Some results on functionally convex sets in real Banach spaces

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\textbf{Abstract}

We use of two notions functionally convex (briefly, F–convex) and functionally closed (briefly, F–closed) in functional analysis and obtain more results. We show that if \( \{A_a\}_{a \in \mathcal{I}} \) is a family F–convex subsets with non empty intersection of a Banach space \( X \), then \( \bigcup_{a \in \mathcal{I}} A_a \) is F–convex. Moreover, we introduce new definition of notion F–convexity.

\textbf{1. Introduction}

In [5], M. Eshahgi, H. R. Reisi and A. R. Moazzen introduced two new notions in functional analysis. By defining functionally convex (briefly, F–convex) and functionally closed (briefly, F–

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closed) sets, they improved some basic theorems in functional analysis. Among other things, the Krein-Milman theorem has been generalized on finite dimensional Banach spaces. Hence, they have proved that, the set of extreme points of every bounded, $F$-convex and $F$-closed subset of a finite dimensional space is nonempty. Additionally, they partially proved the famous Chebyshev open problem (which asks whether or not every Chebyshev set in a Hilbert space is convex?). Hence, they have shown that, if $A$ is a Chebyshev subset of a Hilbert space and the metric projection $P_A$ is continuous, then $A$ is $F$-convex.

From now on, we suppose that all normed spaces and Banach spaces are real.

**Definition 1.1.** [5] In a normed space $X$, we say that $K(\subseteq X)$ is functionally convex (briefly, $F$-convex) if for every bounded linear transformation $T \in B(X, \mathbb{R})$, the subset $T(K)$ of $\mathbb{R}$ is convex.

**Proposition 1.2.** [5] If $T$ is a bounded linear mapping from a normed space $X$ into a normed space $Y$, and $K$ is $F$-convex in $X$, then $T(K)$ is $F$-convex in $Y$.

**Corollary 1.3.** [5] Let $A, B$ be two $F$-convex subsets of a normed space $X$ and $\lambda$ be a real number, then

$$A + B = \{a + b : a \in A, b \in B\}, \quad \lambda A = \{\lambda a : a \in A\}$$

are $F$-convex.

**Proposition 1.4.** [5] Let $A$ and $B$ be $F$-convex subsets of a linear space $X$, which have nonempty intersection. Then $A \cup B$ is $F$-convex.

**Definition 1.5.** [5] Let $X$ be a normed space and let $A \subseteq X$. $A$ is functionally closed (briefly, $F$-closed), if $f(A)$ is closed for all $f \in X^*$.

Note that every compact set is $F$-closed. Also, every closed subset of real numbers $\mathbb{R}$ is $F$-closed. In $X = \mathbb{R}^2$, the set $A = \{(x, y) : x, y \geq 0\}$ is (non-compact) $F$-closed whereas, the set $A = \mathbb{Z} \times \mathbb{Z}$ is closed but it is not $F$-closed (by taking $f(x, y) = x + \sqrt{2}y$, the set $f(A)$ is not closed in $\mathbb{R}$). By taking $A = \{(x, y) : 1 \leq x^2 + y^2 \leq 4\}$ a nonconvex $F$-closed and $F$–convex set is obtained. Also, the set $B = \{(x, y) : x \in [0, \frac{\pi}{2}), y \geq \tan(x)\}$ is a closed convex set which is not $F$–closed. On the other hand, $A = \{(x, y) : 1 < x^2 + y^2 \leq 4\}$ is a non-compact and $F$–closed set. The two last examples show that weakly closed (weakly compact) and $F$–closed sets are different.

