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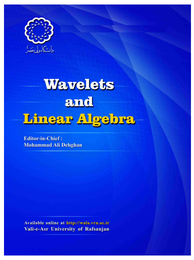

Hermite-Hadamard Type Inequalities for Sub-Topical Functions

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Abstract

In this paper, we study Hermite-Hadamard type inequalities for sub-topical (increasing and plus sub-homogeneous) functions in the framework of abstract convexity. Some examples of such inequalities for functions defined on special domains are given.

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1. Introduction

Let f be a convex function defined on the segment $[a, b]$ of the real line. Then the following inequality holds:

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leq \frac{1}{2}(f(a)+f(b)). \tag{1.1}$$

These inequalities are well known as the Hermite-Hadamard inequalities (see [4]). There are many generalizations of these inequalities for classes of non-convex functions such as quasiconvex functions [8, 9], p-functions [8], ICAR (increasing and convex-along-rays) functions [3], IPH (increasing and positively homogeneous) functions [1] and $\mathbb{B}$-convex and $\mathbb{B}^{-1}$-convex functions [11].
For instance [9], if $f : [0, 1] \longrightarrow \mathbb{R}$ is an arbitrary nonnegative quasiconvex function, then for any $u \in (0, 1)$ one has

$$f(u) \leq \frac{1}{\min(u, 1-u)}\int_0^1 f(x)\,\mathrm{d}x. \tag{1.2}$$

If

$$D = \left\{(x, y) \in \mathbb{R}^2_+ \mid 0 \leq x \leq a,\ 0 \leq \frac{y}{x} \leq v\right\}$$

that $a > 0$ and $v > 0$, then for each ICAR function f we have:

$$f(\frac{a}{3}, \frac{va}{3}) \leq \frac{1}{A(D)}\int_D f(x, y)\,\mathrm{d}x\mathrm{d}y,$$

where $A(D)$ is the area of D.
The class of topical functions is another class of abstract convex functions that some Hermite-Hadamard inequalities for these functions were presented in [2]. For example, if $f : D \longrightarrow \mathbb{R}$ is a topical function that

$$D = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq a+\delta,\ 0 \leq y \leq x-a\},$$

where $a, \delta \in \mathbb{R}$ and $\delta \geq 3$, then

$$f(\frac{1}{3}\delta + a, \frac{1}{3}\delta) \leq \frac{2}{\delta^2}\int_D f(x, y)\,\mathrm{d}x\mathrm{d}y.$$

The class of sub-topical functions is a natural extension of topical functions. These functions were introduced and examined in [5, 6, 7, 10]. In the present paper some Hermite-Hadamard type inequalities for sub-topical functions are given. Examples for particular domains are also presented.
This article has the following structure: In Section 2, we provide some preliminaries, definitions and results relative to sub-topical functions. In Section 3, we consider Hermite-Hadamard type inequalities for the class of sub-topical functions. Finally, some examples of such inequalities for functions defined on $\mathbb{R}^2$ are given in Section 4.

2. Preliminaries

We assume that $\mathbb{R}^n$ is equipped with coordinate-wise order relation. A function $f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}} = [-\infty, +\infty]$ is said to be increasing if $f(x) \leq f(y)$ for each $x, y \in \mathbb{R}^n$ such that $x \leq y$. The function f is called plus sub-homogeneous if $f(x + \lambda \mathbf{1}) \leq f(x) + \lambda$ for all $x \in \mathbb{R}^n$ and all $\lambda \geq 0$, where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$. It is easy to see that f is plus sub-homogeneous if and only if $f(x + \lambda \mathbf{1}) \geq f(x) + \lambda$ for all $x \in \mathbb{R}^n$ and all $\lambda \leq 0$. The following definitions and results can be found in [9, 10].

Definition 2.1. A function $f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ is called sub-topical if it is increasing and plus sub-homogeneous.

Remark 2.2. A function $f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ is called topical if it is increasing and $f(x + \lambda \mathbf{1}) = f(x) + \lambda$ for all $x \in \mathbb{R}^n$ and all $\lambda \in \mathbb{R}$. It is clear that any topical function is sub-topical.

Lemma 2.3. *Let $f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ be a sub-topical function.*
(i) If there exists $x \in \mathbb{R}^n$ such that $f(x) = +\infty$, then $f \equiv +\infty$.
(ii) If there exists $x \in \mathbb{R}^n$ such that $f(x) = -\infty$, then $f \equiv -\infty$.

It follows from Lemma 2.3 that a sub-topical function is either finite (i.e., finite-valued at each $x \in \mathbb{R}^n$) or identically $+\infty$ or $-\infty$. Now, we present the following simple examples.

