Wavelets and Linear Algebra 3(2) (2016) 33 - 42

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Wavelets and Linear Algebra

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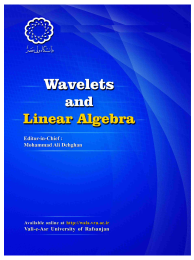

Cartesian decomposition of matrices and some norm inequalities

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Article Info

Article history:
Received 26 February 2016
Accepted 3 October 2016
Available online 25 December 2016
Communicated by Ali Armandnejad

Keywords:
joint $C-$numerical radius, $C-$spectral norm

2000 MSC:
15A60
15A18

Abstract

Let X be an $n-$square complex matrix with the Cartesian decomposition $X = A + iB$, where A and B are $n \times n$ Hermitian matrices. It is known that $\|X\|_p^2 \leq 2(\|A\|_p^2 + \|B\|_p^2)$, where $p \geq 2$ and $\|.\|_p$ is the Schatten $p-$norm. In this paper, this inequality and some of its improvements are studied and investigated for the joint $C-$numerical radius, joint spectral radius, and for the $C-$spectral norm of matrices.

1. Introduction and preliminaries

Let $\mathbb{M}_n$ be the algebra of all $n \times n$ complex matrices and $\mathcal{U}_n$ be the group of unitary matrices in $\mathbb{M}_n$. For every $X \in \mathbb{M}_n$ with singular values $s_1 \geqslant s_2 \geqslant \cdots \geq s_n$, and for every $2 \leq p < \infty$,

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http://doi.org/10.22072/wala.2016.23238

let $\|X\|_p = (\sum_{j=1}^n s_j^p)^{\frac{1}{p}}$ be the Schatten $p-$norm of X. Also, the spectral norm of X is defined as $\|X\| = \max_{\|x\|_2=1} \|Xx\|_2$, where $\|.\|_2$ is the Euclidean vector norm on $\mathbb{C}^n$. It is known that $\|X\| = s_1$ which is usually denoted by $\|X\|_\infty$. Also, it is evident that these norms are unitarily invariant. For more information, see [8].

Let $X, C \in \mathbb{M}_n$ have eigenvalues $\alpha_1, \ldots, \alpha_n$ and $\gamma_1, \ldots, \gamma_n$, respectively. The $C-$spectral radius, $C-$numerical radius and the $C-$spectral norm of X are defined and denoted, respectively, by:

$$\rho_C(X) = \max\{|\sum_{j=1}^n \alpha_j \gamma_{\sigma(j)}| \ : \ \sigma \text{ is a permutation of } \{1, 2, \ldots, n\}\},$$

$$r_C(X) = \max\{|tr\,(CUXU^*)| \ : \ U \in \mathcal{U}_n\} \tag{1.1}$$

and

$$\|X\|_C = \max\{|tr\,(CUXV^*)| \ : \ U, V \in \mathcal{U}_n\}.$$

It is known that

$$\rho_C(X) \leqslant r_C(X) \leqslant \|X\|_C = \sum_{j=1}^n s_j(X) s_j(C). \tag{1.2}$$

For the case $C = diag(1, 0, \ldots, 0) \in \mathbb{M}_n$, the inequality (1.2) reduces to

$$\rho(X) \leqslant r(X) \leqslant \|X\|,$$

where $\rho(X) = \max\{|\alpha_j| : j = 1, \ldots, n\}$ and $r(X) = \max\{|x^* X x| \ : \ x \in \mathbb{C}^n, \ x^* x = 1\}$ are the spectral radius and the numerical radius of X, respectively. One should note that the $C-$numerical radius for the case that $C \in \mathbb{M}_n$ is a nonscalar matrix, is a norm on $\mathbb{M}_n$ which is not necessarily invariant under unitary transformations. Also, the spectral radius is not a norm on $\mathbb{M}_n$. For more information about the $C-$numerical radius, $C-$spectral radius and the $C-$spectral norm of matrices and their applications, see [10, 12] and references therein.

