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Classical wavelet systems over finite fields

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ABSTRACT

This article presents an analytic approach to study admissibility conditions related to classical full wavelet systems over finite fields using tools from computational harmonic analysis and theoretical linear algebra. It is shown that for a large class of non-zero window signals (wavelets), the generated classical full wavelet systems constitute a frame whose canonical dual is classical full wavelet frame as well, and hence each vector defined over a finite field can be represented as a finite coherent sum of classical wavelet coefficients as well.

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1. Introduction

Throughout this article, we extend our recent results on classical wavelet systems over prime fields (finite Abelian groups of prime order) [8, 12, 14] for finite fields [7, 15]. The mathematical theory of finite fields has significant roles and applications in computer science, information theory, communication engineering, coding theory, cryptography, finite quantum systems and number theory [16, 22, 23]. Discrete exponentiation can be computed quickly using techniques of fast

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exponentiation such as binary exponentiation within a finite field operations and also in coding theory, many codes are constructed as subspaces of vector spaces over finite fields, see [17, 19] and references therein.

Wavelet analysis is a coherent state analysis which employ time-scale representations [9, 10, 11, 13]. The mathematical theory of wavelet analysis is originated from the abstract harmonic analysis of quasi-regular representations [3, 4, 5]. In a nutshell, wavelet analysis of periodic data classically rely on embedding the vector space of finite size data in the Hilbert space of all complex valued sequences with finite $\|.\|_2$-norm. This method is not on finite dimensional analogous to the continuous setting as is the case in Gabor analysis [1, 2, 18].

In this article we introduce the abstract notion of classical wavelet group $\mathrm{W}_{\mathbb{F}}$ associated to the finite field $\mathbb{F}$, as the group consists of classical dilations and translations. Then we present basic properties of the classical wavelet systems over the finite field $\mathbb{F}$. It is also shown that for a large class of non-zero window signals (wavelets), the generated classical full wavelet system constitute a frame whose canonical dual are classical full wavelet frames as well.

2. Preliminaries and Notations

Let $\mathbb{H}$ be a finite dimensional complex Hilbert space and $\dim\mathbb{H} = N$. A finite system (sequence) $\mathfrak{A} = \{\mathbf{y}_j : 0 \leq j \leq M-1\} \subset \mathbb{H}$ is called a frame (or finite frame) for $\mathbb{H}$, if there exist positive constants $0 < A \leq B < \infty$ such that

$$A\|\mathbf{x}\|^2 \leq \sum_{j=0}^{M-1} |\langle \mathbf{x}, \mathbf{y}_j\rangle|^2 \leq B\|\mathbf{x}\|^2, \text{ for all } \mathbf{x} \in \mathbb{H}. \tag{2.1}$$

If $\mathfrak{A} = \{\mathbf{y}_j : 0 \leq j \leq M-1\}$ is a frame for $\mathbb{H}$, the synthesis operator $F : \mathbb{C}^M \to \mathbb{H}$ is $F\{c_j\}_{j=0}^{M-1} = \sum_{j=0}^{M-1} c_j\mathbf{y}_j$ for all $\{c_j\}_{j=0}^{M-1} \in \mathbb{C}^M$. The adjoint (analysis) operator $F^* : \mathbb{H} \to \mathbb{C}^M$ is $F^*\mathbf{x} = \{\langle \mathbf{x}, \mathbf{y}_j\rangle\}_{j=0}^{M-1}$ for all $\mathbf{x} \in \mathbb{H}$. By composing F and F^*, we get the positive and invertible frame operator $S : \mathbb{H} \to \mathbb{H}$ given by

$$\mathbf{x} \mapsto S\mathbf{x} = FF^*\mathbf{x} = \sum_{j=0}^{M-1}\langle \mathbf{x}, \mathbf{y}_j\rangle\mathbf{y}_j \text{ for all } \mathbf{x} \in \mathbb{H}, \tag{2.2}$$

In terms of the analysis operator we have $A\|\mathbf{x}\|_2^2 \leq \|F^*\mathbf{x}\|_2^2 \leq B\|\mathbf{x}\|_2^2$ for $\mathbf{x} \in \mathbb{H}$. If $\mathfrak{A}$ is a finite frame for $\mathbb{H}$, the set $\mathfrak{A}$ spans the complex Hilbert space $\mathbb{H}$ which implies $M \geq N$, where $M = |\mathfrak{A}|$. It should be mentioned that each finite spanning set in $\mathbb{H}$ is a finite frame for $\mathbb{H}$. The ratio between M and N is called as redundancy of the finite frame $\mathfrak{A}$ (i.e. $\mathrm{red}_{\mathfrak{A}} = M/N$), where $M = |\mathfrak{A}|$. If $\mathfrak{A} = \{\mathbf{y}_j : 0 \leq j \leq M-1\}$ is a finite frame for $\mathbb{H}$, each $\mathbf{x} \in \mathbb{H}$ satisfies the following reconstruction formulas

$$\mathbf{x} = \sum_{j=0}^{M-1}\langle \mathbf{x}, S^{-1}\mathbf{y}_j\rangle\mathbf{y}_j = \sum_{j=0}^{M-1}\langle \mathbf{x}, \mathbf{y}_j\rangle S^{-1}\mathbf{y}_j. \tag{2.3}$$

In this case, the complex numbers $\langle \mathbf{x}, S^{-1}\mathbf{y}_j\rangle$ are called frame coefficients and the finite sequence $\mathfrak{A}^\bullet := \{S^{-1}\mathbf{y}_j : 0 \leq j \leq M-1\}$ which is a frame for $\mathbb{H}$ as well, is called the canonical dual frame of $\mathfrak{A}$. A finite frame $\mathfrak{A} = \{\mathbf{y}_j : 0 \leq j \leq M-1\}$ for $\mathbb{H}$ is called tight if we have $A = B$. If

$\mathfrak{A} = \{\mathbf{y}_j : 0 \leq j \leq M-1\}$ is a tight frame for $\mathbb{H}$ with frame bound A, then the canonical dual frame $\mathfrak{A}^\bullet$ is exactly $\{A^{-1}\mathbf{y}_j : 0 \leq j \leq M-1\}$ and for $\mathbf{x} \in \mathbb{H}$ we have

$$\mathbf{x} = \frac{1}{A}\sum_{j=0}^{M-1}\langle \mathbf{x}, \mathbf{y}_j\rangle \mathbf{y}_j. \tag{2.4}$$

For a finite group G, the finite dimensional complex vector space $\mathbb{C}^G = \{\mathbf{x} : G \to \mathbb{C}\}$ is a $|G|$-dimensional Hilbert space with complex vector entries indexed by elements in the finite group G. [1] The inner product of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{C}^G$ is $\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{g\in G}\mathbf{x}(g)\overline{\mathbf{y}(g)}$, and the induced norm is the $\|.\|_2$-norm of $\mathbf{x}$, that is $\|\mathbf{x}\|_2 = \sqrt{\langle \mathbf{x}, \mathbf{x}\rangle}$. For $\mathbb{C}^{\mathbb{Z}_N}$, where $\mathbb{Z}_N$ denotes the cyclic group of N elements $\{0, \ldots, N-1\}$, we simply write $\mathbb{C}^N$ at times.

Time-scale analysis and time-frequency analysis on finite Abelian group G as modern computational harmonic analysis tools are based on three basic operations on $\mathbb{C}^G$. The translation operator $T_k : \mathbb{C}^G \to \mathbb{C}^G$ given by $T_k\mathbf{x}(g) = \mathbf{x}(g-k)$ with $g, k \in G$. The modulation operator $M_\ell : \mathbb{C}^G \to \mathbb{C}^G$ given by $M_\ell\mathbf{x}(g) = \overline{\ell(g)}\mathbf{x}(g)$ with $g \in G$ and $\ell \in \widehat{G}$, where $\widehat{G}$ is the character/dual group of G. As the fundamental theorem of finite Abelian groups provides a factorization of G into cyclic groups, that is, $G \cong \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \ldots \times \mathbb{Z}_{N_d}$ as groups, which implies $\widehat{G} \cong G$, we can assume that the action of $\ell = (\ell_1, \ldots, \ell_d) \in \widehat{G}$ on $g = (g_1, ..., g_d) \in G$ is given by

$$\ell(g) = ((\ell_1, \ell_2, \ldots, \ell_d), (g_1, \ldots, g_d)) = \prod_{j=1}^{d}\mathbf{e}_{\ell_j}(g_j),$$

where $\mathbf{e}_{\ell_j}(g_j) = e^{2\pi i \ell_j g_j/N_j}$ for all $1 \leq j \leq d$. Thus

$$\ell(g) = ((\ell_1, \ell_2, \ldots, \ell_d), (g_1, \ldots, g_d)) = e^{2\pi i(\ell_1 g_1/N_1 + \ell_2 g_2/N_2 + \ldots + \ell_d g_d/N_d)}.$$

The character/dual group $\widehat{G}$ of any finite Abelian group G is isomorphic with G via the canonical group isomorphism $\ell \mapsto \mathbf{e}_\ell$, where the character $\mathbf{e}_\ell : G \to \mathbb{T}$ is given by $\mathbf{e}_\ell(g) = \ell(g)$ for all $g \in G$. The third fundamental operator is the discrete Fourier transform (DFT) $\mathcal{F}_G : \mathbb{C}^G \to \mathbb{C}^{\widehat{G}} = \mathbb{C}^G$ which allows us to pass from time representations to frequency representations. It is defined as a function on $\widehat{G}$ by

$$\mathcal{F}_G(\mathbf{x})(\ell) = \widehat{\mathbf{x}}(\ell) = \frac{1}{\sqrt{|G|}}\sum_{g\in G}\mathbf{x}(g)\overline{\ell(g)} \tag{2.5}$$

for all $\ell \in \widehat{G}$ and $\mathbf{x} \in \mathbb{C}^G$.

