G-dual function-valued frames in $L_2(0, \infty)$

M. A. Hasankhanifard$^{a, *}$, M. A. Dehghan$^a$

$^a$Department of Mathematics, Vali-E-Asr University, Rafsanjan, Islamic Republic of Iran, P.O.Box: 546

**Abstract**

In this paper, g-dual function-valued frames in $L_2(0, \infty)$ are introduced. We can achieve more reconstruction formulas to obtain signals in $L_2(0, \infty)$ by applying g-dual function-valued frames in $L_2(0, \infty)$.

**Keywords:**
g-dual frame, function-valued frame,

**2000 MSC:**
46L99, 42C15

1. Introduction

Given a separable Hilbert space $\mathcal{H}$ with inner product $\langle ., . \rangle$, a sequence $\{f_k\}_{k=1}^{\infty}$ is called a frame for $\mathcal{H}$ if there exist constants $A > 0$, $B < \infty$ such that for all $f \in \mathcal{H}$,

$$A\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B\|f\|^2,$$

(1.1)
where $A$ and $B$ are the lower and upper frame bounds, respectively. The second inequality of the frame condition (1.1) is also known as the Bessel condition for $(f_k)_{k=1}^n$. For more information concerning frames refer to [1, 2, 4, 11].

We consider three classes of operators on $L_2(\mathbb{R})$. Their definitions are as follows:

- Translation by $a \in \mathbb{R}$, $T_a : L_2(\mathbb{R}) \to L_2(\mathbb{R})$, $(T_ag)(x) = g(x-a)$.
- Modulation by $b \in \mathbb{R}$, $E_b : L_2(\mathbb{R}) \to L_2(\mathbb{R})$, $(E_bg)(x) = e^{2\pi ibx}g(x)$.
- Dilation by $c \neq 0$, $D_c : L_2(\mathbb{R}) \to L_2(\mathbb{R})$, $(D_cg)(x) = \frac{1}{\sqrt{|c|}}g(\frac{x}{c})$.

A Gabor frame is a frame for $L_2(\mathbb{R})$ of the form $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$, where $a, b > 0$, $g$ is a fixed function in $L_2(\mathbb{R})$. This frame is special case of shift-invariant systems (up to an irrelevant complex factor). Casazza and Lammers define a function-valued inner product of two function $f, g \in L_2(\mathbb{R})$ as

$$\langle f, g \rangle_a(x) = \sum_{n \in \mathbb{Z}} f(x-na)\overline{g(x-na)} \quad \forall x \in \mathbb{R},$$

where $a$ is a fixed positive real number [3]. They called it as a-inner product and used it in the study of Gabor frames.

The dilation-invariant system generated by the sequence $(g_k)_{k \in \mathbb{Z}}$ in $L_2(\mathbb{R})$ and $a > 1$ is the sequence $(D_ag_k)_{k \in \mathbb{Z}}$, where $D_ag_k$ is the dilation operator by $a^j$. Dilation-invariant systems contain wavelet frames and hence they will play an important role in the analysis of wavelet frames. A function-valued inner product on $L_2(0, \infty)$ by using of the dilation operator has been introduced in [8]. The authors use of a function-valued inner product in the study of the dilation-invariant systems:

Fix $a > 1$. For each pair $f, g \in L_2(0, \infty)$, the function $\langle f, g \rangle_a$ on $(0, \infty)$ is defined by

$$\langle f, g \rangle_a(x) := \sum_{j \in \mathbb{Z}} a^j f(a^jx)\overline{g(a^jx)}$$

and is called function-valued inner product on $L_2(0, \infty)$ with respect to $a$. It is easy to show that $\langle f, g \rangle_a = \int_1^a \langle f, g \rangle_{a^j}(x)dx$, where $\langle \cdot, \cdot \rangle$ is the original inner product in $L_2(0, \infty)$. Also, the function-valued norm on $L_2(0, \infty)$ with respect to $a$ is defined by

$$||f||_a(x) := \sqrt{\langle f, f \rangle_a(x)} \quad \forall f \in L_2(0, \infty) \quad \text{and} \quad \forall x \in (0, \infty).$$

The function $\phi$ on $(0, \infty)$ is called dilation periodic function with period $a$ if $\phi(ax) = \phi(x)$ for all $x \in (0, \infty)$. The set of bounded dilation periodic functions on $(0, \infty)$ is denoted by $B_a$.