Let $C \in \mathbb{M}_n$. The joint $C-$numerical radius of $(X_1, \ldots, X_k) \in \mathbb{M}_n^k$ is

$$r_C(X_1, \ldots, X_k) := \sup\{\ell_2(a_1, \ldots, a_k) : (a_1, \ldots, a_k) \in W_C(X_1, \ldots, X_k)\},$$

where $\ell_2(a_1, \ldots, a_k) = \left(\sum_{j=1}^k |a_j|^2\right)^{1/2}$ is the usual Euclidean norm, and

$$W_C(X_1, \ldots, X_k) = \{(tr(CUX_1U^*), \ldots, tr(CUX_kU^*)) : U \in \mathcal{U}_n\}$$

is the joint $C-$numerical range of $(X_1, \ldots, X_k)$; for more information, see [1] and its references. By setting $C = diag(1, 0, \ldots, 0) \in \mathbb{M}_n$, we see that $r_C(X_1, \ldots, X_k)$ reduces to the joint numerical radius of $(X_1, \ldots, X_k)$; i.e.,

$$\begin{aligned} r_C(X_1, \ldots, X_k) &= r(X_1, \ldots, X_k) \\ &:= \sup\{\ell_2(a_1, \ldots, a_k) : (a_1, \ldots, a_k) \in W(X_1, \ldots, X_k)\}, \end{aligned}$$

where $W(X_1, \ldots, X_k) = \{(x^* X_1 x, \ldots, x^* X_k x) \ : \ x \in \mathbb{C}^n, \ x^* x = 1\}$ is the joint numerical range of $(X_1, \ldots, X_k)$. Note that for the case $k = 1$, this notion reduces to relation (1.1). For other classes of norms ν on $\mathbb{C}^k$, we can extend our results to the $\nu-$joint numerical radius of $(X_1, \dots, X_k) \in \mathbb{M}_n^k$, which is defined as

$$r_\nu(X_1, \dots, X_k) = \sup\{\nu(a_1, \dots, a_k) : (a_1, \dots, a_k) \in W(X_1, \dots, X_k)\}.$$

So, $r_{\ell_2}(X_1, \dots, X_k) = r(X_1, \dots, X_k)$.

Let $(X_1, \dots, X_k) \in \mathbb{M}_n^k$. The joint spectrum $\sigma(X_1, \dots, X_k)$ is the set of all points $(\lambda_1, \dots, \lambda_k) \in \mathbb{C}^k$ for which there exists a nonzero vector $x \in \mathbb{C}^n$ such that

$$X_j x = \lambda_j x, \qquad j = 1, \dots, k.$$

The joint spectrum of matrices may be an empty set; for an example, see [13, p. 226]. It can be a nonempty set; see [2, Proposition 2.3(iii)]. So, we assume, to avoid of trivial cases, that $X_1, \dots, X_k$ are the matrices such that $\sigma(X_1, \dots, X_k)$ is a nonempty set. In this sense, the geometric joint spectral radius of $(X_1, \dots, X_k)$ is defined, e.g., see [4], as

$$\rho(X_1, \dots, X_k) = \max\{\ell_2(\lambda_1, \dots, \lambda_k) : (\lambda_1, \dots, \lambda_k) \in \sigma(X_1, \dots, X_k)\}.$$

Let $X \in \mathbb{M}_n$ be a matrix with the Cartesian decomposition $X = Re\,X + iIm\,X$, where $Re\,X = \frac{1}{2}(X + X^*)$ and $Im\,X = \frac{1}{2i}(X - X^*)$. Corollary 1 in [5] is an improvement of the following inequality:

$$\|X\|_p^2 \leq 2(\|Re\,X\|_p^2 + \|Im\,X\|_p^2), \tag{1.3}$$

where $p \geq 2$. In the two last decades, some interesting norm inequalities involving the Cartesian decomposition of matrices have been obtained; see for example [9] and its references. These kinds of inequalities have applications in the analysis of operators [3], and in mathematical physics [14]. In the next section of this paper, we study the inequality (1.3) and some of its improvements for the joint $C-$numerical radius, joint spectral radius, and for the $C-$spectral norm of matrices.