That is equivalently

$$\mathcal{F}_G(\mathbf{x})(\ell) = \widehat{\mathbf{x}}(\ell) = \frac{1}{\sqrt{|G|}}\sum_{g_1=0}^{N_1-1}\cdots\sum_{g_d=0}^{N_d-1}\mathbf{x}(g_1, \ldots, g_d)\overline{((\ell_1, \ldots, \ell_d), (g_1, \ldots, g_d))},$$

[1] $|G|$ denotes the order of the group G, or, more generally, the cardinality of a set G.

for all $\ell = (\ell_1, \ldots, \ell_d) \in \widehat{G}$ and $\mathbf{x} \in \mathbb{C}^G$. Translation, modulation, and the Fourier transform on the Hilbert space $\mathbb{C}^G = \mathbb{C}^{\widehat{G}}$ are unitary operators with respect to the $\|.\|_2$-norm. For $\ell, k \in G \cong \widehat{G}$ we have $(T_k)^* = (T_k)^{-1} = T_{-k}$ and $(M_\ell)^* = (M_\ell)^{-1} = M_{-\ell}$. The circular convolution of $\mathbf{x}, \mathbf{y} \in \mathbb{C}^G$ is defined by

$$\mathbf{x} * \mathbf{y}(k) = \frac{1}{\sqrt{|G|}} \sum_{g \in G} \mathbf{x}(g)\mathbf{y}(k-g), \quad \text{for } k \in G.$$

In terms of the translation operators we have $\mathbf{x}*\mathbf{y}(k) = \frac{1}{\sqrt{|G|}} \sum_{g\in G} \mathbf{x}(g) T_g \mathbf{y}(k)$ for $k \in G$. The circular involution or circular adjoint of $\mathbf{x} \in \mathbb{C}^G$ is given by $\mathbf{x}^*(k) = \overline{\mathbf{x}(-k)}$. The complex linear space $\mathbb{C}^G$ equipped with the $\|.\|_1$-norm, that is $\|\mathbf{x}\|_1 = \sum_{g\in G} |\mathbf{x}(g)|$, the circular convolution, and involution is a Banach $*$-algebra, which means that for all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^G$ we have

$$\|\mathbf{x} * \mathbf{y}\|_1 \leq \frac{1}{\sqrt{|G|}} \|\mathbf{x}\|_1 \|\mathbf{y}\|_1, \text{ and } \|\mathbf{x}^*\|_1 = \|\mathbf{x}\|_1.$$

The unitary DFT (2.5) satisfies

$$\widehat{T_k \mathbf{x}} = M_k \widehat{\mathbf{x}}, \quad \widehat{M_\ell \mathbf{x}} = T_{-\ell} \widehat{\mathbf{x}}, \quad \widehat{\mathbf{x}^*} = \overline{\widehat{\mathbf{x}}}, \quad \widehat{\mathbf{x} * \mathbf{y}} = \widehat{\mathbf{x}}.\widehat{\mathbf{y}},$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^G$, $k \in G$ and $\ell \in \widehat{G}$. See standard references of harmonic analysis such as [20] and references therein.

3. Harmonic Analysis over Finite Fields

Throughout this section, we present a summary of basic and classical results concerning harmonic analysis over finite fields. For proofs we refer readers to see [16, 21, 23] and references therein.

Let $\mathbb{F} = \mathbb{F}_q$ be a finite field of order q. Then there is a prime number p and an integer number $d \geq 1$ in which $q = p^d$. Every finite field of order $q = p^d$ is isomorphic as a field to every other field of order q. From now on, when it is necessary we denote any finite field of order $q = p^d$ by $\mathbb{F}_q$ otherwise we just denote it by $\mathbb{F}$. The prime number p is called the characteristic of $\mathbb{F}$, which means that

$$p.\tau = \sum_{k=1}^{p} \tau = 0 \quad \text{for all } \tau \in \mathbb{F}.$$

The absolute trace map $\mathbf{t} : \mathbb{F} \to \mathbb{Z}_p$ is given by $\tau \mapsto \mathbf{t}(\tau)$ where

$$\mathbf{t}(\tau) = \sum_{k=0}^{d-1} \tau^{p^k} \quad \text{for all } \tau \in \mathbb{F}.$$

The absolute trace map $\mathbf{t}$ is a $\mathbb{Z}_p$-linear transform from $\mathbb{F}$ onto $\mathbb{Z}_p$. It should be mentioned that in the case of prime fields, the trace map is readily the identity map.

There exists an irreducible polynomial $P \in \mathbb{Z}_p[t]$ of degree d and a root $\theta \in \mathbb{F}$ of P such that the set

$$\mathcal{B}_\theta := \{\theta^j : j = 0, ..., d-1\} = \{1, \theta, \theta^2, ..., \theta^{d-2}, \theta^{d-1}\},$$

is a linear basis of $\mathbb{F}$ over $\mathbb{Z}_p$. Then $\mathcal{B}_\theta$ is called as a polynomial basis of $\mathbb{F}$ over $\mathbb{Z}_p$ and θ is called as a defining element of $\mathbb{F}$ over $\mathbb{Z}_p$. Let $\mathbf{H} = \mathbf{H}_\theta \in \mathbb{Z}_p^{d\times d}$ be the $d \times d$ matrix with entries in the field $\mathbb{Z}_p$ given by $\mathbf{H}_{jk} := \mathbf{t}(\theta^{j+k})$ for all $0 \leq j, k \leq d-1$, which is invertible with the inverse $\mathbf{S} \in \mathbb{Z}_p^{d\times d}$. Then the dual polynomial basis

$$\widetilde{\mathcal{B}_\theta} := \{\Theta_k : k = 0, ..., d-1\}, \tag{3.1}$$

given by

$$\Theta_k = \sum_{j=0}^{d-1} \mathbf{S}_{kj}\theta^j, \tag{3.2}$$

satisfies the following orthogonality relation

$$\mathbf{t}(\theta^k \Theta_j) = \delta_{k,j}, \tag{3.3}$$

for all $j, k = 0, ..., d-1$.

Proposition 3.1. *Let $\mathbb{F}$ be a finite field of order $q = p^d$ with trace map $\mathbf{t} : \mathbb{F} \to \mathbb{Z}_p$. Then*

1. *For $\tau \in \mathbb{F}$ we have the following decompositions*

$$\tau = \sum_{k=0}^{d-1} \tau_{(k)}\theta^k = \sum_{k=0}^{d-1} \tau_{[k]}\Theta_k,$$

where for all $k = 0, ..., d-1$ we have

$$\tau_{(k)} := \mathbf{t}(\tau\Theta_k), \qquad \tau_{[k]} := \mathbf{t}(\tau\theta^k).$$

2. *For $\tau \in \mathbb{F}$ the coefficients (components) $\{\tau_{(k)} : k = 0, ..., d-1\}$ and $\{\tau_{[k]} : k = 0, ..., d-1\}$ satisfy*

$$\tau_{(k)} = \sum_{j=0}^{d-1} \mathbf{S}_{kj}\tau_{[j]}, \qquad \tau_{[k]} = \sum_{j=0}^{d-1} \mathbf{H}_{kj}\tau_{(j)},$$

for all $k = 0, ..., d-1$.

Let $\theta \in \mathbb{F}$ be a defining element of $\mathbb{F}$ over $\mathbb{Z}_p$. Then θ defines a $\mathbb{Z}_p$-linear isomorphism $J_\theta : \mathbb{F} \to \mathbb{Z}_p^d$ by

$$\gamma \mapsto J_\theta(\tau) := \tau_\theta = (\tau_{(k)})_{k=1}^d, \quad \text{for all } \tau \in \mathbb{F}. \tag{3.4}$$

Then the additive group of the finite field $\mathbb{F}$, $\mathbb{F}^+$, is isomorphic with the finite elementary group $\mathbb{Z}_p^d$ via J_θ. Thus, using classical dual theory on the ring $\mathbb{Z}_p^d$ we get

$$\mathbf{e}_{\tau_\theta}(\tau'_\theta) = \mathbf{e}_{1,p}\left(\tau_\theta.\tau'_\theta\right) = \mathbf{e}_{1,p}\left(\sum_{k=1}^{d} \tau_{(k)}\tau'_{(k)}\right), \quad \text{for all } \tau, \tau' \in \mathbb{F}.$$

Remark 3.2. The dual (character) group of the finite elementary group $\mathbb{Z}_p^d$, that is $\widehat{\mathbb{Z}_p^d}$, is precisely

$$\left\{\mathbf{e}_\ell : \ell = (\ell_1, ..., \ell_d) \in \mathbb{Z}_p^d\right\},$$

where the additive character $\mathbf{e}_\ell : \mathbb{Z}_p^d \to \mathbb{T}$ is given by

$$\mathbf{e}_\ell(g) = \mathbf{e}_{1,p}(\ell \cdot g) = \exp\left(\frac{2\pi i\, \ell \cdot g}{p}\right) = \prod_{k=1}^{d} \mathbf{e}_{\ell_k,p}(g_k) \quad \text{for all } g = (g_1, ..., g_d) \in \mathbb{Z}_p^d,$$

with $\ell \cdot g = \sum_{k=1}^d \ell_k g_k$.

