.1), and using Lemma (2.1) we obtain:

$$aI_t^\beta u(x,t) - aI_t^\beta g_0(t) + bI_x^\alpha u(x,t) - bI_x^\alpha g_1(x) = I_t^\beta I_x^\alpha f(x,t). \tag{4.3}$$

By assuming

$$g_1(x) \approx \tilde{\Phi}_m^T(x) A \tilde{\Phi}_m(t)\ ,\ g_0(t) \approx \tilde{\Phi}_m^T(x) B \tilde{\Phi}_m(t), \tag{4.4}$$

and substituting (4.2) and (4.4) in (4.3) we have:

$$\begin{aligned} &a\Phi_m^T(x) C F_m^\beta \Phi_m(t) - a\Phi_m^T(x)(P_m^{-1})^T B P_m^{-1} F_m^\beta \Phi_m(t) + b\Phi_m^T(x)(F_m^\alpha)^T C \Phi_m(t) \\ &- b\Phi_m^T(x)(F_m^\alpha)^T (P_m^{-1})^T A P_m^{-1} \Phi_m(t) = \Phi_m^T(x)(F_m^\alpha)^T (P_m^{-1})^T Z P_m^{-1} F_m^\beta \Phi_m(t), \end{aligned} \tag{4.5}$$

where $f(x,t) \approx \tilde{\Phi}_m^T(x) Z \tilde{\Phi}_m(t)$. Multiplying (4.5) by $\tilde{\Phi}_m(x)$ from the left and $\tilde{\Phi}_m^T(t)$ from the right and then integrating from 0 to 1, we have:

$$\begin{aligned} &aCF_m^\beta - a(P_m^{-1})^T B P_m^{-1} F_m^\beta + b(F_m^\alpha)^T C - b(F_m^\alpha)^T (P_m^{-1})^T A P_m^{-1} \\ &= (F_m^\alpha)^T (P_m^{-1})^T Z P_m^{-1} F_m^\beta. \end{aligned} \tag{4.6}$$

Equation (4.6) gives a system of $2^j - r$, $(r = -m + 1)$ unknown equations, which can be solved to find C and then $u(x, t)$ can be calculated by (4.2).

For convergence analysis we refer readers to the following theorem from [13].

Theorem 4.1. *If $\{\varphi_{m,j,k}\}_{j,k\in\mathbb{Z}}$ are B-spline functions, then for an arbitrary function u we have:*

$$\|u - u_j\| \leqslant C_m 2^{-jm} \|f^{(m)}\|,$$

where $u_j(x) = \sum_{k\in\mathbb{Z}} \langle f, \tilde{\varphi}_{m,j,k}\rangle \varphi_{m,j,k}(x)$ and $C_m = \sqrt{\frac{|B_{2m}|}{(2m)!}}$ and B_{2m} is Bernouilli's number of order $2m$.

5. Numerical Examples

In this section, we show the efficiency of the mentioned method on some partial differential equations. Note that L_2 error is obtained as:

$$\|e_j\|_2 = \left(\int_9^1 e_j^2(x)\mathrm{d}x\right)^{1/2} \cong \left(\frac{1}{N}\sum_{i=0}^{N} e_j^2(x_i)\right)^{1/2}, \tag{5.1}$$

where $e_j(x_i) = u(x_i) - u_j(x_i)$, $i = 0, 1, \cdots, N (N \in \mathbb{N})$. $u(x)$ is the exact solution and $u_j(x)$ is the approximate solution which is obtained by equation (4.2).

Example 5.1. We consider the boundary conditions for this example as given in [14]:

$$f(x, t) = 0; a = b = 1,$$
$$g_0(t) = t, \quad g_1(x) = x^2.$$

In this example the exact solution is $u(x, t) = [t - x - (t - x)^2]U(t - x) + (t - x)^2$ where U is unit step function. Table 1 shows a comparison between the results of splines of order $2, 3, 4$ and 5. We studied splines of order 2 and 3 in [3], and presented the result here. The figures of $\|e_5\|_2$ for $m = 2, 3, 4, 5$ are given in the figures 1.

L_2-error	m=2[3]	m=3 [3]	m=4	m=5
j=3	6.7462e-03	4.5540e-03	4.4520e-03	3.8894e-03
j=4	2.9740e-03	2.1866e-03	2.1225e-03	1.9792e-03
j=5	1.4583e-03	1.0820e-03	1.0589e-03	9.9558e-04

Table 1: L_2-error of Example 5.1.