**Remark 1.6.** Note that we can not reduce definition of $F$–convexity to a basis of $X^*$, in the sense that a set in $X$ is $F$–convex whenever its image under elements of a basis is convex. For instance, by taking the Euclidean space $\mathbb{R}^2$ and the set

$$A = \{(0, \alpha) : \alpha \in \mathbb{R} - \mathbb{Q} \cap [-\sqrt{2}, 1]\} \cup \{(\beta, 1) : \beta \in \mathbb{R} - \mathbb{Q} \cap [0, \sqrt{2}]\}$$

$$\cup \{(r, -\sqrt{2}) : r \in \mathbb{Q} \cap [0, \sqrt{2}]\} \cup \{\sqrt{2}, s) : s \in \mathbb{Q} \cap [-\sqrt{2}, 1]\}$$

$$\cup \{(0, 1), (0, \sqrt{2}), (\sqrt{2}, -\sqrt{2}), (\sqrt{2}, 1)\}$$

$p_1(x, y) = x$ and $p_2(x, y) = y$, projections on axis, is a base for $X = \mathbb{R}^2$ and $P_1(A) = [0, 1]$ also, $p_2(A) = [-\sqrt{2}, 1]$ but $f(x, y) = x + y$ is an element of $X^*$ and $f(A)$ is not convex.
In [5], we prove the following theorem, which help us to find a big class of F–convex sets.

**Theorem 1.7.** Every arcwise connected subset of a normed space $X$ is F–convex.

**Remark 1.8.** The converse of the above theorem is not valid. Hence, by taking $S = \{(x, \sin(\frac{1}{x}) : 0 < x \leq 1\}$, the set $\overline{S}$ which is called the sine’s curve of topologist is connected and so for any linear functional $f \in (\mathbb{R} \times \mathbb{R})'$, the set $f(\overline{S})$ is an interval. Thus, $\overline{S}$ is an F–convex set which is not arcwise connected.

**2. Main Results**

In this section, we show, how construct new subset F–convex one of given ones.

**Proposition 2.1.** Let $A, B$ be subsets of Banach space $X$. If $A$ is F–convex and $A \subset B \subset \overline{A}$ then, $B$ is F–convex.

**Proof.** For every $f \in X^*$, we have $f(A) \subseteq f(B) \subseteq f(\overline{A}) \subseteq \overline{f(A)}$. Hence, by assumption, $f(\overline{A})$ is an interval. This completes the proof.

**Remark 2.2.** In contrary the case of convex sets, interior of an F–convex set, necessarily is not F–convex. For instance, take $X = \mathbb{R} \times \mathbb{R}$ and let $B = \{(x, y) : x^2 + y^2 \leq 1\}$. Then if $A$ is all elements surrounded by $B$ and $B + \frac{1}{2}$ is F–convex, but the interior of $A$ is not F–convex. Since, by taking $f$ as projection on $x$-axis we have $f(A^o) = (-\frac{1}{2}, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{2})$, which is not convex.

**Theorem 2.3.** Let $\{A_\alpha\}_{\alpha \in \mathcal{I}}$ be collection of F–convex subsets in Banach space $X$. If $\bigcap_{\alpha \in \mathcal{I}} A_\alpha \neq \emptyset$ then, $\bigcup_{\alpha \in \mathcal{I}} A_\alpha$ is F–convex.

**Proof.** For each $f \in X^*$ and $\alpha \in \mathcal{I}$, we know, $f(A_\alpha)$ is an interval and $\bigcap_{\alpha \in \mathcal{I}} f(A_\alpha) \neq \emptyset$. Thus, $f(\bigcup_{\alpha \in \mathcal{I}} A_\alpha) = \bigcup_{\alpha \in \mathcal{I}} f(A_\alpha)$ is convex.

We know that, if $\{A_\alpha\}_{\alpha \in \mathcal{I}}$ be a collection of connected subsets in $X$, $A$ is connected and $A \cap A_\alpha \neq \emptyset$ for all $\alpha \in \mathcal{I}$, then $A \cup (\bigcup_{\alpha \in \mathcal{I}} A_\alpha)$ is connected. Now, we have the following theorem;

**Theorem 2.4.** Let $\{A_\alpha\}_{\alpha \in \mathcal{I}}$ be a collection of F–convex subsets in Banach space $X$. If $A$ is F–convex and $A \cap A_\alpha \neq \emptyset$ for evrey $\alpha \in \mathcal{I}$, then $A \cup (\bigcup_{\alpha \in \mathcal{I}} A_\alpha)$ is F–convex.