2. Main results

At first, we investigate the inequality (1.3) for the joint $C-$numerical radius of matrices.

Theorem 2.1. *Let $(X_1, \dots, X_k) \in \mathbb{M}_n^k$, and $C \in \mathbb{M}_n$. Then*

$$r_C^2(X_1, \dots, X_k) \leqslant 2\left(r_C^2(Re\,X_1, \dots, Re\,X_k) + r_C^2(Im\,X_1, \dots, Im\,X_k)\right).$$

Proof. For every $U \in \mathcal{U}_n$, we have

$$\begin{aligned}
&2\left(r_C^2(Re\,X_1,\ldots,Re\,X_k)+r_C^2(Im\,X_1,\ldots,Im\,X_k)\right)\\
&\geqslant 2\ell_2^2(tr(CU(Re\,X_1)U^*),\ldots,tr(CU(Re\,X_k)U^*))\\
&\quad+2\ell_2^2(tr(CU(Im\,X_1)U^*),\ldots,tr(CU(Im\,X_k)U^*))\\
&=2\sum_{i=1}^{k}(|tr(CU(Re\,X_i)U^*)|^2+|tr(CU(Im\,X_i)U^*)|^2)\\
&=\frac{1}{2}\sum_{i=1}^{k}(|tr(CU(X_i+X_i^*)U^*)|^2+|tr(CU(X_i-X_i^*)U^*)|^2)\\
&\geqslant\frac{1}{4}\sum_{i=1}^{k}(|tr(CU(X_i+X_i^*)U^*)|+|tr(CU(X_i-X_i^*)U^*)|)^2\\
&\geqslant\frac{1}{4}\sum_{i=1}^{k}|tr(CU(X_i+X_i^*)U^*)+tr(CU(X_i-X_i^*)U^*)|^2\\
&=\sum_{i=1}^{k}|tr(CUX_iU^*)|^2\\
&=\ell_2^2(tr(CUX_1U^*),\ldots,tr(CUX_kU^*)).
\end{aligned}$$

Taking now the maximum over all $U \in \mathcal{U}_n$, we obtain the result. □

By setting $k = 1$ in Theorem 2.1, we obtain the following result.

Corollary 2.2. *Let $X, C \in \mathbb{M}_n$. Then*

$$r_C^2(X) \leqslant 2\left(r_C^2(Re\,X)+r_C^2(Im\,X)\right).$$

Now, we state one of the interesting improvements of Theorem 2.1.

Theorem 2.3. *Let $(X_1,\ldots,X_k) \in \mathbb{M}_n^k$, and $C \in \mathbb{M}_n$ be a multiple of a Hermitian matrix. Then*

$$r_C^2(X_1,\ldots,X_k) \leqslant r_C^2(Re\,X_1,\ldots,Re\,X_k)+r_C^2(Im\,X_1,\ldots,Im\,X_k).$$

Consequently,

$$r^2(X_1,\ldots,X_k) \leqslant r^2(Re\,X_1,\ldots,Re\,X_k)+r^2(Im\,X_1,\ldots,Im\,X_k).$$

Proof. We assume, without loss of generality, that C is Hermitian. For every $U \in \mathcal{U}_n$, we have