Example 5.2. We consider the boundary conditions for this example as given in [6]:

$$f(x, t) = \frac{8}{3\sqrt{\pi}}(x^{\frac{3}{2}} + t^{\frac{3}{2}}); a = b = 1,$$
$$g_0(t) = t^2, \quad g_1(x) = x^2.$$

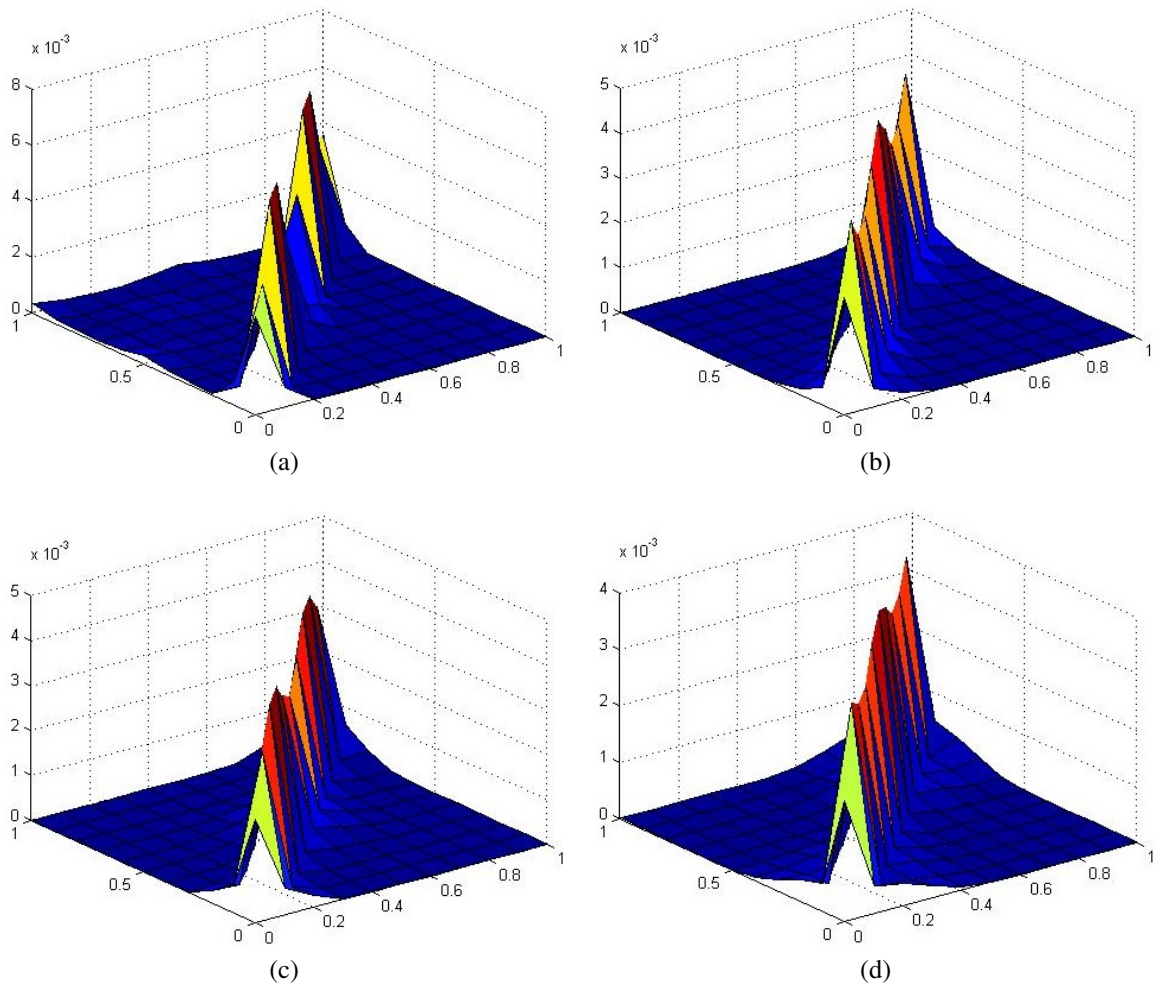

Figure 1: $\|e_5\|_2$ of example 5.1 for (a) $m = 2$ (b) $m = 3$ (c) $m = 4$ and (d) $m = 5$.

The exact solution of this problem is given as $u(x,t) = x^2 + t^2$. Table 2 shows a comparison between the results of splines of order 2, 3, 4 and 5. We studied splines of order 2 and 3 in [3], and presented the result here. Also we show the $\|e_4\|_2$ for $m = 2, 3, 4, 5$ in the figures 2.