Let $\chi : \mathbb{F} \to \mathbb{T}$ be given by

$$\chi(\tau) := \exp\left(\frac{2\pi i \mathbf{t}(\tau)}{p}\right) = \mathbf{e}_{1,p}(\mathbf{t}(\tau)), \quad \text{for all } \tau \in \mathbb{F}.$$

Since the trace map is $\mathbb{Z}_p$-linear, we deduce that χ is a character on the additive group of $\mathbb{F}$ (i.e $\chi \in \widehat{\mathbb{F}^+}$).

Proposition 3.3. *Let $\mathbb{F}$ be a finite field of order $q = p^d$ with trace map $\mathbf{t} : \mathbb{F} \to \mathbb{Z}_p$. Then*

1. *For $\tau, \tau' \in \mathbb{F}$ we have*

$$\mathbf{t}(\tau\tau') = \sum_{j=0}^{d-1}\sum_{k=0}^{d-1} \mathbf{H}_{jk}\tau_{(j)}\tau'_{(k)} = \sum_{j=0}^{d-1}\sum_{k=0}^{d-1} \mathbf{S}_{jk}\tau_{[j]}\tau'_{[k]} = \sum_{k=0}^{d-1} \tau_{(k)}\tau'_{[k]} = \sum_{k=0}^{d-1} \tau_{[k]}\tau'_{(k)}.$$

2. *For $\tau, \tau' \in \mathbb{F}$ we have*

$$\chi(\tau\tau') = \mathbf{e}_{1,p}\left(\sum_{k=1}^{d} \tau_{(k)}\tau'_{[k]}\right) = \mathbf{e}_{1,p}\left(\sum_{k=1}^{d} \tau_{[k]}\tau'_{(k)}\right).$$

For $\gamma \in \mathbb{F}$, let $\chi_\gamma : \mathbb{F} \to \mathbb{T}$ be given by

$$\chi_\gamma(\tau) := \chi(\gamma\tau) = \exp\left(\frac{2\pi i \mathbf{t}(\gamma\tau)}{p}\right) = \mathbf{e}_{1,p}(\mathbf{t}(\gamma\tau)), \quad \text{for all } \tau \in \mathbb{F}.$$

Then χ_γ is a character on the additive group of $\mathbb{F}$ (i.e $\chi_\gamma \in \widehat{\mathbb{F}^+}$). For $\gamma = 1$ we get $\chi = \chi_1$.

If $\alpha \in \mathbb{F}^*$ the character χ_α is called as a non-principal character. The interesting property of non-principal characters is that any non-principal character can parametrize the full character group of the additive group of $\mathbb{F}$. In details, if $\alpha \in \mathbb{F}^*$, then we have

$$\widehat{\mathbb{F}^+} = \left\{\chi_{\alpha\gamma} \; : \; \gamma \in \mathbb{F}\right\}.$$

Thus, the mapping $\gamma \mapsto \chi_{\alpha\gamma}$ is group isomorphism of $\mathbb{F}$ onto $\widehat{\mathbb{F}^+}$. Then for $\alpha = 1$ we get

$$\widehat{\mathbb{F}^+} = \left\{\chi_\gamma \; : \; \gamma \in \mathbb{F}\right\}. \tag{3.5}$$

Remark 3.4. The characterization (3.5) for the character group of finite fields is a consequence of applying the trace map in duality theory over finite fields. This characterization plays significant role in structure of dual action and hence wave packet groups over finite fields, see Section 4.

Then the Fourier transform of a vector $\mathbf{x} \in \mathbb{C}^{\mathbb{F}}$ at $\gamma \asymp \chi_\gamma \in \widehat{\mathbb{F}^+}$ is

$$\widehat{\mathbf{x}}(\chi_\gamma) = \frac{1}{\sqrt{p^d}} \sum_{\tau\in\mathbb{F}} \mathbf{x}(\tau)\overline{\chi_\gamma(\tau)} = \frac{1}{\sqrt{p^d}} \sum_{\tau\in\mathbb{F}} \mathbf{x}(\tau)\overline{\mathbf{F}(\gamma,\tau)},$$

where the matrix $\mathbf{F} : \mathbb{F} \times \mathbb{F} \to \mathbb{C}$ is given by

$$\mathbf{F}(\gamma,\tau) := \chi(\gamma\tau) = \exp\left(\frac{2\pi i\mathbf{t}(\gamma\tau)}{p}\right), \quad \text{for all } \gamma,\tau \in \mathbb{F}.$$

Remark 3.5. (i) For $\beta \in \mathbb{F}$, the translation operator $T_\beta : \mathbb{C}^{\mathbb{F}} \to \mathbb{C}^{\mathbb{F}}$ is

$$T_\beta\mathbf{x}(\tau) := \mathbf{x}(\tau - \beta), \quad \text{for all } \tau \in \mathbb{F} \text{ and } \mathbf{x} \in \mathbb{C}^{\mathbb{F}}.$$

(ii) For $\gamma \asymp \chi_\gamma \in \widehat{\mathbb{F}^+}$, the modulation operator $M_\gamma : \mathbb{C}^{\mathbb{F}} \to \mathbb{C}^{\mathbb{F}}$ is

$$M_\gamma\mathbf{x}(\tau) := \overline{\chi_\gamma(\tau)}\mathbf{x}(\tau), \quad \text{for all } \tau \in \mathbb{F} \text{ and } \mathbf{x} \in \mathbb{C}^{\mathbb{F}}.$$

4. Classical Wavelet Groups over Finite Fields

The abstract notion of wave packet groups over prime fields (finite Abelian groups of prime order) introduced in [8, 12, 14].

Let $\mathbb{F} = \mathbb{F}_q$ be a finite field of order $q = p^d$. The finite multiplicative group

$$\mathbb{F}^* := \mathbb{F} - \{0\} = \{\alpha \in \mathbb{F} : \alpha \neq 0\}, \tag{4.1}$$

of nonzero elements of $\mathbb{F}$ is a finite cyclic group of order $q - 1 = p^d - 1$. Any generator of the finite cyclic group $\mathbb{F}^*$ is called a primitive element or primitive root of $\mathbb{F}$ over $\mathbb{Z}_p$.

For $\alpha \in \mathbb{F}^*$, define the *dilation operator* $D_\alpha : \mathbb{C}^{\mathbb{F}} \to \mathbb{C}^{\mathbb{F}}$ by

$$D_\alpha\mathbf{x}(\tau) := \mathbf{x}(\alpha^{-1}\tau),$$

for all $\tau \in \mathbb{F}$ and $\mathbf{x} \in \mathbb{C}^{\mathbb{F}}$.

Hence we state basic algebraic properties of dilation operators.

Proposition 4.1. *Let $\mathbb{F}$ be a finite field. Then*

1. *For $(\alpha,\beta) \in \mathbb{F}^* \times \mathbb{F}$ we have $D_\alpha T_\beta = T_{\alpha\beta} D_\alpha$.*
2. *For $\alpha,\alpha' \in \mathbb{F}^*$ we have $D_{\alpha\alpha'} = D_\alpha D_{\alpha'}$.*
3. *For $(\alpha,\beta),(\alpha',\beta') \in \mathbb{F}^* \times \mathbb{F}$ we have $T_{\beta+\alpha\beta'} D_{\alpha\alpha'} = T_\beta D_\alpha T_{\beta'} D_{\alpha'}$.*

Proof. Let $\mathbb{F}$ be a finite field and $\mathbf{x} \in \mathbb{C}^{\mathbb{F}}$. Then
(1) For $(\alpha, \beta) \in \mathbb{F}^* \times \mathbb{F}$ and $\tau \in \mathbb{F}$, we can write

$$\begin{aligned} D_\alpha T_\beta \mathbf{x}(\tau) &= T_\beta \mathbf{x}(\alpha^{-1}\tau) \\ &= \mathbf{x}(\alpha^{-1}\tau - \beta) \\ &= \mathbf{x}(\alpha^{-1}\tau - \alpha^{-1}\alpha\beta) \\ &= \mathbf{x}(\alpha^{-1}(\tau - \alpha\beta)) \\ &= D_\alpha \mathbf{x}(\tau - \alpha\beta) = T_{\alpha\beta} D_\alpha \mathbf{x}(\tau). \end{aligned}$$

(2) For $\alpha, \alpha' \in \mathbb{F}^*$ and $\tau \in \mathbb{F}$, we can write

$$\begin{aligned} D_{\alpha\alpha'} \mathbf{x}(\tau) &= \mathbf{x}((\alpha\alpha')^{-1}\tau) \\ &= \mathbf{x}(\alpha'^{-1}\alpha^{-1}\tau) \\ &= D_{\alpha'} \mathbf{x}(\alpha^{-1}\tau) = D_\alpha D_{\alpha'} \mathbf{x}(\tau). \end{aligned}$$

(3) It is straightforward from (1) and (2).