Example 1.1. Let $f$ be a bounded function on $(0, \infty)$ and let $(G, +)$ be a finite group (for example $G = \mathbb{Z}_n, n \in \mathbb{N}, n \geq 2$). Then the function $\phi$ defined by $\phi(x) = \sum_{j \in G} f(a^jx)$, for all $x \in (0, \infty)$ is in $B_a$.

For any function $\phi$ on $[1, a]$, the function $\tilde{\phi}$ defined by $\tilde{\phi}(a^jx) = \phi(x)$, for all $j \in \mathbb{Z}$ and $x \in [1, a]$ is dilation periodic. Throughout this paper, let $\phi$ be the dilation periodic function defined as above for any complex function $\phi$ on $[1, a]$. 
Proposition 1.2. [8] Let \( f, g \in L_2(0, \infty) \) and \( \phi \in B_a \). Then
\[
\langle \phi f, g \rangle_a = \phi \langle f, g \rangle_a \quad \text{and} \quad \langle f, \phi g \rangle_a = \bar{\phi} \langle f, g \rangle_a
\]

For any \( f, g \in L_2(0, \infty) \), \( f \) and \( g \) are function-valued orthogonal with respect to \( a \), or simply function-valued orthogonal if \( \langle f, g \rangle_a = 0 \) a.e. on \([1, a]\).

A sequence \( \{e_n\}_{n \in \mathbb{Z}} \) in \( L_2(0, \infty) \) is called function-valued orthogonal with respect to \( a \) if \( e_n \perp_a e_m \), for all \( n \neq m \in \mathbb{Z} \). If also \( ||e_n||_a = 1 \) a.e. on \([1, a]\), then \( \{e_n\}_{n \in \mathbb{Z}} \) is called a function-valued orthonormal sequence with respect to \( a \), or simply function-valued orthonormal sequence, in \( L_2(0, \infty) \).

A sequence \( \{e_n\}_{n \in \mathbb{Z}} \) is called function-valued orthonormal basis with respect to \( a \), or simply function-valued orthonormal basis, for \( L_2(0, \infty) \) if it is a function-valued orthonormal sequence and \( \sum_{m \in \mathbb{Z}} |\psi_m(x)|^2 = L_2(0, \infty) \), where \( \psi_m \) is defined by \( \psi_m(x) = \frac{1}{\sqrt{2\pi}} e^{2\pi i \frac{m}{a} (a-x)} \) for all \( m \in \mathbb{Z} \) and \( x \in [1, a] \).

Proposition 1.3. [8] If \( \{e_n\}_{n \in \mathbb{Z}} \) is a function-valued orthonormal basis in \( L_2(0, \infty) \), then \( \{\bar{\psi}_m e_n\}_{m,n \in \mathbb{Z}} \) is an orthonormal basis in \( L_2(0, \infty) \) and \( f = \sum_{n \in \mathbb{Z}} (\langle f, e_n \rangle_a) \bar{e}_n \) on \( (0, \infty) \).

Let \( E \) be a measurable subset of \((0, \infty)\) and \( 1 \leq p \leq \infty \). A linear operator \( L : L_2(0, \infty) \to L_p(E) \), is called a function-valued factorable operator with respect to \( a \), or simply function-valued factorable operator if \( L(\phi f) = \phi L(f) \) for all \( f \in L_2(0, \infty) \) and \( \phi \in B_a \).

Proposition 1.4. [8] If \( L : L_2(0, \infty) \to L_2(0, \infty) \) is a bounded function-valued factorable operator, then for all \( f, g \in L_2(0, \infty) \) we have
\[
\langle L(f), g \rangle_a(x) = \langle f, L^*(g) \rangle_a(x), \quad \text{for all } x \in (0, \infty),
\]
where \( L^* \) is the adjoint operator of \( L \).

A sequence \( \{f_n\}_{n \in \mathbb{Z}} \) in \( L_2(0, \infty) \) is called a function-valued frame with respect to \( a \) for \( L_2(0, \infty) \), or simply function-valued frame for \( L_2(0, \infty) \) if there exist constants \( A > 0, B < \infty \) such that
\[
A ||f||_a^2(x) \leq \sum_{n \in \mathbb{Z}} |\langle f, f_n \rangle_a(x)|^2 \leq B ||f||_a^2(x),
\]
for a.e. \( x \in [1, a] \) and for all \( f \in L_2(0, \infty) \).