Example 2.4. Let $a \in \mathbb{R}^n$ be such that $a \geq \mathbf{0}$ and $\langle a, \mathbf{1} \rangle \leq 1$. Then the linear function

$$f(x) = \langle a, x \rangle, \quad (x \in \mathbb{R}^n),$$

is sub-topical.

Example 2.5. Functions of the form

$$f(x) = \frac{1}{\theta} \ln \Big(\sum_{i=1}^{n} e^{\langle a_i, x \rangle} \Big), \quad (x \in \mathbb{R}^n),$$

where $a_i \in \mathbb{R}^n$, $a_i \geq \mathbf{0}$, $i = 1, 2, \ldots, n$, and $\theta \geq max_{1 \leq i \leq n} \langle a_i, \mathbf{1} \rangle$, are sub-topical. Indeed, since the functions ln and exp are increasing, it is clear that the function f is increasing. To see that f is plus sub-homogeneous, let $x \in \mathbb{R}^n$ and $\lambda \geq 0$. Then

$$\begin{aligned}
f(x + \lambda \mathbf{1}) &= \frac{1}{\theta} \ln \Big(\sum_{i=1}^{n} e^{\langle a_i, x + \lambda \mathbf{1} \rangle} \Big) \\
&= \frac{1}{\theta} \ln \Big(\sum_{i=1}^{n} e^{\langle a_i, x \rangle} e^{\lambda \langle a_i, \mathbf{1} \rangle} \Big) \\
&\leq \frac{1}{\theta} \ln \Big(e^{\lambda \theta} \sum_{i=1}^{n} e^{\langle a_i, x \rangle} \Big) \\
&= \frac{1}{\theta} \Big(\ln (e^{\lambda \theta}) + \ln \Big(\sum_{i=1}^{n} e^{\langle a_i, x \rangle} \Big) \Big) \\
&= \lambda + \frac{1}{\theta} \ln \Big(\sum_{i=1}^{n} e^{\langle a_i, x \rangle} \Big) \\
&= \lambda + f(x).
\end{aligned}$$

Example 2.6. Let $\{f_i\}_{1\le i\le k}$ be a set of real valued sub-topical functions. Put

$$f(x) = \min\{f_1(x), \dots, f_k(x)\}, \quad F(x) = \max\{f_1(x), \dots, f_k(x)\}, \quad (x \in \mathbb{R}^n).$$

Then the functions f and F are sub-topical.

Let us mention some properties of the set $\mathbf{\Gamma}$ of all sub-topical functions $f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$.
(1) We have $\mathbf{\Gamma} + \mathbb{R} = \mathbf{\Gamma}$, that is, if $f \in \mathbf{\Gamma}$ and $c \in \mathbb{R}$, then $f + c \in \mathbf{\Gamma}$.
(2) $\mathbf{\Gamma}$ is a convex set.
(3) $\mathbf{\Gamma}$ is a complete lattice, that is, if $\{f_\beta\}_{\beta\in B}$ is an arbitrary family of elements of $\mathbf{\Gamma}$ and

$$f(x) = \sup_{\beta\in B} f_\beta(x), \quad (x \in \mathbb{R}^n),$$

then the function f belongs to $\mathbf{\Gamma}$.
(4) $\mathbf{\Gamma}$ is closed under the pointwise convergence of functions.

Remark 2.7. Every finite sub-topical function f is continuous on $\mathbb{R}^n$. Indeed, let $\{x_k\} \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$, $x_k \longrightarrow x$ and $\epsilon > 0$. Then, for sufficiently large k we have $x - \epsilon\mathbf{1} \le x_k \le x + \epsilon\mathbf{1}$, whence, since f is increasing and plus sub-homogeneous, we obtain

$$f(x) - \epsilon \le f(x - \epsilon\mathbf{1}) \le f(x_k) \le f(x + \epsilon\mathbf{1}) \le f(x) + \epsilon.$$

These inequalities imply the continuity of f at x.