**Proof.** For evrey $f \in X^*$ and all $\alpha \in \mathcal{I}$, $f(A_\alpha)$ and $f(A)$ are intervals such that $f(A) \cap f(A_\alpha) \neq \emptyset$. Therefore, $f(A \cup (\bigcup_{\alpha \in \mathcal{I}} A_\alpha)) = \bigcup_{\alpha \in \mathcal{I}} f(A_\alpha) \cup f(A)$ is interval for evrey $f \in X^*$. So, $A \cup (\bigcup_{\alpha \in \mathcal{I}} A_\alpha)$ is F–convex.

We know that, if $\{A_n\}_{n \in \mathbb{N}}$ be a collection of connected subsets in $X$ such that $A_n \cap A_{n+1} \neq \emptyset$ for all $n \in \mathbb{N}$, then $\bigcup_{n \in \mathbb{N}} A_n$ is connected. Now, we have the following theorem;

**Theorem 2.5.** Let $\{A_n\}_{n \in \mathbb{N}}$ be a collection of F–convex subsets in Banach space $X$. If $A_n \cap A_{n+1} \neq \emptyset$ for evrey $n \in \mathbb{N}$, then $\bigcup_{n \in \mathbb{N}} A_n$ is F–convex.

**Proof.** For evrey $f \in X^*$ and all $n \in \mathbb{N}$, $f(A_n)$ is interval and $f(A_n) \cap f(A_{n+1}) \neq \emptyset$. Therefore, $f(\bigcup_{n \in \mathbb{N}} A_n) = \bigcup_{n \in \mathbb{N}} f(A_n)$ is interval for evrey $f \in X^*$. So, $\bigcup_{n \in \mathbb{N}} A_n$ is F–convex.
Let $A$ be a subset of linear space $X$. We define an equivalence relation on $A$ as: $x \sim y$ if and only if both lie in a $F$–convex subset of $A$. The relation $\sim$ actually is an equivalence relation. For transitivity, note that if $x \sim y$ and $y \sim z$ then there are weakly convex subsets $A$ and $B$ such that $x, y \in A$ and $y, z \in B$. Proposition 1.4 asserts that $A \cup B$ is $F$–convex subset of $X$ and so $x \sim z$.

**Theorem 2.6.** Let $(X_i, ||.||)$ be norm linear spaces, then $A_i \subset X_i$ are $F$–convex if and only if, $\prod_{i=1}^n A_i$ is $F$–convex in $\prod_{i=1}^n X_i$ equipted by the norm

\[ ||(x_1, x_2, \cdots , x_n)|| = \left\{ \sum_{i=1}^n ||x_i||^2 \right\}^{1/2}. \]

**Proof.** We Know that

\[ (\prod_{i=1}^n X_i)^* = \prod_{i=1}^n X_i^*. \]

So, for every $g \in (\prod_{i=1}^n X_i)^*$ there are unique $f_i \in X_i^*$, $i = 1, 2, \cdots , n$ such that, $g = \sum_{i=1}^n f_i$. Now we have

\[ g(\prod_{i=1}^n A_i) = \sum_{i=1}^n f_i(A_i). \]

Since, every $A_i$ is $F$–convex so, $f_i(A_i)$ and their sum is an interval. Conversely, for every $f_i \in X_i^*$, taking $g = 0 + 0 + \cdots + f_i + \cdots + 0$, we have $f_i(A_i) = g(\prod_{i=1}^n A_i)$ so, $A_i$ is $F$–convex.

**Theorem 2.7.** Let $Y$ be a subspace of the norm linear space $X$. If $A \subset Y$ is $F$–convex then, $A$ is $F$–convex in $X$.

**Proof.** Let $Y$ be a subspace of $X$. There exists subspace $Y^\perp$ of $X$ such that $X = Y \oplus Y^\perp$. Thus, for every $f \in X^*$ we have, $f|_Y \in Y^*$. Now, if $A$ is $F$–convex in $Y$, Therefore, $f(A) = f|_Y(A) + f(Y^\perp)$. By assumption, $f|_Y(A)$ is $F$–convex also, since $Y^\perp$ is a subspace, so $Y^\perp$ is $F$–convex in $X$. Thus, By using 1.3 $f(A)$ is $F$–convex in $X$.

**Definition 2.8.** Let $A$ be a subset of linear space $X$. Let $\sim A = \{ A_\alpha \}_{\alpha \in I}$ be the set of all equivalence classes. For each $\alpha \in I, A_\alpha$ is called $F$–convex component of $A$.