Let \( \{f_n\}_{n \in \mathbb{Z}} \) be a function-valued frame for \( L_2(0, \infty) \). A function-valued frame \( \{g_n\}_{n \in \mathbb{Z}} \) is called a dual function-valued frame of \( \{f_n\}_{n \in \mathbb{Z}} \) with respect to \( a \), or simply dual function-valued frame of \( \{f_n\}_{n \in \mathbb{Z}} \) for \( L_2(0, \infty) \) if for all \( f \in L_2(0, \infty) \)
\[
f = \sum_{n \in \mathbb{Z}} \langle f, g_n \rangle_a f_n. \tag{1.2}
\]

Theorem 1.5. [8] Let \( \{f_n\}_{n \in \mathbb{Z}} \) be a sequence in \( L_2(0, \infty) \). The following statements are equivalent:
1) \( \{f_n\}_{n \in \mathbb{Z}} \) is a function-valued frame for \( L_2(0, \infty) \).
2) \( \{\psi_m f_n\}_{m,n \in \mathbb{Z}} \) is a frame for \( L_2(0, \infty) \).
Two frames \( \{ f_k \}_{k=1}^{\infty} \) and \( \{ g_k \}_{k=1}^{\infty} \) are dual frames for \( \mathcal{H} \) if
\[
f = \sum_{k=1}^{\infty} < f, g_k > f_k, \quad \forall f \in \mathcal{H}.
\]

Dual frames have an important role for the reconstruction of signals. From this point of view, dual frames have been generalized. Pseudo duals \([10]\), Oblique dual frames \([5, 9]\) and approximately dual frames \([6]\) are some generalizations of dual frames.

G-duals of a frame in a separable Hilbert space \( \mathcal{H} \) are introduced in \([7]\).

Let \( \{ f_k \}_{k=1}^{\infty} \) be a frame for \( \mathcal{H} \). A frame \( \{ g_k \}_{k=1}^{\infty} \) is called a generalized dual frame or g-dual frame of \( \{ f_k \}_{k=1}^{\infty} \) for \( \mathcal{H} \) if there exists an invertible operator \( A \in B(\mathcal{H}) \) such that for all \( f \in \mathcal{H} \),
\[
f = \sum_{k=1}^{\infty} < Af, g_k > f_k.
\]

In this paper, g-dual function-valued frames in \( L_2(0, \infty) \) are introduced. Also an application of g-dual function-valued frames in \( L_2(0, \infty) \) for characterizing g-dual frame of a dilation-invariant system in \( L_2(0, \infty) \) is given.

2. G-dual function-valued frames in \( L_2(0, \infty) \)

**Definition 2.1.** Let \( \{ f_n \}_{n \in \mathbb{Z}} \) be a function-valued frame for \( L_2(0, \infty) \). A function-valued frame \( \{ g_n \}_{n \in \mathbb{Z}} \) is called a **g-dual function-valued frame** of \( \{ f_n \}_{n \in \mathbb{Z}} \) with respect to \( a \), or simply g-dual function-valued frame of \( \{ f_n \}_{n \in \mathbb{Z}} \) for \( L_2(0, \infty) \) if there exists a bounded invertible function-valued factorable operator \( L \) on \( L_2(0, \infty) \) such that for all \( f \in L_2(0, \infty) \)
\[
f = \sum_{n \in \mathbb{Z}} \langle L f, g_n \rangle_a f_n.
\]

The function-valued factorable operator \( L \) in \((2.1)\) is unique. Indeed, if \( L_1 \) and \( L_2 \) are two bounded invertible function-valued factorable operators which satisfy in \((2.1)\), then for all \( f \in L_2(0, \infty) \)
\[
L_1^{-1} f = \sum_{n \in \mathbb{Z}} \langle f, g_n \rangle_a f_n = L_2^{-1} f.
\]

Also, we say the function-valued frame \( \{ g_n \}_{n \in \mathbb{Z}} \) is a g-dual function-valued frame of \( \{ f_n \}_{n \in \mathbb{Z}} \) with the corresponding bounded invertible function-valued factorable operator \( L \) (or with bounded invertible function-valued factorable operator \( L \)).