Now, we recall some definitions from abstract convexity. Consider a set X and a set H of functions $h : X \longrightarrow \overline{\mathbb{R}}$. The function $f : X \longrightarrow \overline{\mathbb{R}}$ is called abstract convex with respect to H (or H-convex) if there exists a subset U of H such that

$$f(x) = \sup_{h\in U} h(x), \quad (x \in X).$$

The set H is called the set of elementary functions. Consider the function $\varphi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\varphi(x, y, \alpha) = \min_{1\le i\le n}\{\alpha, x_i + y_i\}, \quad (x,\ y \in \mathbb{R}^n,\ \alpha \in \mathbb{R}).$$

Let $(y, \alpha) \in \mathbb{R}^n \times \mathbb{R}$ be arbitrary. Denote by $\varphi_{(y,\alpha)}$ the function defined on $\mathbb{R}^n$ by the formula $\varphi_{(y,\alpha)}(x) = \varphi(x, y, \alpha)$. It is clear that $\varphi_{(y,\alpha)}$ is a sub-topical function on $\mathbb{R}^n$. Let $X_\varphi = \{\varphi_{(y,\alpha)} \mid y \in \mathbb{R}^n,\ \alpha \in \mathbb{R}\}$, then it is known that any function f defined on $\mathbb{R}^n$ is sub-topical if and only if f is X_φ-convex. It is also known that the function $f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ is sub-topical if and only if

$$f(x) \ge \varphi_{(-y,\alpha)}(x) + f(y + \alpha\mathbf{1}) - \alpha, \quad \forall x,\ y \in \mathbb{R}^n,\ \forall\alpha \in \mathbb{R}. \tag{2.1}$$

Formula (2.1) implies the following statement.

Proposition 2.8. *Let f be a sub-topical function defined on $\mathbb{R}^n$, $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}$ such that $0 \in V$, and $\Delta = U \times V$. Then the function*

$$f_\Delta(x) = \sup_{(y,\alpha)\in\Delta} (\varphi_{(-y,\alpha)}(x) + f(y + \alpha\mathbf{1}) - \alpha), \quad (x \in \mathbb{R}^n)$$

is sub-topical and it possesses the properties:
(i) $f_\Delta(x) \le f(x)$ for all $x \in \mathbb{R}^n$.
(ii) $f_\Delta(x) = f(x)$ for all $x \in U$.

Proof. Since the function $\varphi_{(-y,\alpha)}(x) + f(y + \alpha\mathbf{1}) - \alpha$ is sub-topical for any $(y, \alpha) \in \Delta$, and since the pointwise supremum of a family of sub-topical functions is sub-topical, so the function f_Δ is sub-topical.
According to Equation (2.1), It is clear that $f_\Delta(x) \leq f(x)$ for all $x \in \mathbb{R}^n$.
Now, let $x \in U$. Since $0 \in V$, so $(x, 0) \in \Delta$. Then

$$f_\Delta(x) \geq \varphi_{(-x,0)}(x) + f(x + 0 \times \mathbf{1}) - 0 = f(x).$$

This implies that $f_\Delta(x) = f(x)$. □

3. Hermite-Hadamard Type Inequalities

Let $D \subset \mathbb{R}^n$ be a closed domain, that is, D is a bounded set such that $\mathrm{cl}(\mathrm{int}D) = D$. Let $Q(D)$ be the set of all points $(y, \alpha) \in D \times \mathbb{R}$ such that

$$\frac{1}{A(D)} \int_D \varphi_{(-y,\alpha)}(x)\, \mathrm{d}x = 1,$$

where $A(D) = \int_D \mathrm{d}x$.

Proposition 3.1. *Assume that the set $Q(D)$ is nonempty and let f be a sub-topical function. Then the following inequality holds:*

$$\sup_{(y,\alpha)\in D} (f(y + \alpha\mathbf{1}) - \alpha) \leq \frac{1}{A(D)} \int_D f(x)\, \mathrm{d}x - 1. \tag{3.1}$$

Proof. Since f is sub-topical, it follows from (2.1) that

$$\varphi_{(-y,\alpha)}(x) + f(y + \alpha\mathbf{1}) - \alpha \leq f(x), \quad \forall x,\ y \in D,\ \forall \alpha \in \mathbb{R}.$$

Let $(y, \alpha) \in Q(D)$. It follows from the definition of $Q(D)$ that

$$A(D)(1 + f(y + \alpha\mathbf{1}) - \alpha) = \int_D (\varphi_{(-y,\alpha)}(x) + f(y + \alpha\mathbf{1}) - \alpha)\, \mathrm{d}x \leq \int_D f(x)\, \mathrm{d}x.$$

Therefore

$$f(y + \alpha\mathbf{1}) + 1 - \alpha \leq \frac{1}{A(D)} \int_D f(x)\, \mathrm{d}x.$$

This completes the proof. □

Remark 3.2. For each $(y, \alpha) \in Q(D)$ we have also the following inequality, which is weaker than (3.1)

$$f(y + \alpha\mathbf{1}) + 1 - \alpha \leq \frac{1}{A(D)} \int_D f(x)\, \mathrm{d}x. \tag{3.2}$$