L_2-error	m=2 [3]	m=3 [3]	m=4	m=5	Ref. [6]
j=3	2.2409e-03	1.2590e-05	4.0128e-06	3.3318e-06	1.4e-03
j=4	5.6041e-04	2.9704e-06	9.95995e-07	2.7034e-06	5.4e-03
j=5	1.4150e-04	6.8505e-07	3.5661e-07	4.4447e-05	2.0e-03

Table 2: L_2-error of Example 5.2.

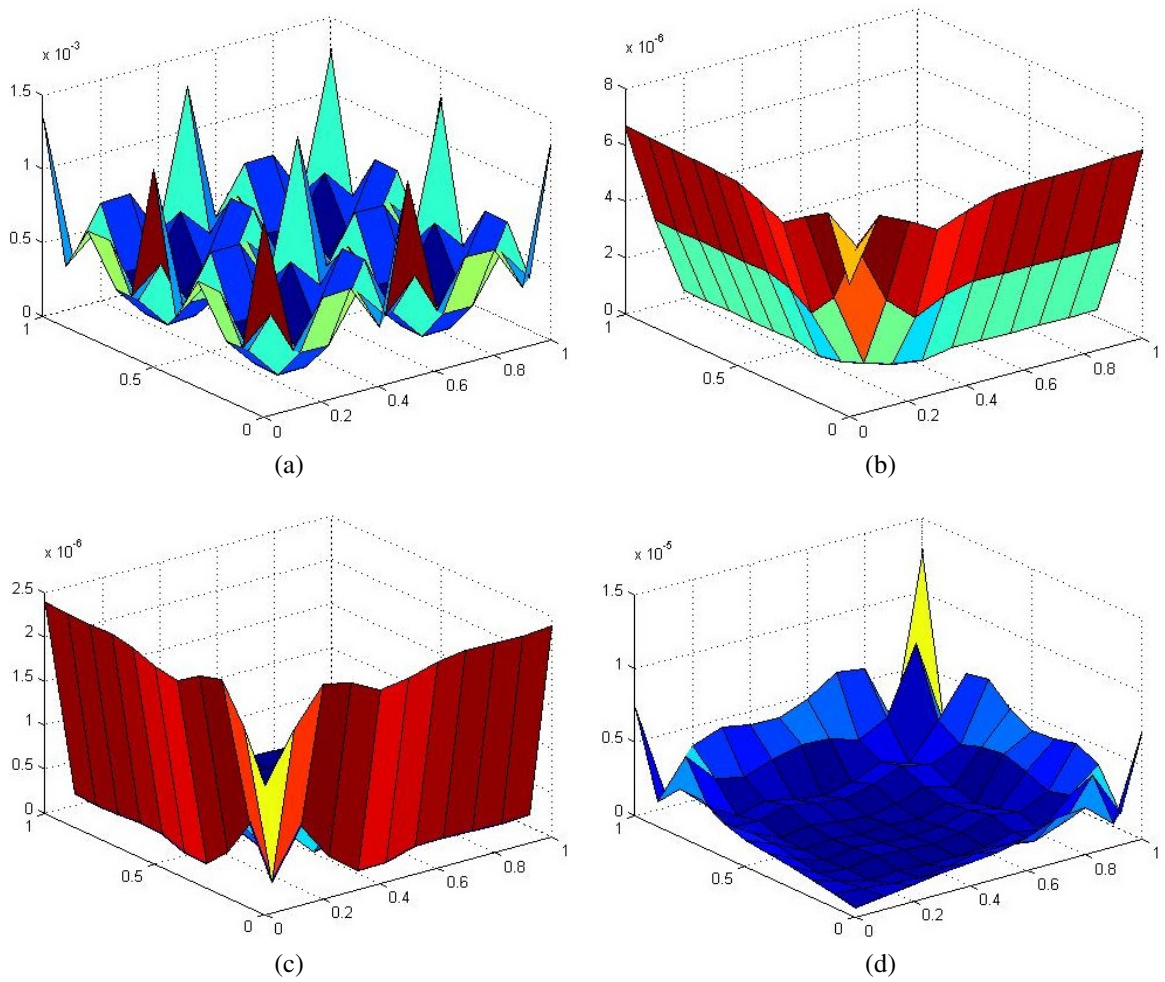

Figure 2: $\|e_4\|_2$ of example 5.2 for (a) $m = 2$ (b) $m = 3$ (c) $m = 4$ and (d) $m = 5$.

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