**Proposition 2.2.** Let \( \{ f_n \}_{n \in \mathbb{Z}} \) and \( \{ g_n \}_{n \in \mathbb{Z}} \) be function-valued frames for \( L_2(0, \infty) \). Then \( \{ g_n \}_{n \in \mathbb{Z}} \) is a g-dual function-valued frame of \( \{ f_n \}_{n \in \mathbb{Z}} \) for \( L_2(0, \infty) \) with bounded invertible function-valued factorable operator \( L \) if and only if \( \{ f_n \}_{n \in \mathbb{Z}} \) is a g-dual function-valued frame of \( \{ g_n \}_{n \in \mathbb{Z}} \) for \( L_2(0, \infty) \) with bounded invertible function-valued factorable operator \( L' \).
Example 2.3. Assume that \( \{g_n\}_{n \in \mathbb{Z}} \) is a dual function-valued frame of \( \{f_n\}_{n \in \mathbb{Z}} \) for \( L_2(0, \infty) \) and assume that \( \phi \) is a non zero constant function on \( (0, \infty) \). Then \( \{\phi g_n\}_{n \in \mathbb{Z}} \) is a g-dual function-valued frame of \( \{f_n\}_{n \in \mathbb{Z}} \) for \( L_2(0, \infty) \) with bounded invertible function-valued factorable operator \( L \) defined on \( L_2(0, \infty) \) by \( L f = \frac{1}{\phi} f \), for all \( f \in L_2(0, \infty) \).

Every function-valued frame is a g-dual function-valued frame of itself.

Theorem 2.4. Let \( \{f_n\}_{n \in \mathbb{Z}} \) be a function-valued frame for \( L_2(0, \infty) \) and

\[
S f = \sum_{n \in \mathbb{Z}} \langle \widetilde{f}, f_n \rangle_a f_n, \quad \forall f \in L_2(0, \infty). \tag{2.2}
\]

1) \( S \) is a well define bounded invertible function-valued factorable operator on \( L_2(0, \infty) \).
2) \( \{f_n\}_{n \in \mathbb{Z}} \) is a g-dual function-valued frame of itself with bounded invertible function-valued factorable operator \( S^{-1} \).
Proof. Let \( \{f_n\}_{n \in \mathbb{Z}} \) be a function-valued frame for \( L_2(0, \infty) \) with bounds \( A \) and \( B \). Then for all \( f \in L_2(0, \infty) \),

\[
A|f|_a^2(x) \leq \sum_{n \in \mathbb{Z}} \langle f, f_n \rangle_a(x)^2 \leq B|f|_a^2(x), \quad \text{for a.e. } x \in [1, a].
\]

(2.3)

Thus

\[
\left\| \sum_{n \in N} \langle f, f_n \rangle_a f_n \right\|_{L_2(0, \infty)}^2 = \int_1^\infty \left\| \sum_{|n| = N} \langle f, f_n \rangle_a f_n \right\|_{L_a^2}^2 \, dx
\]

\[
= \int_1^\infty \sup_{|n| = 1} \left\| \sum_{|n| = N} \langle f, f_n \rangle_a f_n \right\|_{L_a^2}^2 \, dx
\]

\[
= \int_1^\infty \sup_{|n| = 1} \left\| \sum_{|n| = N} \langle f, f_n \rangle_a \right\|_{L_a^2}^2 \, dx
\]

\[
\leq \int_1^\infty \sup_{|n| = 1} \left\| \sum_{|n| = N} \langle f, f_n \rangle_a \right\|_{L_a^2}^2 \, dx
\]

\[
\leq B \int_1^\infty \sum_{|n| = N} \langle f, f_n \rangle_a \, dx \to 0,
\]

as \( M, N \to \infty \), since, the second inequality in (2.3) and Monotone Convergence Theorem imply that \( \sum_{n \in \mathbb{Z}} |\langle f, f_n \rangle_a|^2 \) converges in \( L_1[1, a] \). Thus \( S \) is well define and \( \|S\| < B \). It is easy to show that \( S \) is linear. If \( \phi \in B_a \), then for all \( f \in L_2(0, \infty) \)

\[
S(\phi f) = \sum_{n \in \mathbb{Z}} \langle \phi f, f_n \rangle_a f_n
\]

\[
= \phi \sum_{n \in \mathbb{Z}} \langle f, f_n \rangle_a f_n
\]

\[
= \phi S(f)
\]

by Proposition 1.2. Also inequality (2.3) shows that

\[
A|f|_a^2(x) \leq \langle S f, f \rangle_a(x) \leq B|f|_a^2(x), \quad \text{for a.e. } x \in [1, a].
\]

By integration of above inequality on \([1, a]\) we have \( A|f|^2 \leq \langle S f, f \rangle \leq B|f|^2 \). Thus \( \|I - B^{-1}S\| < 1 \) and so \( S \) is invertible.