Note that if $f(x) = \varphi_{(-y,\alpha)}(x)$, then in (3.2) the equality holds. Indeed,

$$\varphi_{(-y,\alpha)}(y + \alpha\mathbf{1}) + 1 - \alpha = \alpha + 1 - \alpha = 1 = \frac{1}{A(D)} \int_D \varphi_{(-y,\alpha)}(x)\, \mathrm{d}x.$$

Remark 3.3. We can generalize the inequality from the right-hand side of (1.1). Indeed, let f be a sub-topical function and $D \subset \mathbb{R}^n$ be a convex closed domain. By setting $\alpha = 0$ in (2.1), we have $\varphi_{(-x,0)}(y) + f(x) \leq f(y)$ for all $x,\ y \in D$. Now, let $y \in D$ be a minimal element of the set D (note that the point $y \in D$ is called a minimal point of the set D, if $x \in D$ and $x \leq y$ implies that $x = y$). So we get the following inequality:

$$\int_D f(x)\,\mathrm{d}x \leq f(y)A(D) + \int_D \max_{1\leq i\leq n}\{0, x_i - y_i\}\,\mathrm{d}x. \tag{3.3}$$

In the following, we characterize the set $Q(D)$, that D is a bounded closed interval of $\mathbb{R}$. Let $D = [a, b]$, that $a < b$, $l = b - a$ and $y \in D$. Let $(y, \alpha) \in Q(D)$, then $a < y + \alpha$. Indeed, if $y + \alpha \leq a$, then $\varphi_{(-y,\alpha)}(x) = \alpha$ for all $x \in D$. So,

$$1 = \frac{1}{A(D)}\int_D \varphi_{(-y,\alpha)}(x)\,\mathrm{d}x = \frac{1}{b-a}\int_a^b \alpha\,\mathrm{d}x = \frac{1}{b-a}(b-a)\alpha = \alpha.$$

We conclude that $y + 1 \leq a$. But $a \leq y$, which yields $y + 1 \leq y$, that is a contradiction. Hence $a < y + \alpha$.
Now, based on whether the point $y + \alpha$ belongs to the interval D or not, we consider two case:
case (i): $y + \alpha \geq b$.
In this case,

$$\varphi_{(-y,\alpha)}(x) = x - y, \quad \forall x \in D.$$

Then

$$1 = \frac{1}{b-a}\int_a^b \varphi_{(-y,\alpha)}(x)\,\mathrm{d}x = \frac{1}{b-a}\int_a^b (x-y)\,\mathrm{d}x = \frac{1}{2}(b+a) - y.$$

So, $y = \frac{1}{2}(b + a) - 1$. Since $a \leq y \leq b$, we conclude that $l \geq 2$. On the other hand, $y + \alpha \geq b$. This implies that $\alpha \geq \frac{1}{2}l + 1$. It is easy to see that if $y = \frac{1}{2}(a + b) - 1$, $l \geq 2$ and $\alpha \geq \frac{1}{2}l + 1$, then $y \in D$, $y + \alpha \geq b$ and $\frac{1}{b-a}\int_a^b \varphi_{(-y,\alpha)}(x)\,\mathrm{d}x = 1$, so $(y, \alpha) \in Q(D)$.
case (ii): $a < y + \alpha < b$.
We get

$$\varphi_{(-y,\alpha)}(x) = \begin{cases} x - y, & a \leq x \leq y + \alpha, \\ \alpha, & y + \alpha \leq x \leq b. \end{cases}$$

Then

$$\begin{aligned} 1 &= \frac{1}{b-a}\int_a^b \varphi_{(-y,\alpha)}(x)\,\mathrm{d}x \\ &= \frac{1}{b-a}\Big(\int_a^{y+\alpha} (x-y)\,\mathrm{d}x + \int_{y+\alpha}^b \alpha\,\mathrm{d}x\Big) \\ &= \frac{-1}{2(b-a)}(y^2 + 2(\alpha - a)y + (\alpha^2 + a^2 - 2\alpha b)). \end{aligned}$$

This implies that $\alpha > 1$ and $y = a - \alpha + \sqrt{2l(\alpha - 1)}$. But $a \leq y \leq b$, so we must have $l > 2$ and $l - \sqrt{l(l-2)} \leq \alpha \leq l + \sqrt{l(l-2)}$. Also, in this case $a < y + \alpha < b$. Therefore we get $\alpha < \frac{1}{2}l + 1$.

Since $l > 2$, it is easy to check that $\frac{1}{2}l + 1 < l + \sqrt{l(l-2)}$. Also $l > 2$ implies that $\alpha > 1$. Hence, in this case, $(y, \alpha) \in Q(D)$ if and only if $l > 2$, $y = a - \alpha + \sqrt{2l(\alpha - 1)}$ and $l - \sqrt{l(l-2)} \leq \alpha < \frac{1}{2}l + 1$. We have proved the following proposition.