□

Next proposition summarizes analytic properties of dilation operators.

Proposition 4.2. *Let $\mathbb{F}$ be a finite field and $\alpha \in \mathbb{F}^*$. Then*

1. $D_\alpha : \mathbb{C}^{\mathbb{F}} \to \mathbb{C}^{\mathbb{F}}$ *is a $*$-isometric isomorphism of the Banach $*$-algebra $\mathbb{C}^{\mathbb{F}}$*
2. $D_\alpha : \mathbb{C}^{\mathbb{F}} \to \mathbb{C}^{\mathbb{F}}$ *is unitary in $\|.\|_2$-norm and satisfies $(D_\alpha)^* = (D_\alpha)^{-1} = D_{\alpha^{-1}}$.*

Proof. (1) Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^{\mathbb{F}}$ and $\tau \in \mathbb{F}$. Then we have

$$D_\alpha(\mathbf{x} * \mathbf{y})(\tau) = \mathbf{x} * \mathbf{y}(\alpha^{-1}\tau) = \frac{1}{\sqrt{q}} \sum_{\tau' \in \mathbb{F}} \mathbf{x}(\tau')\mathbf{y}(\alpha^{-1}\tau - \tau').$$

Replacing τ' with $\alpha^{-1}\tau'$ we get

$$\begin{aligned} \frac{1}{\sqrt{q}} \sum_{\tau' \in \mathbb{F}} \mathbf{x}(\tau')\mathbf{y}(\alpha^{-1}\tau - \tau') &= \frac{1}{\sqrt{q}} \sum_{\tau' \in \mathbb{F}} \mathbf{x}(\alpha^{-1}\tau')\mathbf{y}(\alpha^{-1}\tau - \alpha^{-1}\tau') \\ &= \frac{1}{\sqrt{q}} \sum_{\tau' \in \mathbb{F}} \mathbf{x}(\alpha^{-1}\tau')\mathbf{y}(\alpha^{-1}(\tau - \tau')) \\ &= \frac{1}{\sqrt{q}} \sum_{\tau' \in \mathbb{F}} D_\alpha \mathbf{x}(\tau') D_\alpha \mathbf{y}(\tau - \tau') = (D_\alpha \mathbf{x}) * (D_\alpha \mathbf{y})(\tau), \end{aligned}$$

which implies that $D_\alpha(\mathbf{x} * \mathbf{y}) = (D_\alpha \mathbf{x}) * (D_\alpha \mathbf{y})$.
We can also write

$$\begin{aligned} (D_\alpha \mathbf{x})^*(\tau) &= \overline{D_\alpha \mathbf{x}(-\tau)} \\ &= \overline{\mathbf{x}(-\alpha^{-1}\tau))} \\ &= \mathbf{x}^*(\alpha^{-1}\tau) = D_\alpha \mathbf{x}^*(\tau), \end{aligned}$$

which guarantees $(D_\alpha\mathbf{x})^* = D_\alpha\mathbf{x}^*$.
(2) Let $\mathbf{x} \in \mathbb{C}^{\mathbb{F}}$. Then we can write

$$\begin{aligned}\|D_\alpha\mathbf{x}\|_2^2 &= \sum_{\tau\in\mathbb{F}} |D_\alpha\mathbf{x}(\tau)|^2 \\ &= \sum_{\tau\in\mathbb{F}} |\mathbf{x}(\alpha^{-1}\tau)|^2 \\ &= \sum_{\tau\in\mathbb{F}} |\mathbf{x}(\tau)|^2 = \|\mathbf{x}\|_2^2,\end{aligned}$$

which implies that $D_\alpha : \mathbb{C}^{\mathbb{F}} \to \mathbb{C}^{\mathbb{F}}$ is unitary in $\|.\|_2$-norm and satisfies

$$(D_\alpha)^* = (D_\alpha)^{-1} = D_{\alpha^{-1}}.$$

□

In the remainder of this article, we use the explicit characterization of the character group given by (3.5). Using (3.5), which can be considered as a consequence of analytic and algebraic properties of the trace map, the finite field $\mathbb{F}$ parametrizes the full character group $\widehat{\mathbb{F}^+}$. This parametrization implies a unified labeling on the character group $\widehat{\mathbb{F}^+}$ with $\mathbb{F}$.

Then we can present the following proposition.

Proposition 4.3. *Let $\mathbb{F}$ be a finite field and $\gamma \asymp \chi_\gamma \in \widehat{\mathbb{F}^+}$. Then*

1. *$M_\gamma : \mathbb{C}^{\mathbb{F}} \to \mathbb{C}^{\mathbb{F}}$ is a unitary operator in $\|.\|_2$-norm and satisfies $(M_\gamma)^* = (M_\gamma)^{-1} = M_{-\gamma}$.*
2. *For $\alpha \in \mathbb{F}^*$ we have $D_\alpha M_\gamma = M_{\alpha^{-1}\gamma}D_\alpha$.*
3. *For $\beta \in \mathbb{F}$ we have $T_\beta M_\gamma = \chi_\gamma(\beta)M_\gamma T_\beta$.*

Proof. (1) This statement is evident invoking definition of modulation operators.
(2) Let $\alpha \in \mathbb{F}^*$. Let $\mathbf{x} \in \mathbb{C}^{\mathbb{F}}$ and $\tau \in \mathbb{F}$. Then we can write

$$\begin{aligned}D_\alpha M_\gamma\mathbf{x}(\tau) &= M_\gamma\mathbf{x}(\alpha^{-1}\tau) \\ &= \overline{\chi_\gamma(\alpha^{-1}\tau)}\mathbf{x}(\alpha^{-1}\tau) \\ &= \overline{\chi(\gamma\alpha^{-1}\tau)}\mathbf{x}(\alpha^{-1}\tau) \\ &= \overline{\chi(\alpha^{-1}\gamma\tau)}\mathbf{x}(\alpha^{-1}\tau) \\ &= \overline{\chi_{\alpha^{-1}\gamma}(\tau)}\mathbf{x}(\alpha^{-1}\tau) \\ &= \overline{\chi_{\alpha^{-1}\gamma}(\tau)}D_\alpha\mathbf{x}(\tau) = M_{\alpha^{-1}\gamma}D_\alpha\mathbf{x}(\tau),\end{aligned}$$

which implies $D_\alpha M_\gamma = M_{\alpha^{-1}\gamma}D_\alpha$.

(3) Let $\beta \in \mathbb{F}$. Let $\mathbf{x} \in \mathbb{C}^{\mathbb{F}}$ and $\tau \in \mathbb{F}$. Then we have

$$\begin{aligned}T_\beta M_\gamma\mathbf{x}(\tau) &= M_\gamma\mathbf{x}(\tau-\beta) \\ &= \overline{\chi_\gamma(\tau-\beta)}\mathbf{x}(\tau-\beta) \\ &= \overline{\chi_\gamma(-\beta)\chi_\gamma(\tau)}\mathbf{x}(\tau-\beta) \\ &= \overline{\chi_\gamma(-\beta)\chi_\gamma(\tau)}T_\beta\mathbf{x}(\tau) = \chi_\gamma(\beta)M_\gamma T_\beta\mathbf{x}(\tau),\end{aligned}$$

which implies $T_\beta M_\gamma = \chi_\gamma(\beta)M_\gamma T_\beta$. □

For $\alpha \in \mathbb{F}^*$, let $\widehat{D}_\alpha : \mathbb{C}^{\widehat{\mathbb{F}^+}} \to \mathbb{C}^{\widehat{\mathbb{F}^+}}$ be given by

$$\widehat{D}_\alpha \mathbf{x}(\chi_\gamma) := \mathbf{x}(\chi_{\alpha^{-1}\gamma}),$$

for all $\gamma \asymp \chi_\gamma \in \widehat{\mathbb{F}^+}$ and $\mathbf{x} \in \mathbb{C}^{\widehat{\mathbb{F}}}$. Since $\mathbb{F}$ and $\widehat{\mathbb{F}^+}$ are isomorphic as finite Abelian groups, we may use D_α instead of $\widehat{D}_\alpha$ at times.

The following proposition presents some analytic properties of dilation operators on the frequency domain.

Proposition 4.4. *Let $\mathbb{F}$ be a finite field and $\alpha \in \mathbb{F}^*$. Then*

1. $D_\alpha : \mathbb{C}^{\widehat{\mathbb{F}^+}} \to \mathbb{C}^{\widehat{\mathbb{F}^+}}$ *is a $*$-isometric isomorphism of the Banach $*$-algebra* $\mathbb{C}^{\widehat{\mathbb{F}^+}}$
2. $D_\alpha : \mathbb{C}^{\widehat{\mathbb{F}^+}} \to \mathbb{C}^{\widehat{\mathbb{F}^+}}$ *is unitary in $\|.\|_2$-norm and satisfies* $(D_\alpha)^* = (D_\alpha)^{-1} = D_{\alpha^{-1}}$.

Next result states analytic properties of dilation operators and also connections with the Fourier transform.