Also by replacing \( f \) with \( S^{-1}f \) in (2.2),

\[
f = \sum_{n \in \mathbb{Z}} \langle S^{-1}f, f_n \rangle_a f_n, \quad \forall f \in L_2(0, \infty).
\]

Thus \( \{f_n\}_{n \in \mathbb{Z}} \) is a g-dual function-valued frame of itself with invertible function-valued factorable operator \( S^{-1} \). \( \square \)
The function-valued factorable operator \( S \) defined by (2.2) is called **function-valued frame operator** of \( \{f_n\}_{n \in \mathbb{Z}} \). Now we are going to give a simple way for construction of infinitely many \( g \)-dual function-valued frames of a given function-valued frame (with common bounded invertible function-valued factorable operator).

**Proposition 2.5.** Assume that \( \{g_n\}_{n \in \mathbb{Z}} \) is a \( g \)-dual function-valued frame of \( \{f_n\}_{n \in \mathbb{Z}} \) for \( L_2(0, \infty) \) with bounded invertible function-valued factorable operator \( L \) and \( \phi \in B_a \). Then the sequence \( \{h_n\}_{n \in \mathbb{Z}} \) defined by \( h_n = \phi g_n + (1 - \phi) (L^{-1})^* S^{-1} f_n \), is a \( g \)-dual function-valued frame of \( \{f_n\}_{n \in \mathbb{Z}} \) for \( L_2(0, \infty) \) with bounded invertible function-valued factorable operator \( L \), where \( S \) is the function-valued frame operator of \( \{f_n\}_{n \in \mathbb{Z}} \).

**Proof.** For all \( f \in L_2(0, \infty) \) we have

\[
\sum_{n \in \mathbb{Z}} \langle Lf, h_n \rangle_a f_n = \phi \sum_{n \in \mathbb{Z}} \langle Lf, g_n \rangle_a f_n + (1 - \phi) \sum_{n \in \mathbb{Z}} \langle f, S^{-1} f_n \rangle_a f_n \\
= \phi f + (1 - \phi) f = f
\]

\( \square \)

**Example 2.6.** If \( \{g_n\}_{n \in \mathbb{Z}} \) is a dual function-valued frame of \( \{f_n\}_{n \in \mathbb{Z}} \), then \( \frac{1}{2} g_n + \frac{1}{2} S^{-1} f_n \), is a \( g \)-dual function-valued frame of \( \{f_n\}_{n \in \mathbb{Z}} \) for \( L_2(0, \infty) \) with bounded invertible function-valued factorable operator \( I \), where \( S \) is the function-valued frame operator of \( \{f_n\}_{n \in \mathbb{Z}} \) and \( I \) is the identity operator on \( L_2(0, \infty) \).

**Definition 2.7.** A sequence \( \{g_n\}_{n \in \mathbb{Z}} \) is called a function-valued Riesz basis with respect to \( a \), or simply function-valued Riesz basis for \( L_2(0, \infty) \) if there exist function-valued orthonormal basis \( \{e_n\}_{n \in \mathbb{Z}} \) and bounded invertible function-valued factorable operator \( L \) on \( L_2(0, \infty) \) such that \( g_n = L e_n \), for all \( n \in \mathbb{Z} \).

Not only function-valued orthonormal bases, but also function-valued Riesz bases are \( g \)-dual function-valued frames.