Proposition 3.4. *Let $D = [a, b]$ that $-\infty < a < b < \infty$, and $l = b - a$. We have the following assertions:*
(i) $Q(D) = \emptyset$ if and only if $l < 2$.
(ii) If $l = 2$, then $Q(D) = \{(y, \alpha) \mid y = a,\ \alpha \geq 2\}$.
(iii) If $l > 2$, then $Q(D) = \{(y, \alpha) \mid y = \frac{1}{2}(a+b)-1,\ \alpha \geq \frac{1}{2}l+1\} \cup \{(y, \alpha) \mid y = a-\alpha+\sqrt{2l(\alpha-1)},\ l - \sqrt{l(l-2)} \leq \alpha < \frac{1}{2}l + 1\}$.

Remark 3.5. Let f be a sub-topical function and consider the bounded closed interval $D = [a, b]$. If we set $y = a$, then by (3.3) we have

$$\int_a^b f(x)\,\mathrm{d}x \leq f(a)(b-a) + \frac{1}{2}(b-a)^2.$$

Now, we describe the set $Q(D)$, that D is a convex closed domain in $\mathbb{R}^2$. Let $(\bar{x}, \bar{y}) \in D$ and $\bar{\alpha} \in \mathbb{R}$. Consider the line $R = \{(x, y) \in \mathbb{R}^2 \mid y = x + \gamma\}$, that $\gamma = \bar{y} - \bar{x}$. Set $S = \{(x, y) \in \mathbb{R}^2 \mid x \geq \bar{x} + \bar{\alpha},\ y \geq \bar{y} + \bar{\alpha}\}$, $S' = \mathbb{R}^2 \setminus int(S)$, $D_1 = D \cap S$, $D_2 = D \cap S' \cap R^+$ and $D_3 = D \cap S' \cap R^-$, that R^+ and R^- are upper half-plane and lower half-plane defined by the line R, respectively; i.e., $R^+ = \{(x, y) \in \mathbb{R}^2 \mid y \geq x + \gamma\}$ and $R^- = \{(x, y) \in \mathbb{R}^2 \mid y \leq x + \gamma\}$. Then we conclude that $D = D_1 \cup D_2 \cup D_3$ and $int(D_i) \cap int(D_j) = \emptyset$, for $i \neq j$. See Figure 1.
Now, we define the function $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$g(x, y) = \min\{\bar{\alpha}, x - \bar{x}, y - \bar{y}\}, \quad ((x, y) \in \mathbb{R}^2).$$

Then we conclude that

$$g(x, y) = \begin{cases} \bar{\alpha}, & (x, y) \in D_1, \\ x - \bar{x}, & (x, y) \in D_2, \\ y - \bar{y}, & (x, y) \in D_3. \end{cases}$$

Indeed, if $(x, y) \in D_1$, then $(x, y) \in D$, $x \geq \bar{x} + \bar{\alpha}$ and $y \geq \bar{y} + \bar{\alpha}$. So, $\bar{\alpha} \leq x - \bar{x}$ and $\bar{\alpha} \leq y - \bar{y}$. Therefore, we obtain that $g(x, y) = \bar{\alpha}$. If $(x, y) \in D_2$, then $(x, y) \in D$, $y \geq x + \gamma$ and $(x, y) \notin int(S)$. Since $\gamma = \bar{y} - \bar{x}$, so

$$y - \bar{y} \geq x - \bar{x}. \tag{3.4}$$

On the other hand, $(x, y) \notin int(S)$, so we obtain that either $x \leq \bar{x} - \bar{\alpha}$ or $y \leq \bar{y} - \bar{\alpha}$. By using (3.4), in both cases we get $g(x, y) = x - \bar{x}$. If $(x, y) \in D_3$, by an argument similar to the previous case, we obtain that $g(x, y) = y - \bar{y}$.
In the sequel, for the convex closed domain $D \subset \mathbb{R}^2$, we need to define the following notations:

$$x_m^D = \min_{(x,y)\in D} x, \quad x_M^D = \max_{(x,y)\in D} x,$$

and

$$y_m^D = \min_{(x,y)\in D} y, \quad y_M^D = \max_{(x,y)\in D} y.$$

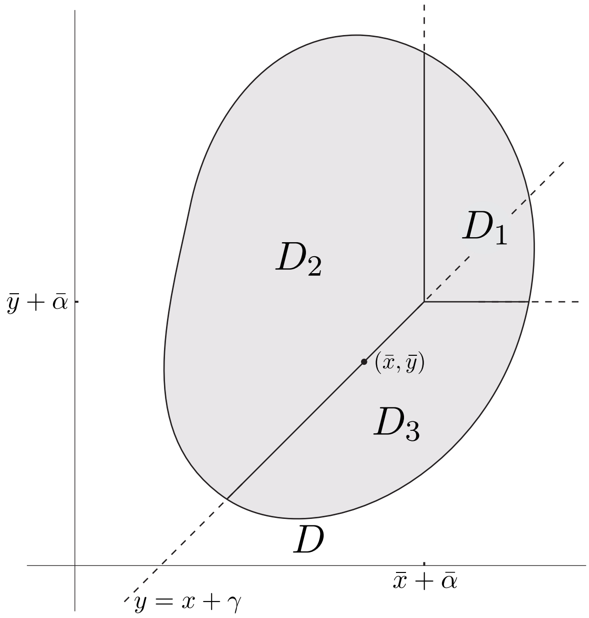

Figure 1: D is a convex closed domain in $\mathbb{R}^2$

Note that since D is a compact set and the functions x and y are continuous on D, these functions attain their minimum and maximum values on D. For example, $(x_m^D, y_0) \in D$, for some $y_0 \in \mathbb{R}$.

Lemma 3.6. *Let* $((\bar{x}, \bar{y}), \bar{\alpha}) \in Q(D)$. *Then* $\bar{\alpha} > \min\{x_m^D - \bar{x}, y_m^D - \bar{y}\}$.

Proof. By contradiction, suppose that $\bar{\alpha} \leq \min\{x_m^D - \bar{x}, y_m^D - \bar{y}\}$. So, $\bar{\alpha} \leq x_m^D - \bar{x}$ and $\bar{\alpha} \leq y_m^D - \bar{y}$. By definition of x_m^D and y_m^D, we get

$$\bar{\alpha} \leq x - \bar{x} \quad \text{and} \quad \bar{\alpha} \leq y - \bar{y}, \quad \forall (x, y) \in D. \tag{3.5}$$

This implies that $g(x, y) = \bar{\alpha}$ for all $(x, y) \in D$. Since $((\bar{x}, \bar{y}), \bar{\alpha}) \in Q(D)$, we have

$$1 = \frac{1}{A(D)} \int_D g(x, y)\, \mathrm{d}x\mathrm{d}y = \frac{1}{A(D)} \int_D \bar{\alpha}\, \mathrm{d}x\mathrm{d}y = \frac{A(D)}{A(D)}\bar{\alpha} = \bar{\alpha}.$$

On the other hand, by setting $(x, y) = (\bar{x}, \bar{y})$ in (3.5), we obtain that $\bar{\alpha} \leq 0$, which is a contradiction. This completes the proof. □

Remark 3.7. If we set

$$\beta_i = \frac{A(D_i)}{A(D)}, \quad i = 1, 2, 3,$$

then we have $0 \leq \beta_i \leq 1$ $(i = 1, 2, 3)$ and $\beta_1 + \beta_2 + \beta_3 = 1$. From Lemma 3.6, it is clear that if $((\bar{x}, \bar{y}), \bar{\alpha}) \in Q(D)$, then $\beta_2 + \beta_3 > 0$.

In the following theorems, we characterize the set $Q(D)$, that $D \subset \mathbb{R}^2$ is a convex closed domain.
Note that for the set D, we assume that

$$X_D = \frac{1}{A(D)} \int_D x \,\mathrm{d}x\mathrm{d}y \quad \text{and} \quad Y_D = \frac{1}{A(D)} \int_D y \,\mathrm{d}x\mathrm{d}y.$$

Theorem 3.8. *Let $((\bar{x}, \bar{y}), \bar{\alpha}) \in Q(D)$ and set $\gamma = \bar{y} - \bar{x}$. Then we have*

$$(\beta_2 + \beta_3)\bar{x} = \beta_1\bar{\alpha} + \beta_2 X_{D_2} + \beta_3(Y_{D_3} - \gamma) - 1. \tag{3.6}$$