$$\begin{aligned}
r_C^2(Re\,X_1, \ldots, Re\,X_k) + r_C^2(Im\,X_1, \ldots, Im\,X_k) \\
&\geqslant \ell_2^2(tr(CU(Re\,X_1)U^*), \ldots, tr(CU(Re\,X_k)U^*)) \\
&\quad + \ell_2^2(tr(CU(Im\,X_1)U^*), \ldots, tr(CU(Im\,X_k)U^*)) \\
&= \sum_{i=1}^{k} (|tr(CU(Re\,X_i)U^*)|^2 + |tr(CU(Im\,X_i)U^*)|^2) \\
&= \frac{1}{4} \sum_{i=1}^{k} (|tr(CU(X_i + X_i^*)U^*)|^2 + |tr(CU(X_i - X_i^*)U^*)|^2) \\
&\geqslant \frac{1}{4} \sum_{i=1}^{k} |[tr(CU(X_i + X_i^*)U^*)]^2 - [tr(CU(X_i - X_i^*)U^*)]^2| \\
&= \sum_{i=1}^{k} |tr(CUX_iU^*)||tr(CUX_i^*U^*)| \\
&= \sum_{i=1}^{k} |tr(CUX_iU^*)||tr((UX_iU^*C)^*)| \\
&= \sum_{i=1}^{k} |tr(CUX_iU^*)||\overline{tr(CUX_iU^*)}| \\
&= \sum_{i=1}^{k} |tr(CUX_iU^*)|^2 \\
&= \ell_2^2(tr(CUX_1U^*), \ldots, tr(CUX_kU^*)).
\end{aligned}$$

Taking now the maximum over all $U \in \mathcal{U}_n$, we obtain the result. By setting $C = diag(1, 0, \ldots, 0) \in \mathbb{M}_n$, the second assertion also holds. So, the proof is complete. □

By setting $k = 1$ in Theorem 2.3, we obtain the following result.

Corollary 2.4. *Let $X, C \in \mathbb{M}_n$ and let C be a multiple of a Hermitian matrix. Then*

$$r_C^2(X) \leqslant r_C^2(Re\,X) + r_C^2(Im\,X).$$

Consequently,

$$r^2(X) \leqslant r^2(Re\,X) + r^2(Im\,X).$$

Next, we are going to investigate an improvement of the inequality (1.3) for the $C-$spectral radius of matrices. For this, we need the following lemma.

Lemma 2.5. *[12, Theorem 4.1] Let $C \in \mathbb{M}_n$. Then C is normal if and only if $\rho_C(X) = r_C(X)$ for all normal matrices $X \in \mathbb{M}_n$.*

Theorem 2.6. *Let $X, C \in \mathbb{M}_n$ and let C be a multiple of a Hermitian matrix. Then*

$$\rho_C^2(X) \leqslant \rho_C^2(Re\, X) + \rho_C^2(Im\, X).$$

Consequently,

$$\rho^2(X) \leqslant \rho^2(Re\, X) + \rho^2(Im\, X).$$

Proof. Since $Re X$ and $Im X$ are Hermitian, the result follows from (1.2), Corollary 2.4 and Lemma 2.5. By setting $C = diag(1, 0, \dots, 0) \in \mathbb{M}_n$, the second assertion also holds. So, the proof is complete. □

The following example shows that if C is an arbitrary normal matrix, then the results in Corollary 2.4 and Theorem 2.6 are not true.

Example 2.7. Let $C = X = diag(1,\ 1+i) \in \mathbb{M}_2$. Then by Lemma 2.5, we have:

$$r_C^2(X) = \rho_C^2(X) = 8,\ r_C^2(Re\, X) = \rho_C^2(Re\, X) = 5,\ \textit{and}\ r_C^2(Im\, X) = \rho_C^2(Im\, X) = 2.$$

Hence,

$$r_C^2(X) = \rho_C^2(X) = 8 > 7 = r_C^2(Re\, X) + r_C^2(Im\, X) = \rho_C^2(Re\, X) + \rho_C^2(Im\, X).$$

By the same manner as in the proof of Theorem 2.1, we have the following result in which we investigate the inequality (1.3) for the $C-$spectral norm.

Proposition 2.8. *Let $X, C \in \mathbb{M}_n$. Then*

$$\|X\|_C^2 \leqslant 2\left(\|Re\, X\|_C^2 + \|Im\, X\|_C^2\right).$$

Consequently,

$$\|X\|^2 \leqslant 2\left(\|Re\, X\|^2 + \|Im\, X\|^2\right).$$

Now, we state an improvement of Proposition 2.8. For this, we need the following lemma.