**Proposition 2.8.** Every two function-valued Riesz bases are \( g \)-dual function-valued frames.

**Proof.** Let \( \{g_n\}_{n \in \mathbb{Z}} \) and \( \{h_n\}_{n \in \mathbb{Z}} \) be two function-valued Riesz bases for \( L_2(0, \infty) \). There exist function-valued orthonormal basis \( \{e_n\}_{n \in \mathbb{Z}} \) and bounded invertible function-valued factorable operators \( L_q \) and \( L_b \) on \( L_2(0, \infty) \) such that \( g_n = L_q e_k \) and \( h_n = L_b e_k \). Since \( L_q \) and \( L_b \) are invertible, there exists a bounded invertible function-valued factorable operator \( L \) on \( L_2(0, \infty) \) such that \( L_b L^*_q L = I \) and hence for all \( f \in L_2(0, \infty) \) we have

\[
f = L_b L^*_q L f = L_b \left( \sum_{n \in \mathbb{Z}} \langle L^*_q Lf, e_n \rangle_a e_n \right) \\
= \sum_{n \in \mathbb{Z}} \langle Lf, L^*_q e_n \rangle_a L_b e_n \\
= \sum_{n \in \mathbb{Z}} \langle Lf, g_n \rangle_a h_n.
\]

\( \square \)
The relation between g-dual frames and g-dual function-valued frames for \( L_2(0, \infty) \) is given in the next theorem.

**Theorem 2.9.** Let \( \{ f_n \}_{n \in \mathbb{Z}} \) and \( \{ g_n \}_{n \in \mathbb{Z}} \) be function-valued frames in \( L_2(0, \infty) \). The following statements are equivalent:

1) \( \{ g_n \}_{n \in \mathbb{Z}} \) is a g-dual function-valued frame of \( \{ f_n \}_{n \in \mathbb{Z}} \) with bounded invertible function-valued factorable operator \( L \).

2) \( \{ \widetilde{\psi}_m f_n \}_{m,n \in \mathbb{Z}} \) is a g-dual frame of \( \{ f_n \}_{n \in \mathbb{Z}} \) with bounded invertible operator \( L \).

**Proof.** The sequences \( \{ \widetilde{\psi}_m g_n \}_{m,n \in \mathbb{Z}} \) and \( \{ \widetilde{\psi}_m f_n \}_{m,n \in \mathbb{Z}} \) are frames in \( L_2(0, \infty) \) by Theorem 1.5. Let \( \{ e_n \}_{n \in \mathbb{Z}} \) be a function-valued orthonormal basis for \( L_2(0, \infty) \). A similar argument as the proof of Theorem 2.4 shows that the operators \( T_1 : L_2(0, \infty) \to L_2(0, \infty) \) defined by \( T_1 f = \sum_{n \in \mathbb{Z}} \langle f, e_n \rangle g_n \) and \( T_2 : L_2(0, \infty) \to L_2(0, \infty) \) defined by \( T_2 f = \sum_{n \in \mathbb{Z}} \langle f, e_n \rangle a_f n \) are well defined bounded function-valued factorable operators. Also \( T_1 e_n = g_n, T_2 e_n = f_n, T_1(\widetilde{\psi}_m e_n) = \widetilde{\psi}_m g_n \) and \( T_2(\widetilde{\psi}_m e_n) = \widetilde{\psi}_m f_n \).

Now let \( \{ g_n \}_{n \in \mathbb{Z}} \) is a g-dual function-valued frame of \( \{ f_n \}_{n \in \mathbb{Z}} \) with bounded invertible function-valued factorable operator \( L \). Then for all \( f \in L_2(0, \infty) \)

\[
f = \sum_{n \in \mathbb{Z}} \langle Lf, g_n \rangle a_f n = \sum_{n \in \mathbb{Z}} \langle Lf, T_1 e_n \rangle T_2 e_n
\]

\[
= T_2 \left( \sum_{n \in \mathbb{Z}} \langle T_1^* Lf, e_n \rangle a_f n \right)
\]

\[
= T_2 T_1^* Lf,
\]

by Proposition 1.3. Now \( \{ \widetilde{\psi}_m e_n \}_{m,n \in \mathbb{Z}} \) is an orthonormal basis in \( L_2(0, \infty) \) by Proposition 1.3 and hence for all \( f \in L_2(0, \infty) \)

\[
\sum_{n \in \mathbb{Z}} \langle Lf, \widetilde{\psi}_m g_n \rangle \widetilde{\psi}_m f_n = \sum_{n \in \mathbb{Z}} \langle Lf, T_1(\widetilde{\psi}_m e_n) \rangle T_2(\widetilde{\psi}_m e_n)
\]

\[
= T_2 \left( \sum_{n \in \mathbb{Z}} \langle T_1^* Lf, \widetilde{\psi}_m e_n \rangle \widetilde{\psi}_m e_n \right)
\]

\[
= T_2 T_1^* Lf = f.
\]

Therefore \( \{ \widetilde{\psi}_m g_n \}_{m,n \in \mathbb{Z}} \) is a g-dual frame of \( \{ \widetilde{\psi}_m f_n \}_{m,n \in \mathbb{Z}} \) with bounded invertible operator \( L \).