**Theorem 2.9.** Let $A$ be a subset of linear space $X$. The $F$–convex components of $A$ are disjoint $F$–convex subsets of $A$ whose their union is $A$, such that any non empty $F$–convex subset of $A$ contains only one of them.

**Proof.** Being equivalence classes, the $F$–convex component of $A$ are disjoint and their union is $A$. Each $F$–convex subset of $A$ contains only one of them. For if, $A$ intersects the components $A_1, A_2$ of $A$ say, in points $x_1, x_2$ respectively, then $x_1 \sim x_2$. this means $A_1 = A_2$. To show the $F$–convex component $B$ is $F$–convex, choose a point $x$ of $B$. For each $y \in B$, we know that $x_1 \sim x_2$, so there is a $F$–convex subset $A_y$ containing $x, y$. By the result just proved $A_y \subset A$. Thus, $B = \bigcup_{y \in A} A_y$. Since subsets $A_y$ are $F$–convex and the point $x$ is in their intersection, by 2.3 $B$ is $F$–convex.
Remark 2.10. Let $A$ be a subset of linear space $X$. $A$ is F–convex if and only if it has one F–convex component.

In the following theorem, for a subset $A$ of a Banach space $X$, a necessary and sufficient condition for F–convexity is proved.

**Theorem 2.11.** Let $X$ be a Banach space, $A \subseteq X$ is F–convex if and only if

$$co(A) \subseteq \bigcap_{f \in X^*} A + Ker(f).$$

**Proof.** The set $A \subseteq X$ is F–convex iff for all $f \in X^*$, the element $\sum_{i=1}^{n} \lambda_i a_i$ belongs to $f(A)$ which, $\lambda_i \geq 0$, $a_i \in A$ and $\sum_{i=1}^{n} \lambda_i = 1$. This is equivalent that for all $f \in X^*$, there is $a \in A$ such that $a - \sum_{i=1}^{n} \lambda_i a_i \in Ker(f)$. \qed

**Remark 2.12.** Note that in special case $X = \mathbb{R}$, since every nonzero functional is one to one so we have $\bigcap_{f \in X^*} A + Ker(f) = A$. Thus $A \subseteq \mathbb{R}$ is F–convex iff $co(A) \subseteq A$. Also, we have $A \subseteq co(A)$. Then we obtain $A \subseteq \mathbb{R}$ is F–convex iff $A$ is convex.

Let $X$ be a vector space. A hyperplane in $X$ (through $x_0 \in X$) is a set of the form $H = x_0 + Ker(f) \subseteq X$, where $f$ is a non-zero linear functional on $X$. Equivalently, $H = f^{-1}(\gamma)$, where $\gamma = f(x_0)$. So, we have

$$\bigcap_{f \in X^*} A + Ker(f) = \bigcap_{f \in X^*} \bigcup_{a \in A} a + Ker(f) = \bigcap_{f \in X^*} f^{-1}(f(A)).$$

Hence, $A \subseteq X$ is F–convex if and only if

$$co(A) \subseteq \bigcap_{f \in X^*} f^{-1}(f(A)).$$

**Proposition 2.13.** Let $A$ be a subset of Banach space $X$. The set $U = \bigcap_{B \in \Gamma} \bigcap_{f \in X^*} f^{-1}(f(B))$ is F–convex, where $\Gamma = \{ B : A \subseteq B, \ B \text{ is F–convex} \}$.

**Proof.** By discussion ago, we have $co(B) \subseteq \bigcap_{f \in X^*} f^{-1}(f(B))$. Intersecting on all $B \in \Gamma$, implies that

$$co(A) = \bigcap_{B \in \Gamma} co(B) \subseteq U \subseteq \bigcap_{f \in X^*} f^{-1}(f(co(A))).$$