Proof. We have

$$\begin{aligned}
\frac{1}{A(D)} \int_D g(x,y) \,\mathrm{d}x\mathrm{d}y &= \frac{1}{A(D)}\Big(\int_{D_1} \bar{\alpha} \,\mathrm{d}x\mathrm{d}y + \int_{D_2} (x - \bar{x}) \,\mathrm{d}x\mathrm{d}y + \int_{D_3} (y - \bar{y}) \,\mathrm{d}x\mathrm{d}y\Big) \\
&= \frac{1}{A(D)}\Big(A(D_1)\bar{\alpha} + \int_{D_2} x \,\mathrm{d}x\mathrm{d}y - A(D_2)\bar{x} + \int_{D_3} y \,\mathrm{d}x\mathrm{d}y - A(D_3)\bar{y}\Big) \\
&= \frac{1}{A(D)}\Big(A(D_1)\bar{\alpha} + A(D_2)X_{D_2} - A(D_2)\bar{x} + A(D_3)Y_{D_3} - A(D_3)\bar{y}\Big) \\
&= \beta_1\bar{\alpha} + \beta_2 X_{D_2} - (\beta_2 + \beta_3)\bar{x} + \beta_3(Y_{D_3} - \gamma).
\end{aligned}$$

But, $\frac{1}{A(D)} \int_D g(x,y) \,\mathrm{d}x\mathrm{d}y = 1$, so $(\beta_2 + \beta_3)\bar{x} = \beta_1\bar{\alpha} + \beta_2 X_{D_2} + \beta_3(Y_{D_3} - \gamma) - 1$. □

In the following, we present the converse of Theorem 3.8.

Theorem 3.9. *Let $(\bar{x}, \bar{y}) \in D$ and set $\gamma = \bar{y} - \bar{x}$. Consider the line $R = \{(x,y) \in \mathbb{R}^2 \mid y = x + \gamma\}$. Assume that $\bar{\alpha} > \min\{x_m^D - \bar{x}, y_m^D - \bar{y}\}$ is such that $(\beta_2 + \beta_3)\bar{x} = \beta_1\bar{\alpha} + \beta_2 X_{D_2} + \beta_3(Y_{D_3} - \gamma) - 1$. Then $((\bar{x}, \bar{y}), \bar{\alpha}) \in Q(D)$.*

Proof. By an argument similar to the proof of Theorem 3.8, we have

$$\frac{1}{A(D)} \int_D g(x,y) \,\mathrm{d}x\mathrm{d}y = \beta_1\bar{\alpha} + \beta_2 X_{D_2} - (\beta_2 + \beta_3)\bar{x} + \beta_3(Y_{D_3} - \gamma).$$

So by hypothesis, we get $\frac{1}{A(D)} \int_D g(x,y) \,\mathrm{d}x\mathrm{d}y = 1$, thus $((\bar{x}, \bar{y}), \bar{\alpha}) \in Q(D)$. □

Based on that some of β_i may be zero or not, we can deduce some special cases from Theorem 3.9, that in the following we present these cases.

Corollary 3.10. *Let $\gamma \in \mathbb{R}$ and set $\bar{x} = Y_D - \gamma - 1$ and $\bar{y} = Y_D - 1$. Let $D \subset R^- = \{(x,y) \in \mathbb{R}^2 \mid y \leq x + \gamma\}$. If $(\bar{x}, \bar{y}) \in D$, then $((\bar{x}, \bar{y}), \bar{\alpha}) \in Q(D)$ for all $\bar{\alpha} \geq y_M^D - \bar{y}$.*

Proof. Since $D \subset R^-$, we obtain that $int(D_2) = \emptyset$. This implies that $\beta_2 = 0$. Let $\bar{\alpha} \geq y_M^D - \bar{y}$. So we get $int(D_1) = \emptyset$ and therefore $\beta_1 = 0$. On the other hand, we have $\beta_1 + \beta_2 + \beta_3 = 1$. So $\beta_3 = 1$. Hence we obtain that $((\bar{x}, \bar{y}), \bar{\alpha})$ satisfies in Equation (3.6). Note that since $int(D) \neq \emptyset$, we get $y_M^D > y_m^D$. This implies that $y_M^D - \bar{y} > \min\{x_m^D - \bar{x}, y_m^D - \bar{y}\}$. By Theorem 3.9, we deduce that $((\bar{x}, \bar{y}), \bar{\alpha}) \in Q(D)$. □

The proof of the following corollary is similar to the Corollary 3.10, so we omit it.

Corollary 3.11. *Let $\gamma \in \mathbb{R}$ and set $\bar{x} = X_D - 1$ and $\bar{y} = X_D + \gamma - 1$. Let $D \subset R^+ = \{(x, y) \in \mathbb{R}^2 \mid y \geq x + \gamma\}$. If $(\bar{x}, \bar{y}) \in D$, then $((\bar{x}, \bar{y}), \bar{\alpha}) \in Q(D)$ for all $\bar{\alpha} \geq x_M^D - \bar{x}$.*