Lemma 2.9. *[12, Theorem 5.11] Let $X \in \mathbb{M}_n$ have singular values $a_1 \geqslant \cdots \geqslant a_n$ and eigenvalues $\alpha_1, \dots, \alpha_n$, where $|\alpha_1| \geqslant \cdots \geqslant |\alpha_n|$. Moreover, let $C = diag(\gamma_1, \dots, \gamma_m, 0, \dots, 0) \in \mathbb{M}_n$, where $\gamma_1 \geqslant \cdots \geqslant \gamma_m > 0$. Then $\|X\|_C = r_C(X)$ if and only if there exists $\theta \in \mathbb{R}$ such that $\alpha_j = a_j e^{i\theta}$ for all $j = 1, \dots, m$.*

Theorem 2.10. *Let $X \in \mathbb{M}_n$ have singular values $a_1 \geqslant \cdots \geqslant a_n$ and eigenvalues $\alpha_1, \dots, \alpha_n$, where $|\alpha_1| \geqslant \cdots \geqslant |\alpha_n|$. If $C = diag(\gamma_1, \dots, \gamma_m, 0, \dots, 0) \in \mathbb{M}_n$, where $\gamma_1 \geqslant \cdots \geqslant \gamma_m > 0$, and there exists a $\theta \in \mathbb{R}$ such that $\alpha_j = a_j e^{i\theta}$ for all $j = 1, \dots, m$, then*

$$\|X\|_C^2 \leqslant \|Re\, X\|_C^2 + \|Im\, X\|_C^2.$$

Proof. By Lemma 2.9, we have $\|X\|_C = r_C(X)$. So, by Corollary 2.4 and relation (1.2), we have:

$$\begin{aligned}\|X\|_C^2 = r_C^2(X) &\leqslant r_C^2(Re\, X) + r_C^2(Im\, X)\\ &\leqslant \|Re\, X\|_C^2 + \|Im\, X\|_C^2.\end{aligned}$$

So, the proof is complete. □

Corollary 2.11. *Let $X \in \mathbb{M}_n$ be a normal matrix with eigenvalues $\alpha_1, \ldots, \alpha_n$, where $|\alpha_1| \geqslant \cdots \geqslant |\alpha_n|$. If $1 \leq m \leq n$ and $\alpha_1, \ldots, \alpha_m$ lie on a line passing through the origin, and $C = diag(\gamma_1, \ldots, \gamma_m, 0, \ldots, 0)$, where $\gamma_1 \geqslant \cdots \geqslant \gamma_m > 0$, then*

$$\|X\|_C^2 \leqslant \|Re\, X\|_C^2 + \|Im\, X\|_C^2.$$

Consequently, if $X \in \mathbb{M}_n$ is a normal matrix, then

$$\|X\|^2 \leqslant \|Re\, X\|^2 + \|Im\, X\|^2.$$

The following example shows that in Theorem 2.10, we can not remove the condition $\alpha_j = a_j e^{i\theta}$ for all $j = 1, \ldots, m$.

Example 2.12. If $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $C = diag(1, 0)$, then $a_1 = 1$ and $\alpha_1 = 0$. So, there is no $\theta \in \mathbb{R}$ such that $\alpha_1 = a_1 e^{i\theta}$. A simple calculation shows that:

$$\|Re\, X\|_C^2 + \|Im\, X\|_C^2 = \frac{1}{2} < 1 = \|X\|_C^2.$$

Since $\sigma(X_1, \ldots, X_k) \subseteq W(X_1, \ldots, X_k)$,

$$\rho(X_1, \ldots, X_k) \leqslant r(X_1, \ldots, X_k). \tag{2.1}$$

Also, it is known, e.g., see [2, Proposition 2.3(iii)], that if $(X_1, X_2, \ldots, X_k)$ is a family of commuting normal matrices, then

$$W(X_1, \ldots, X_k) = conv(\sigma(X_1, \ldots, X_k)), \tag{2.2}$$

where $conv(.)$ denotes the convex hull. Now, we state the following proposition which follows from relations (2.1) and (2.2), and Corollary 2.4. It is an improvement of the inequality (1.3) for the joint spectral radius of matrices.