On the other hand, for every $g \in X^*$,

$$g(co(A)) \subseteq g(U) \subseteq g(g^{-1}(g(co(A)))) \subseteq g(co(A))$$

Hence, for every $g \in X^*$, $g(U) = g(co(A))$. So $U$ is F–convex. \qed

**Theorem 2.14.** [3] If $K_1$ and $K_2$ are disjoint closed convex subsets of a locally convex linear topological space $X$, and if $K_1$ is compact, then there exist constants $c$ and $\epsilon > 0$, and a continuous linear functional $f$ on $X$, such that

$$f(K_2) \leq c - \epsilon < c \leq f(K_1).$$
Lemma 2.15. [5] If $A$ is a subset of a Banach space $X$, then

$$\bigcap_{f \in X^*} f^{-1}(f(A)) \subseteq \overline{co}(A)$$

Corollary 2.16. [5] Let $A$ be an $F$–closed subset of a Banach space $X$. Then $A$ is $F$–convex if and only if

$$\overline{co}(A) = \bigcap_{f \in X^*} f^{-1}(f(A)).$$

Corollary 2.17. A compact subset $A$ in a Banach space $X$ is convex if and only if $A$ is $F$–convex and $X^*$ separates $A$ and every element of $X - A$.

Proof. If $A$ is a compact convex subset of $X$, then by Theorem 2.14, the assertion holds. Conversely, assume that $A$ is a compact $F$–convex subset of $X$. Hence, $\overline{co}(A) = \bigcap_{f \in X^*} f^{-1}(f(A))$. On the other hand, there is $f \in X^*$ such that for every $x \in X - A$, we have $f(A) < f(x)$. This implies that $x$ is outside of $f^{-1}(f(A))$. Thus $f^{-1}(f(A)) = A$ and $\overline{co}(A) = A$. \hfill $\square$

Remark 2.18. If $X$ is a Hilbert space, then by Riesz representation theorem for every $f \in X^*$, there exists a unique $z \in X$ such that for all $x \in X$, $f(x) = \langle x, z \rangle$, the inner product of $x$ and $z$. Then

$$Ker(f) = \{x \in X : \langle x, z \rangle = 0\} = z^\perp.$$

In this case, we have

$$\bigcap_{f \in X^*} f^{-1}(f(A)) = \bigcap_{f \in X^*} A + Ker(f) = \bigcap_{z \in X} A + z^\perp. \quad (2.1)$$

Thus, in a Hilbert space $X$, every $F$–closed subset $A$ of $X$ is $F$–convex iff

$$\overline{co}(A) = \bigcap_{z \in X} A + z^\perp.$$

Corollary 2.19. Let $A$ and $B$ be $F$–closed and $F$–convex subsets of a Banach space $X$ which have nonempty intersection. Then

$$\overline{co}(A \cup B) = \overline{co}(A) \cup \overline{co}(B).$$

Proof. By Proposition 1.4, $A \cup B$ is $F$–convex. Then we have

$$\overline{co}(A \cup B) = \bigcap_{f \in X^*} f^{-1}(f(A \cup B))$$

$$= \left( \bigcap_{f \in X^*} f^{-1}(f(A)) \right) \bigcup \left( \bigcap_{f \in X^*} f^{-1}(f(A)) \right)$$

$$= \overline{co}(A) \cup \overline{co}(B).$$

\hfill $\square$
Corollary 2.20. Let $A$ and $B$ be $F$–closed and $F$–convex subsets of a Banach space $X$. Then

$$
\overline{co}(A + B) = \overline{co}(A) + \overline{co}(B).
$$

Proof. Obviously, we have

$$
\overline{co}(A + B) \subseteq \overline{co}(A) + \overline{co}(B).
$$

Let $x$ be an arbitrary element of $\overline{co}(A) + \overline{co}(B)$. Then there are $x_1 \in \overline{co}(A)$ and $x_2 \in \overline{co}(B)$ such that $x = x_1 + x_2$. Then for every $f \in X^*$, we have $f(x_1) \in f(A)$ and $f(x_2) \in f(B)$. This implies that $f(x_1 + x_2) \in f(A + B)$ and hence, $x \in f^{-1}(f(A + B))$. It follows that

$$
\overline{co}(A) + \overline{co}(B) \subseteq \bigcap_{f \in X^*} f^{-1}(f(A + B)) = \overline{co}(A + B).
$$

Note that if $A$ and $B$ are $F$–convex and $F$–closed then, $A + B$ is $F$–closed. \hfill \Box

References