Proposition 4.5. *Let $\mathbb{F}$ be a finite field of order q. Then*

1. *For $\beta \in \mathbb{F}$ we have $\mathcal{F}_{\mathbb{F}} T_\beta = M_\beta \mathcal{F}_{\mathbb{F}}$.*
2. *For $\gamma \asymp \chi_\gamma \in \widehat{\mathbb{F}^+}$ we have $\mathcal{F}_{\mathbb{F}} M_\gamma = T_{-\gamma} \mathcal{F}_{\mathbb{F}}$.*
3. *For $\alpha \in \mathbb{F}^*$ we have $\mathcal{F}_{\mathbb{F}} D_\alpha = \widehat{D}_{\alpha^{-1}} \mathcal{F}_{\mathbb{F}}$.*

Proof. (1) Let $\beta \in \mathbb{F}$ and $\mathbf{x} \in \mathbb{C}^{\mathbb{F}}$. Then for $\gamma \asymp \chi_\gamma \in \widehat{\mathbb{F}^+}$ we have

$$\mathcal{F}_{\mathbb{F}}(T_\beta \mathbf{x})(\chi_\gamma) = \frac{1}{\sqrt{q}} \sum_{\tau \in \mathbb{F}} T_\beta \mathbf{x}(\tau) \overline{\chi_\gamma(\tau)} = \frac{1}{\sqrt{q}} \sum_{\tau \in \mathbb{F}} \mathbf{x}(\tau - \beta) \overline{\chi_\gamma(\tau)}.$$

Replacing τ with $\tau + \beta$ we get

$$\frac{1}{\sqrt{q}} \sum_{\tau \in \mathbb{F}} \mathbf{x}(\tau - \beta) \overline{\chi_\gamma(\tau)} = \frac{1}{\sqrt{q}} \sum_{\tau \in \mathbb{F}} \mathbf{x}(\tau) \overline{\chi_\gamma(\tau + \beta)} = \frac{\overline{\chi_\gamma(\beta)}}{\sqrt{q}} \sum_{\tau \in \mathbb{F}} \mathbf{x}(\tau) \overline{\chi_\gamma(\tau)}.$$

Then we can write

$$\begin{aligned} \mathcal{F}_{\mathbb{F}}(T_\beta \mathbf{x})(\chi_\gamma) &= \overline{\chi_\gamma(\beta)} \mathcal{F}_{\mathbb{F}}(\mathbf{x})(\chi_\gamma) \\ &= \overline{\chi_\gamma(\beta)} \mathcal{F}_{\mathbb{F}}(\mathbf{x})(\chi_\gamma) = \overline{\chi_\beta(\gamma)} \mathcal{F}_{\mathbb{F}}(\mathbf{x})(\chi_\gamma), \end{aligned}$$

implying $\mathcal{F}_{\mathbb{F}} T_\beta = M_\beta \mathcal{F}_{\mathbb{F}}$.

(2) Let $\gamma \asymp \chi_\gamma \in \widehat{\mathbb{F}^+}$ and $\mathbf{x} \in \mathbb{C}^{\mathbb{F}}$. Then for all $\gamma' \asymp \chi_{\gamma'} \in \widehat{\mathbb{F}^+}$ we have

$$\begin{aligned} \mathcal{F}_{\mathbb{F}}(M_\gamma \mathbf{x})(\gamma') &= \frac{1}{\sqrt{q}} \sum_{\tau \in \mathbb{F}} M_\gamma \mathbf{x}(\tau) \overline{\chi_\gamma(\tau)} \\ &= \frac{1}{\sqrt{q}} \sum_{\tau \in \mathbb{F}} \chi_\gamma(\tau) \mathbf{x}(\tau) \overline{\chi_{\gamma'}(\tau)} \\ &= \frac{1}{\sqrt{q}} \sum_{\tau \in \mathbb{F}} \mathbf{x}(\tau) \overline{\chi_{\gamma+\gamma'}(\tau)} \\ &= \mathcal{F}_{\mathbb{F}}(\mathbf{x})(\gamma + \gamma') = T_{-\gamma} \mathcal{F}_{\mathbb{F}}(\mathbf{x})(\gamma'). \end{aligned}$$

(3) Let $\mathbf{x} \in \mathbb{C}^{\mathbb{F}}$ and $\gamma \asymp \chi_\gamma \in \widehat{\mathbb{F}^+}$. Then we have

$$\mathcal{F}_{\mathbb{F}}(D_\alpha \mathbf{x})(\gamma) = \frac{1}{\sqrt{q}} \sum_{\tau \in \mathbb{F}} D_\alpha \mathbf{x}(\tau)\overline{\chi_\gamma(\tau)} = \frac{1}{\sqrt{q}} \sum_{\tau \in \mathbb{F}} \mathbf{x}(\alpha^{-1}\tau)\overline{\chi_\gamma(\tau)}.$$

Replacing τ with $\alpha\tau$ we achieve

$$\begin{aligned}\frac{1}{\sqrt{q}} \sum_{\tau \in \mathbb{F}} \mathbf{x}(\alpha^{-1}\tau)\overline{\chi_\gamma(\tau)} &= \frac{1}{\sqrt{q}} \sum_{\tau \in \mathbb{F}} \mathbf{x}(\tau)\overline{\chi_\gamma(\alpha\tau)} \\ &= \frac{1}{\sqrt{q}} \sum_{\tau \in \mathbb{F}} \mathbf{x}(\tau)\overline{\chi_{\alpha\gamma}(\tau)} = \mathcal{F}_{\mathbb{F}}(\mathbf{x})(\alpha\gamma),\end{aligned}$$

which implies $\mathcal{F}_{\mathbb{F}}(D_\alpha \mathbf{x}) = \widehat{D}_{\alpha^{-1}}(\mathcal{F}_{\mathbb{F}}\mathbf{x})$. □

The underlying set $\mathbb{F}^* \times \mathbb{F}$ equipped with group operations given by

$$(\alpha,\beta) \rtimes (\alpha',\beta') := (\alpha\alpha', \beta + \alpha\beta') \tag{4.2}$$

$$(\alpha,\beta)^{-1} := (\alpha^{-1}, \alpha^{-1}.(-\beta)) \tag{4.3}$$

for all $(\alpha,\beta),(\alpha',\beta') \in \mathbb{F}^* \times \mathbb{F}$, is a finite non-Abelian group of order $q \cdot (q-1)$ which is denoted by $\mathbb{F}^* \rtimes \mathbb{F}$. The group $\mathbb{F}^* \rtimes \mathbb{F}$ is called as **classical wavelet group** over the finite field $\mathbb{F}$. Since any two field of order $q = p^d$ are isomorphic as finite field, we deduce that the notion of $\mathbb{F}^* \rtimes \mathbb{F}$ just depends on q. In details, if $\mathbb{F}$ and $\mathbb{K}$ are two finite field of order q, then the groups $\mathbb{F}^* \rtimes \mathbb{F}$ and $\mathbb{K}^* \rtimes \mathbb{K}$ are isomorphic as finite groups of order $q \cdot (q-1)$.

Next theorem guarantees that the group structure of the wave packet group $\mathbb{F}^* \rtimes \mathbb{F}$ is canonically connected with a group representation.

Theorem 4.6. *Let $\mathbb{F}$ be a finite field of order $q > 2$. Then*

1. *$\mathbb{F}^* \rtimes \mathbb{F}$ is a non-Abelian group of order $q \cdot (q-1)$ which contains $\mathbb{F}$ as a normal Abelian subgroup and $\mathbb{F}^*$ as a non-normal cyclic subgroup.*
2. *The map $\rho : \mathbb{F}^* \rtimes \mathbb{F} \to \mathcal{U}(\mathbb{C}^{\mathbb{F}}) \cong \mathbf{U}_{q\times q}(\mathbb{C})$ defined by*

$$(\alpha,\beta) \mapsto \rho(\alpha,\beta) := T_\beta D_\alpha \quad \text{for } (\alpha,\beta) \in \mathbb{F}^* \rtimes \mathbb{F}, \tag{4.4}$$

is a group representation of the finite classical wavelet group $\mathbb{F}^* \rtimes \mathbb{F}$ on the finite dimensional Hilbert space $\mathbb{C}^{\mathbb{F}}$.

Proof. Let $\mathbb{F}$ be a finite field of order $q > 2$. Then
(1) It is straightforward from the group structure given in (4.2) that $\mathbb{F}$ is a normal Abelian subgroup and $\mathbb{F}^*$ is a non-normal Abelian subgroup of $\mathbb{F}^* \rtimes \mathbb{F}$.
(2) It is evident to check that $\rho(1,0) = I$ and $\rho(\alpha,\beta) : \mathbb{C}^{\mathbb{F}} \to \mathbb{C}^{\mathbb{F}}$ is a unitary operator for all $(\alpha,\beta) \in \mathbb{F}^* \rtimes \mathbb{F}$. Now let $(\alpha,\beta),(\alpha',\beta') \in \mathbb{F}^* \rtimes \mathbb{F}$. Then using Proposition 4.1, we can write

$$T_{\beta+\alpha\beta'} D_{\alpha\alpha'} = T_\beta T_{\alpha\beta'} D_\alpha D_{\alpha'} = T_\beta D_\alpha T_{\beta'} D_{\alpha'}.$$

Thus, we get

$$\begin{aligned}\rho\left((\alpha,\beta)\rtimes(\alpha',\beta')\right) &= \rho(\alpha\alpha',\beta+\alpha\beta') \\ &= T_{\beta+\alpha\beta'}D_{\alpha\alpha'} \\ &= T_\beta D_\alpha T_{\beta'}D_{\alpha'} = \rho(\alpha,\beta)\rho(\alpha',\beta'),\end{aligned}$$

which implies that ρ is a group representation of the finite classical wavelet group $\mathbb{F}^*\rtimes\mathbb{F}$ on the finite dimensional Hilbert space $\mathbb{C}^{\mathbb{F}}$. □

Remark 4.7. In terms of abstract wavelet transforms over locally compact groups, the representation ρ mentioned in Theorem 4.6 is precisely the quasi regular representation generated by the action of the multiplicative group $H = \mathbb{F}^*$ on the finite additive group $K = \mathbb{F}$ on the Hilbert space $\mathbb{C}^{\mathbb{F}}$, see [3, 4, 5] and references therein.