Proposition 2.13. *Let $X_1, X_2, \ldots, X_k \in \mathbb{M}_n$ be such that $\{Re\, X_i : i = 1, \ldots, k\}$ and $\{Im\, X_i : i = 1, \ldots, k\}$ are two commuting families. Then*

$$\rho^2(X_1, \ldots, X_k) \leqslant \rho^2(Re\, X_1, \ldots, Re\, X_k) + \rho^2(Im\, X_1, \ldots, Im\, X_k).$$

Finally, we present some $\nu-$joint numerical radius inequalities. See [6, 7, 11] for related inequalities. For any vectors $x = (x_1, \ldots, x_k)$, $y = (y_1, \ldots, y_k) \in \mathbb{C}^k$, we define $|x| = (|x_1|, \ldots, |x_k|)$, and we say that $|x| \leqslant |y|$ if $|x_i| \leqslant |y_i|$ for all $i = 1, \ldots, k$. A vector norm ν on $\mathbb{C}^k$ is said to be:
(a) monotone if the inequality $|x| \leqslant |y|$ implies that $\nu(x) \leqslant \nu(y)$ for all $x, y \in \mathbb{C}^k$;
(b) absolute if $\nu(x) = \nu(|x|)$ for all $x \in \mathbb{C}^k$.
It is known, e.g., see [8], that the monotonicity of a norm on $\mathbb{C}^k$ is equivalent to its absolutivity. For example, the familiar ℓ_p-vector norms on $\mathbb{C}^k$ which are defined as

$$\ell_p(x_1, \ldots, x_k) = \left(\sum_{j=1}^{k} |x_j|^p \right)^{1/p} \quad and \ \ell_\infty(x_1, \ldots, x_k) = \max_{1 \leqslant i \leqslant k} |x_i|,$$

where $1 \leqslant p < \infty$, are absolute, and consequently are monotone. Next, we state the following proposition.

Proposition 2.14. *If ν is an absolute norm on $\mathbb{C}^k$, then for all $X_1, X_2, \ldots, X_k \in \mathbb{M}_n$,*

$$r_\nu(X_1, \ldots, X_k) \leqslant \nu(r(X_1), \ldots, r(X_k)).$$

Consequently,

$$r(X_1, \ldots, X_k) \leqslant \ell_2(r(X_1), \ldots, r(X_k)).$$

Proof. Let $X_1, X_2, \ldots, X_k \in \mathbb{M}_n$. For any arbitrary $(a_1, \ldots, a_k) \in W(X_1, \ldots, X_k)$, we have that $|a_i| \leqslant r(X_i)$ for all $i = 1, \ldots, k$. So, the absolutivity, and hence, the monotonicity of ν implies that

$$\nu(a_1, \ldots, a_k) = \nu(|a_1|, \ldots, |a_k|) \leqslant \nu(r(X_1), \ldots, r(X_k)).$$

Taking now the maximum over all $(a_1, \ldots, a_k) \in W(X_1, \ldots, X_k)$, we obtain the result. By setting $\nu = \ell_2$, the second assertion also holds. So, the proof is complete. □

The following example shows that the absolutivity of the norm in the Proposition 2.14 is essential.

Example 2.15. Define $\nu(x_1, x_2) = |x_1 - x_2| + |x_2|$, which is not absolute. Now, by considering $X_1 = diag(-1, -2)$ and $X_2 = diag(1, 0)$, it is clear that

$$r_\nu(X_1, X_2) = \max\{\nu(a_1, a_2) : (a_1, a_2) \in W(X_1, X_2)\} = 3.$$

But, $\nu(r(X_1), r(X_2)) = \nu(2, 1) = 2$. Hence, $\nu(r(X_1), r(X_2)) < r_\nu(X_1, X_2)$.

The following definition is related to the study of the converse of Proposition 2.14.