Conversely let \( \{ \widetilde{\psi}_m g_n \}_{m,n \in \mathbb{Z}} \) be a g-dual frame of \( \{ \widetilde{\psi}_m f_n \}_{m,n \in \mathbb{Z}} \) with bounded invertible operator \( L \). Then \( L = (T_2 T_1^*)^{-1} \) is bounded invertible function-valued factorable operator, since \( T_1 \) and \( T_2 \) are function-valued factorable operator. Also for all \( f \in L_2(0, \infty) \)

\[
\sum_{n \in \mathbb{Z}} \langle Lf, g_n \rangle a_f n = \sum_{n \in \mathbb{Z}} \langle Lf, T_1 e_n \rangle a_f n T_2 e_n
\]

\[
= T_2 \left( \sum_{n \in \mathbb{Z}} \langle T_1^* Lf, e_n \rangle a_f n \right)
\]

\[
= T_2 T_1^* Lf = f,
\]

by Proposition 1.3. \( \square \)
Example 2.10. Let $\{e_n\}_{n \in \mathbb{Z}}$ be an orthonormal basis for $L_2(0, \infty)$ and $\lambda \neq 0$. Then $\{\lambda e_n\}_{n \in \mathbb{Z}}$ is a g-dual frame of $\{e_n\}_{n \in \mathbb{Z}}$ with bounded invertible operator $L$ defined on $L_2(0, \infty)$ by $Lf = \frac{1}{\lambda}f$, for all $f \in L_2(0, \infty)$. Therefore $\{\lambda \bar{\psi}_m e_n\}_{m,n \in \mathbb{Z}}$ is a g-dual function-valued frame of $\{\bar{\psi}_m e_n\}_{m,n \in \mathbb{Z}}$ with bounded invertible function-valued factorable operator $L$.

Let $\phi \in L_2(0, \infty)$. Then for all $x \in (0, \infty)$ we have

\[
\bar{\psi}_k D_{a^j} \phi(x) = \bar{\psi}_k(x) D_{a^j} \phi(x) \\
= \frac{1}{\sqrt{a^j}} \bar{\psi}_k(x) \phi(a^{-j} x) \\
= \frac{1}{\sqrt{a^j}} \bar{\psi}_k(a^{-j} x) \phi(a^{-j} x) \\
= D_{a^j} \bar{\psi}_k \phi(x)
\]

and hence $\bar{\psi}_k$ commute with $D_{a^j}$. Thus the following corollary is immediate from Theorem 2.9.

Corollary 2.11. Let $\{D_{a^j} \phi_1\}_{j \in \mathbb{Z}}$ and $\{D_{a^j} \phi_2\}_{j \in \mathbb{Z}}$ be function-valued frames for $L_2(0, \infty)$, where $\phi_1, \phi_2 \in L_2(0, \infty)$. The following are equivalent.

1) $\{D_{a^j} \phi_1\}_{j \in \mathbb{Z}}$ is a g-dual function-valued frame of $\{D_{a^j} \phi_2\}_{j \in \mathbb{Z}}$ with bounded invertible function-valued factorable operator $L$.

2) The dilation invariant system generated by $\{\bar{\psi}_k \phi_1\}_{k \in \mathbb{Z}}$ and $\alpha$ is a g-dual frame of the dilation invariant system generated by $\{\bar{\psi}_k \phi_2\}_{k \in \mathbb{Z}}$ and $\alpha$ with bounded invertible operator $L$.

References

[8] M. A. Hasankhani Fard and M. A. Dehghan, A new function-valued inner product and corresponding function-valued frames in $L_2(0, \infty)$, Linear Multilinear Algebra, (Published online: 01 Jul 2013).