5. Classical Wavelet Systems over Finite Fields

In this section we present abstract theory of classical wavelet systems over finite fields and we study analytic properties of these finite systems. Throughout this section, it is still assumed that $\mathbb{F}$ is a finite field of order $q = p^d$.

A **classical wavelet system** for the complex Hilbert space $\mathbb{C}^{\mathbb{F}}$ is a family or system of the form

$$\mathcal{W}(\mathbf{y},\Delta) := \left\{\rho(\alpha,\beta)\mathbf{y} = T_\beta D_\alpha\mathbf{y} : (\alpha,\beta)\in\Delta\subseteq\mathbb{F}^*\rtimes\mathbb{F}\right\}, \tag{5.1}$$

for some window signal $\mathbf{y}\in\mathbb{C}^{\mathbb{F}}$ and a subset Δ of $\mathbb{F}^*\rtimes\mathbb{F}$. If $\Delta = \mathbb{F}^*\rtimes\mathbb{F}$ we put $\mathcal{W}(\mathbf{y}) := \mathcal{W}(\mathbf{y},\mathbb{F}^*\rtimes\mathbb{F})$, and it is called a **full** classical wavelet system. A classical wavelet system which spans $\mathbb{C}^{\mathbb{F}}$ is a frame and is referred to as a **classical wavelet frame**.

Also, invoking properties of the dilation and translation operators we get

$$\langle\mathbf{x},\rho(\alpha,\beta)\mathbf{y}\rangle = \langle\mathbf{x},T_\beta D_\alpha\mathbf{y}\rangle = \langle T_{-\beta}\mathbf{x},D_\alpha\mathbf{y}\rangle,\quad \text{for } (\alpha,\beta)\in\mathbb{F}^*\rtimes\mathbb{F}. \tag{5.2}$$

The following proposition gives us a Fourier (resp. convolution) representation for the wavelet matrix.

Proposition 5.1. *Let $\mathbb{F}$ be a finite field of order q. Let $\mathbf{x},\mathbf{y}\in\mathbb{C}^{\mathbb{F}}$ and $(\alpha,\beta)\in\mathbb{F}^*\rtimes\mathbb{F}$. Then,*

1. $\langle\mathbf{x},\rho(\alpha,\beta)\mathbf{y}\rangle = \sqrt{q}\mathcal{F}_q(\widehat{\mathbf{x}}.\overline{\widehat{D_\alpha\mathbf{y}}})(\beta)$.
2. $\langle\mathbf{x},\rho(\alpha,\beta)\mathbf{y}\rangle = \mathbf{x}*D_\alpha\mathbf{y}^*(\beta)$.

Proof. Let $\mathbf{x},\mathbf{y} \in \mathbb{C}^{\mathbb{F}}$ and $(\alpha,\beta) \in \mathbb{F}^* \rtimes \mathbb{F}$. (1) Using the Plancherel formula we have

$$\begin{aligned}
\langle \mathbf{x}, \rho(\alpha,\beta)\mathbf{y}\rangle &= \langle \mathbf{x}, T_\beta D_\alpha \mathbf{y}\rangle \\
&= \langle \widehat{\mathbf{x}}, \widehat{T_\beta D_\alpha \mathbf{y}}\rangle \\
&= \sum_{\gamma\in\widehat{\mathbb{F}^+}} \widehat{\mathbf{x}}(\gamma)\overline{\widehat{T_\beta D_\alpha \mathbf{y}}(\gamma)} \\
&= \sum_{\gamma\in\widehat{\mathbb{F}^+}} \widehat{\mathbf{x}}(\gamma)\overline{M_\beta \widehat{D_\alpha \mathbf{y}}(\gamma)} \\
&= \sum_{\gamma\in\widehat{\mathbb{F}^+}} \widehat{\mathbf{x}}(\gamma)\overline{\widehat{D_\alpha \mathbf{y}}(\gamma)}\chi_\beta(\gamma) \\
&= \sum_{\gamma\in\widehat{\mathbb{F}^+}} \left(\widehat{\mathbf{x}}.\overline{\widehat{D_\alpha \mathbf{y}}}\right)(\gamma)\overline{\chi_\gamma(-\beta)} = \sqrt{q}\mathcal{F}_q(\widehat{\mathbf{x}}.\overline{\widehat{D_\alpha \mathbf{y}}})(-\beta).
\end{aligned}$$

(2) Similarly using the Plancherel formula we can write

$$\begin{aligned}
\langle \mathbf{x}, \rho(\alpha,\beta)\mathbf{y}\rangle &= \sum_{\gamma\in\widehat{\mathbb{F}^+}} \widehat{\mathbf{x}}(\gamma)\overline{\widehat{D_\alpha \mathbf{y}}(\gamma)}\chi_\beta(\gamma) \\
&= \sum_{\gamma\in\widehat{\mathbb{F}^+}} \widehat{\mathbf{x}}(\gamma)(\widehat{D_\alpha \mathbf{y}})^*(\gamma)\chi_\beta(\gamma) \\
&= \sum_{\gamma\in\widehat{\mathbb{F}^+}} \widehat{\mathbf{x}}(\gamma)(\widehat{D_\alpha \mathbf{y}^*})(\gamma)\chi_\beta(\gamma) \\
&= \sum_{\gamma\in\widehat{\mathbb{F}^+}} \widehat{\mathbf{x} * D_\alpha \mathbf{y}^*}(\gamma)\chi_\beta(\gamma) = \mathbf{x} * D_\alpha \mathbf{y}^*(\beta).
\end{aligned}$$

□

The following theorem presents a formula for computational aspects of the wavelet coefficients over finite fields.

Theorem 5.2. *Let $\mathbb{F}$ be a finite field of order q. Let $\mathbf{y} \in \mathbb{C}^{\mathbb{F}}$ be a window vector and $\mathbf{x} \in \mathbb{C}^{\mathbb{F}}$. Then,*

$$\sum_{\alpha\in\mathbb{F}^*}\sum_{\beta\in\mathbb{F}} |\langle \mathbf{x}, \rho(\alpha,\beta)\mathbf{y}\rangle|^2 = q\left((q-1)|\widehat{\mathbf{x}}(0)|^2|\widehat{\mathbf{y}}(0)|^2 + \left(\sum_{\gamma\in\mathbb{F}^*}|\widehat{\mathbf{x}}(\chi_\gamma)|^2\right)\left(\sum_{\alpha\in\mathbb{F}^*}|\widehat{\mathbf{y}}(\chi_\alpha)|^2\right)\right). \tag{5.3}$$

Proof. Let $\mathbf{y} \in \mathbb{C}^{\mathbb{F}}$ be a window function, $\mathbf{x} \in \mathbb{C}^{\mathbb{F}}$, and $\alpha \in \mathbb{F}^*$. Using Proposition 5.1 we have

$$\begin{aligned}
\sum_{\beta\in\mathbb{F}} |\langle \mathbf{x}, \rho(\alpha,\beta)\mathbf{y}\rangle|^2 &= q\sum_{\beta\in\mathbb{F}}\left|\mathcal{F}_q(\widehat{\mathbf{x}}.\overline{\widehat{D_\alpha \mathbf{y}}})(-\beta)\right|^2 \\
&= q\sum_{\beta\in\mathbb{F}}\left|\mathcal{F}_q(\widehat{\mathbf{x}}.\overline{\widehat{D_\alpha \mathbf{y}}})(\beta)\right|^2 \\
&= q\sum_{\gamma\in\mathbb{F}}\left|(\widehat{\mathbf{x}}.\overline{\widehat{D_\alpha \mathbf{y}}})(\chi_\gamma)\right|^2 = q\sum_{\gamma\in\mathbb{F}}\left|\widehat{\mathbf{x}}(\chi_\gamma).\overline{\widehat{D_\alpha \mathbf{y}}(\chi_\gamma)}\right|^2.
\end{aligned}$$