Definition 2.16. A vector norm ν on $\mathbb{C}^k$ is said to be weakly absolute if $\nu(x) \leqslant \nu(|x|)$ for all $x \in \mathbb{C}^k$.

Proposition 2.17. *Let ν be a vector norm on $\mathbb{C}^k$ such that for all $X_1, \ldots, X_k \in \mathbb{M}_n$,*

$$r_\nu(X_1, \ldots, X_k) \leqslant \nu(r(X_1), \ldots, r(X_k)).$$

Then ν is a weakly absolute norm on $\mathbb{C}^k$.

Proof. Let $\gamma_1, \ldots, \gamma_k \in \mathbb{C}$. By setting $X_i = \gamma_i I_n$, where $i = 1, \ldots, k$, and using Proposition 2.14, we have that $|\gamma_i| = r(X_i)$, and

$$\begin{aligned}
\nu(\gamma_1, \ldots, \gamma_k) &= \nu(e_1^* X_1 e_1, \ldots, e_1^* X_k e_1) \\
&\leqslant \sup\{\nu(x^* X_1 x, \ldots, x^* X_k x) : x \in \mathbb{C}^n,\ x^* x = 1\} \\
&= r_\nu(X_1, \ldots, X_k) \\
&\leqslant \nu(r(X_1), \ldots, r(X_k)) \\
&= \nu(|\gamma_1|, \ldots, |\gamma_k|).
\end{aligned}$$

Therefore, the proof is complete. □

In the final result, we state some inequalities about the joint numerical radius of the direct sum of matrices.

Theorem 2.18. *Let $X_1, X_2, \dots, X_k \in \mathbb{M}_n$ and $1 \leqslant p \leqslant \infty$. Then*

$$\begin{aligned} r_{\ell_\infty}(X_1, \dots, X_k) \leqslant \ell_\infty(r(X_1), \dots, r(X_k)) &= r(X_1 \oplus \cdots \oplus X_k) \\ &\leqslant r_{\ell_p}(X_1, \dots, X_k) \\ &\leqslant \ell_p(r(X_1), \dots, r(X_k)). \end{aligned}$$

Moreover, if $X_i \neq 0$ for all $i = 1, \dots, k$, then

$$r(X_1 \oplus \cdots \oplus X_k) = \ell_p(r(X_1), \dots, r(X_k)) \Longleftrightarrow k = 1.$$

Proof. The first and last inequalities follow from Proposition 2.14. It is clear that there exist $j \in \{1, 2, \dots, k\}$ and a unit vector $x \in \mathbb{C}^n$ such that

$$\begin{aligned} \ell_\infty(r(X_1), \dots, r(X_k)) &= r(X_1 \oplus \cdots \oplus X_k) \\ &= \max_{1 \leqslant i \leqslant k} r(X_i) \\ &= |x^* X_j x| \\ &\leqslant \ell_p(x^* X_1 x, \dots, x^* X_k x) \\ &\leqslant r_{\ell_p}(X_1, \dots, X_k). \end{aligned}$$

So, the proof of the first assertion is complete. Now, to prove the second assertion, we assume that $X_i \neq 0$ for all $i = 1, \dots, k$, and $p < \infty$, $r(X_1 \oplus \cdots \oplus X_k) = \ell_p(r(X_1), \dots, r(X_k))$ and it is possible that $k \geqslant 2$. Let $j \in \{1, \dots, k\}$ be such that

$$\max_{1 \leqslant i \leqslant k} r(X_i) = r(X_j).$$

Therefore, we have

$$r(X_j) = \left((r(X_j))^p + \sum_{i=1, i \neq j}^{k} (r(X_i))^p \right)^{1/p}.$$

Hence, $X_i = 0$ for all $i \neq j$, which is a contradiction. The converse of the second assertion, i.e., the result for the case $k = 1$, is easy to investigate. Also, the result is trivial for the case $p = \infty$. So, the proof is complete. □

Acknowledgments

The authors wish to express their gratitude to anonymous referees for helpful comments and useful suggestions which greatly improved the presentation of the original manuscript.

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