Therefore, we can write

$$\begin{aligned}\sum_{\alpha\in\mathbb{F}^*}\sum_{\beta\in\mathbb{F}}|\langle\mathbf{x},\rho(\alpha,\beta)\mathbf{y}\rangle|^2 &= q\sum_{\alpha\in\mathbb{F}^*}\sum_{\gamma\in\mathbb{F}}\left|\widehat{\mathbf{x}}(\chi_\gamma).\overline{\widehat{D_\alpha\mathbf{y}}(\chi_\gamma)}\right|^2\\ &= q\sum_{\alpha\in\mathbb{F}^*}\sum_{\gamma\in\mathbb{F}}\left|\widehat{\mathbf{x}}(\chi_\gamma)|^2\cdot|\overline{\widehat{D_\alpha\mathbf{y}}(\chi_\gamma)}\right|^2\\ &= q\sum_{\gamma\in\mathbb{F}}\sum_{\alpha\in\mathbb{F}^*}\left|\widehat{\mathbf{x}}(\chi_\gamma)|^2\cdot|\overline{\widehat{D_\alpha\mathbf{y}}(\chi_\gamma)}\right|^2\\ &= q\sum_{\gamma\in\mathbb{F}}|\widehat{\mathbf{x}}(\chi_\gamma)|^2\cdot\left(\sum_{\alpha\in\mathbb{F}^*}|\overline{\widehat{D_\alpha\mathbf{y}}(\chi_\gamma)}|^2\right)\\ &= q\sum_{\gamma\in\mathbb{F}}|\widehat{\mathbf{x}}(\chi_\gamma)|^2\cdot\left(\sum_{\alpha\in\mathbb{F}^*}|\widehat{D_\alpha\mathbf{y}}(\chi_\gamma)|^2\right).\end{aligned}$$

Now we can write

$$\sum_{\gamma\in\mathbb{F}}|\widehat{\mathbf{x}}(\chi_\gamma)|^2\cdot\left(\sum_{\alpha\in\mathbb{F}^*}|\widehat{D_\alpha\mathbf{y}}(\chi_\gamma)|^2\right) = |\widehat{\mathbf{x}}(0)|^2\left(\sum_{\alpha\in\mathbb{F}^*}|\widehat{\mathbf{y}}(0)|^2\right)+\sum_{\gamma\in\mathbb{F}^*}|\widehat{\mathbf{x}}(\chi_\gamma)|^2\left(\sum_{\alpha\in\mathbb{F}^*}|\widehat{\mathbf{y}}(\chi_{\alpha\gamma})|^2\right). \tag{5.4}$$

Replacing α with $\gamma^{-1}\alpha$ we have

$$\sum_{\alpha\in\mathbb{F}^*}|\widehat{\mathbf{y}}(\chi_{\alpha\gamma})|^2 = \sum_{\alpha\in\mathbb{F}^*}|\widehat{\mathbf{y}}(\chi_\alpha)|^2,$$

which implies

$$\begin{aligned}\sum_{\gamma\in\mathbb{F}}|\widehat{\mathbf{x}}(\chi_\gamma)|^2\cdot\left(\sum_{\alpha\in\mathbb{F}^*}|\widehat{D_\alpha\mathbf{y}}(\chi_\gamma)|^2\right) &= |\widehat{\mathbf{x}}(0)|^2\left(\sum_{\alpha\in\mathbb{F}^*}|\widehat{\mathbf{y}}(0)|^2\right)+\sum_{\gamma\in\mathbb{F}^*}|\widehat{\mathbf{x}}(\chi_\gamma)|^2\left(\sum_{\alpha\in\mathbb{F}^*}|\widehat{\mathbf{y}}(\chi_{\alpha\gamma})|^2\right)\\ &= (q-1)|\widehat{\mathbf{x}}(0)|^2|\widehat{\mathbf{y}}(0)|^2+\sum_{\gamma\in\mathbb{F}^*}|\widehat{\mathbf{x}}(\chi_\gamma)|^2\left(\sum_{\alpha\in\mathbb{F}^*}|\widehat{\mathbf{y}}(\chi_{\alpha\gamma})|^2\right)\\ &= (q-1)|\widehat{\mathbf{x}}(0)|^2|\widehat{\mathbf{y}}(0)|^2+\left(\sum_{\gamma\in\mathbb{F}^*}|\widehat{\mathbf{x}}(\chi_\gamma)|^2\right)\left(\sum_{\alpha\in\mathbb{F}^*}|\widehat{\mathbf{y}}(\chi_\alpha)|^2\right).\end{aligned}$$

Hence using (5.4) we get

$$\sum_{\alpha\in\mathbb{F}^*}\sum_{\beta\in\mathbb{F}}|\langle\mathbf{x},\rho(\alpha,\beta)\mathbf{y}\rangle|^2 = q\left((q-1)|\widehat{\mathbf{x}}(0)|^2|\widehat{\mathbf{y}}(0)|^2+\left(\sum_{\gamma\in\mathbb{F}^*}|\widehat{\mathbf{x}}(\chi_\gamma)|^2\right)\left(\sum_{\alpha\in\mathbb{F}^*}|\widehat{\mathbf{y}}(\chi_\alpha)|^2\right)\right).$$

□

Then we can present following results concerning a unified characterization for classical full wavelet systems over finite fields. The following theorem shows that for a large class of non-zero window signals the classical full wavelet system $\mathcal{W}(\mathbf{y})$ is a frame for the finite dimensional Hilbert space $\mathbb{C}^{\mathbb{F}}$ with redundancy $q-1$.

Theorem 5.3. *Let $\mathbf{y} \in \mathbb{C}^{\mathbb{F}}$ be a non-zero window signal. The full wavelet system $\mathcal{W}(\mathbf{y})$ constitutes a frame for $\mathbb{C}^{\mathbb{F}}$ with the redundancy $q-1$ if and only if $\widehat{\mathbf{y}}(0) \neq 0$ and $\|\widehat{\mathbf{y}}\|_0 \geq 2$.*

Proof. Let $\mathbf{y}$ be a non-zero window signal with $\widehat{\mathbf{y}}(0) \neq 0$ and $\|\widehat{\mathbf{y}}\|_0 \geq 2$. Let $0 < A \leq B < \infty$ be given by

$$A := \min\left\{(q-1)\left|\sum_{\tau\in\mathbb{F}} \mathbf{y}(\tau)\right|^2, q\sum_{\alpha\in\mathbb{F}^*} |\widehat{\mathbf{y}}(\chi_\alpha)|^2\right\},$$

$$B := \max\left\{(p-1)\left|\sum_{\tau\in\mathbb{F}} \mathbf{y}(\tau)\right|^2, q\sum_{\alpha\in\mathbb{F}^*} |\widehat{\mathbf{y}}(\chi_\alpha)|^2\right\}.$$

Then A, B are readily non-zero. If $\mathbf{x} \in \mathbb{C}^{\mathbb{F}}$, using (5.3) we can write

$$\sum_{\alpha\in\mathbb{F}^*}\sum_{\beta\in\mathbb{F}} |\langle \mathbf{x}, T_\beta D_\alpha \mathbf{y}\rangle|^2 = q\left((q-1)|\widehat{\mathbf{x}}(0)|^2|\widehat{\mathbf{y}}(0)|^2 + \left(\sum_{\gamma\in\mathbb{F}^*} |\widehat{\mathbf{x}}(\chi_\gamma)|^2\right)\left(\sum_{\alpha\in\mathbb{F}^*} |\widehat{\mathbf{y}}(\chi_\alpha)|^2\right)\right).$$

Thus we achieve

$$\sum_{\alpha\in\mathbb{F}^*}\sum_{\beta\in\mathbb{F}} |\langle \mathbf{x}, T_\beta D_\alpha \mathbf{y}\rangle|^2 = (q-1)|\widehat{\mathbf{x}}(0)|^2\left|\sum_{\tau\in\mathbb{F}} \mathbf{y}(\tau)\right|^2 + q\left(\sum_{\gamma\in\mathbb{F}^*} |\widehat{\mathbf{x}}(\chi_\gamma)|^2\right)\left(\sum_{\alpha\in\mathbb{F}^*} |\widehat{\mathbf{y}}(\chi_\alpha)|^2\right). \tag{5.5}$$

Now by (5.5) we get

$$\begin{aligned}\sum_{\alpha\in\mathbb{F}^*}\sum_{\beta\in\mathbb{F}} |\langle \mathbf{x}, T_\beta D_\alpha \mathbf{y}\rangle|^2 &= (q-1)|\widehat{\mathbf{x}}(0)|^2\left|\sum_{\tau\in\mathbb{F}} \mathbf{y}(\tau)\right|^2 + q\left(\sum_{\gamma\in\mathbb{F}^*} |\widehat{\mathbf{x}}(\chi_\gamma)|^2\right)\left(\sum_{\alpha\in\mathbb{F}^*} |\widehat{\mathbf{y}}(\chi_\alpha)|^2\right)\\ &\leq \max\left\{(q-1)\left|\sum_{\tau\in\mathbb{F}} \mathbf{y}(\tau)\right|^2, q\sum_{\alpha\in\mathbb{F}^*} |\widehat{\mathbf{y}}(\chi_\alpha)|^2\right\}\left(\sum_{\gamma\in\mathbb{F}} |\widehat{\mathbf{x}}(\chi_\gamma)|^2\right)\\ &= B\|\mathbf{x}\|_2^2.\end{aligned}$$

Similarly, by (5.5) we also have

$$\begin{aligned}\sum_{\alpha\in\mathbb{F}^*}\sum_{\beta\in\mathbb{F}} |\langle \mathbf{x}, T_\beta D_\alpha \mathbf{y}\rangle|^2 &= (q-1)|\widehat{\mathbf{x}}(0)|^2\left|\sum_{\tau\in\mathbb{F}} \mathbf{y}(\tau)\right|^2 + q\left(\sum_{\gamma\in\mathbb{F}^*} |\widehat{\mathbf{x}}(\chi_\gamma)|^2\right)\left(\sum_{\alpha\in\mathbb{F}^*} |\widehat{\mathbf{y}}(\chi_\alpha)|^2\right)\\ &\geq \min\left\{(q-1)\left|\sum_{\tau\in\mathbb{F}} \mathbf{y}(\tau)\right|^2, q\sum_{\alpha\in\mathbb{F}^*} |\widehat{\mathbf{y}}(\chi_\alpha)|^2\right\}\left(\sum_{\gamma\in\mathbb{F}} |\widehat{\mathbf{x}}(\chi_\gamma)|^2\right)\\ &= A\|\mathbf{x}\|_2^2.\end{aligned}$$

Conversely, let $\mathbf{y} \in \mathbb{C}^{\mathbb{F}}$ be a non-zero window signal such that the full cyclic wavelet system $\mathcal{W}(\mathbf{y})$ be a frame for $\mathbb{C}^{\mathbb{F}}$. Then, using (5.5) we get $\widehat{\mathbf{y}}(0) \neq 0$ and $\sum_{\alpha\in\mathbb{F}^*} |\widehat{\mathbf{y}}(\chi_\alpha)|^2 \neq 0$, which implies that $\widehat{\mathbf{y}}(0) \neq 0$ and $\|\widehat{\mathbf{y}}\|_0 \geq 2$. □

Then we conclude the following result.

Corollary 5.4. *Let* $\mathbf{y} \in \mathbb{C}^{\mathbb{F}}$ *be a window signal with* $\widehat{\mathbf{y}}(0) \neq 0$ *and* $\|\widehat{\mathbf{y}}\|_0 \geq 2$. *The full cyclic wavelet system* $\mathcal{W}(\mathbf{y})$ *is a tight frame for* $\mathbb{C}^{\mathbb{F}}$ *if and only if* $\mathbf{y}$ *satisfies* $\|\mathbf{y}\|_{\mathbf{Fr}} = \sqrt{q}|\widehat{\mathbf{y}}(0)|$. *In this case,*

$$A_{\mathbf{y}} := (q-1)\left|\sum_{\tau\in\mathbb{F}} \mathbf{y}(\tau)\right|^2 = q\cdot(q-1)\cdot|\widehat{\mathbf{y}}(0)|^2 = q\sum_{\alpha\in\mathbb{F}^*}|\widehat{\mathbf{y}}(\chi_\alpha)|^2, \tag{5.6}$$

is the frame bound.

In the abstract theory of frames, a dual (canonical dual) pair of coherent frames gives an expansion of any function/signal as a superposition of wavelet frame (coherent frame) elements or dictionary. The following interesting property of cyclic wavelet frames shows that the canonical dual frame of any full cyclic wavelet frame is again a full cyclic wavelet frame.

Theorem 5.5. *The canonical dual of any full wavelet frame for* $\mathbb{C}^{\mathbb{F}}$ *is a full wavelet frame.*

Proof. Let $\mathbf{y} \in \mathbb{C}^{\mathbb{F}}$ be a non-zero window signal such that the full cyclic wavelet system $\mathcal{W}(\mathbf{y})$ be a frame for $\mathbb{C}^{\mathbb{F}}$. Let S be the frame operator of $\mathfrak{A} := \mathcal{W}(\mathbf{y})$. We claim that

$$\mathfrak{A}^\bullet = \mathcal{W}(\mathbf{y}^\bullet) = \{T_\beta D_\alpha \mathbf{y}^\bullet : (\alpha,\beta) \in \mathbb{F}^* \rtimes \mathbb{F}\}, \tag{5.7}$$

where $\mathbf{y}^\bullet := S^{-1}\mathbf{y}$. Invoking the group structure of $\mathbb{F}^* \rtimes \mathbb{F}$ and since ρ is a unitary representation of $\mathbb{F}^* \rtimes \mathbb{F}$ we have $T_\beta D_\alpha S = S T_\beta D_\alpha$ for all $(\alpha,\beta) \in \mathbb{F}^* \rtimes \mathbb{F}$. Then we get $S^{-1}T_\beta D_\alpha = T_\beta D_\alpha S^{-1}$ for all $(\alpha,\beta) \in \mathbb{F}^* \rtimes \mathbb{F}$ which implies (5.7). □

Corollary 5.6. *The canonical dual of any full wavelet frame* $\mathcal{W}(\mathbf{y})$ *with the frame operator* S *is the full wavelet frame* $\mathcal{W}(S^{-1}\mathbf{y})$ *with the frame operator* S^{-1}.

Remark 5.7. The above property of full wavelet frames (Theorem 5.5) assures that canonical dual of the wavelet systems is again a wavelet system. It should be mentioned that a similar property does not hold for traditional wavelet structured frames, for example, canonical dual frames of infinite dimensional wavelet frames are not in general wavelet frame, see [6] and classical list of references therein.

Then as a consequence of the formula (5.3) we can present an irreducible decomposition for the unitary representation ρ.

Let $\mathcal{B}_q$ be the complex linear subspace in $\mathbb{C}^{\mathbb{F}}$ of dimension $q-1$ which is given by

$$\mathcal{B}_q := \left\{\mathbf{x} \in \mathbb{C}^{\mathbb{F}} : \widehat{\mathbf{x}}(0) = \sum_{\tau\in\mathbb{F}} \mathbf{x}(\tau) = 0\right\}. \tag{5.8}$$

Proposition 5.8. *Let $\mathbb{F}$ be a finite field of order q. Then the linear subspace $\mathcal{B}_q$ is an irreducible subspace of the unitary representation $\rho : \mathbb{F}^* \rtimes \mathbb{F} \to \mathcal{U}(\mathbb{C}^{\mathbb{F}})$.*

Proof. It is straightforward to see that $\mathcal{B}_q$ is an invariant subspace of $\mathbb{C}^{\mathbb{F}}$. Let $\mathcal{H}$ be a nontrivial subspace of $\mathcal{B}_q$. It is enough to show that $\mathcal{H}^\perp = \{0\}$. Let $\mathbf{x} \in \mathcal{H}^\perp$ be arbitrary and pick a nonzero vector $\mathbf{y} \in \mathcal{H}$. Using the assumption that $\mathcal{H}$ is an invariant subspace of $\mathcal{B}_q$ we have $\langle \mathbf{x}, T_\beta D_\alpha \mathbf{y}\rangle = 0$ for all $(\alpha,\beta) \in \mathbb{F}^* \times \mathbb{F}$. Since $\mathbf{y}$ is a non-zero vector and $\widehat{\mathbf{y}}(0) = 0$ we achieve that $\sum_{\alpha\in\mathbb{F}^*} |\widehat{\mathbf{y}}(\chi_\alpha)|^2 \neq 0$. Invoking (5.3) we can write

$$\begin{aligned}\left(\sum_{\alpha\in\mathbb{F}^*} |\widehat{\mathbf{y}}(\chi_\alpha)|^2\right) \|\mathbf{x}\|_2^2 &= \left(\sum_{\ell=1}^{p-1} |\widehat{\mathbf{y}}(\ell)|^2\right) \|\widehat{\mathbf{x}}\|_2^2 \\ &= \left(\sum_{\gamma\in\mathbb{F}^*} |\widehat{\mathbf{x}}(\chi_\gamma)|^2\right)\left(\sum_{\alpha\in\mathbb{F}^*} |\widehat{\mathbf{y}}(\chi_\alpha)|^2\right) \\ &= \sum_{\gamma\in\mathbb{F}^*}\sum_{\alpha\in\mathbb{F}^*} |\langle \mathbf{x}, T_\beta D_\alpha \mathbf{y}\rangle|^2 = 0,\end{aligned}$$

which implies that $\mathbf{x} = 0$. Since $\mathbf{x}$ was arbitrary, we deduce that $\mathcal{H}^\perp = \{0\}$. Thus, $\mathcal{H} = \mathcal{B}_q$, which implies that $\mathcal{B}_q$ is an irreducible subspace of the unitary representation ρ. □

Finally, we present the following corollary.

Corollary 5.9. *Let $\mathbf{y} \in \mathbb{C}^{\mathbb{F}}$ be a window signal with $\widehat{\mathbf{y}}(0) \neq 0$ and $\|\widehat{\mathbf{y}}\|_0 \geq 2$.The full wavelet system $\mathcal{W}(\mathbf{y})$ is a tight frame for the Hilbert space $\mathcal{B}_q$ with the frame bound $B_{\mathbf{y}}$, where*

$$B_{\mathbf{y}} := q \sum_{\alpha\in\mathbb{F}^*} |\widehat{\mathbf{y}}(\chi_\alpha)|^2 = q\left(\|\mathbf{y}\|_2^2 - |\widehat{\mathbf{y}}(0)|^2\right) = q\|\mathbf{y}\|_2^2 - \left|\sum_{\tau\in\mathbb{F}} \mathbf{y}(\tau)\right|^2. \tag{5.9}$$

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