Corollary 3.12. *Let $\gamma \in \mathbb{R}$ and assume the line $R = \{(x, y) \in \mathbb{R}^2 \mid y = x + \gamma\}$ is such that $int(D_i) \neq \emptyset$ for $i = 2, 3$. Set $\bar{x} = \beta_2 X_{D_2} + \beta_3(Y_{D_3} - \gamma) - 1$ and $\bar{y} = \bar{x} + \gamma$. If $(\bar{x}, \bar{y}) \in D$, then we have $((\bar{x}, \bar{y}), \bar{\alpha}) \in Q(D)$ for all $\bar{\alpha} \geq \min\{x_M^D - \bar{x}, y_M^D - \bar{y}\}$.*

Proof. Since $int(D_i) \neq \emptyset$ for $i = 2, 3$, we get $\beta_i \neq 0$ for $i = 2, 3$. Now, let $\bar{\alpha} \geq \min\{x_M^D - \bar{x}, y_M^D - \bar{y}\}$. This implies that $\beta_1 = 0$. So $\beta_2 + \beta_3 = 1$. Therefore by hypothesis, $((\bar{x}, \bar{y}), \bar{\alpha})$ satisfies in Equation (3.6). Hence Theorem 3.9 implies that $((\bar{x}, \bar{y}), \bar{\alpha}) \in Q(D)$. □

Corollary 3.13. *Let $(\bar{x}, \bar{y}) \in D$ and set $\gamma = \bar{y} - \bar{x}$. Let $D \subset R^- = \{(x, y) \in \mathbb{R}^2 \mid y \leq x + \gamma\}$. Assume $\min\{x_m^D - \bar{x}, y_m^D - \bar{y}\} < \bar{\alpha} < y_M^D - \bar{y}$ is such that $\beta_3\bar{x} = \beta_1\bar{\alpha} + \beta_3(Y_{D_3} - \gamma) - 1$. Then $((\bar{x}, \bar{y}), \bar{\alpha}) \in Q(D)$.*

Proof. Since $D \subset R^-$, we get $int(D_2) = \emptyset$. So $\beta_2 = 0$. Now, let $\min\{x_m^D - \bar{x}, y_m^D - \bar{y}\} < \bar{\alpha} < y_M^D - \bar{y}$. This implies that $int(D_i) \neq \emptyset$ for $i = 1, 3$. So $\beta_i > 0$ for $i = 1, 3$. On the other hand $\beta_3\bar{x} = \beta_1\bar{\alpha} + \beta_3(Y_{D_3} - \gamma) - 1$, therefore we conclude that $((\bar{x}, \bar{y}), \bar{\alpha})$ satisfies in Equation (3.6). Hence Theorem 3.9 implies that $((\bar{x}, \bar{y}), \bar{\alpha}) \in Q(D)$. □

The proof of the following corollary is similar to the Corollary 3.13, so we omit it.

Corollary 3.14. *Let $(\bar{x}, \bar{y}) \in D$ and set $\gamma = \bar{y} - \bar{x}$. Let $D \subset R^+ = \{(x, y) \in \mathbb{R}^2 \mid y \geq x + \gamma\}$. Assume $\min\{x_m^D - \bar{x}, y_m^D - \bar{y}\} < \bar{\alpha} < x_M^D - \bar{x}$ be such that $\beta_2\bar{x} = \beta_1\bar{\alpha} + \beta_2 X_{D_2} - 1$. Then $((\bar{x}, \bar{y}), \bar{\alpha}) \in Q(D)$.*

4. Examples

In this section, we present some examples.

Example 4.1. Let $D \subset \mathbb{R}^2$ be the square with vertices $(a, 0)$, $(0, a)$, $(a, 2a)$ and $(2a, a)$, that is

$$D = \{(x, y) \mid 0 \leq x \leq a,\ -x + a \leq y \leq x + a\} \cup \{(x, y) \mid a \leq x \leq 2a,\ x - a \leq y \leq -x + 3a\},$$

where $a \geq 4$. Consider the line $R = \{(x, y) \in \mathbb{R}^2 \mid y = x + \gamma\}$ that $|\gamma| \leq \sqrt{a^2 - 4a}$ This line passes through the interior of the set D and divides D into two parts.

We are looking for a point $((\bar{x}, \bar{y}), \bar{\alpha}) \in Q(D)$ that $(\bar{x}, \bar{y}) \in D \cap R$ and $int(D_1) = \emptyset$. First, we must calculate X_{D_2} and Y_{D_3}. It is clear that $A(D) = 2a^2$, $A(D_2) = a(a - \gamma)$ and $A(D_3) = a(a + \gamma)$. We have

$$X_{D_2} = \frac{1}{A(D_2)} \int_{D_2} x\, \mathrm{d}x\mathrm{d}y = \frac{1}{4}(3a - \gamma)$$

and

$$Y_{D_3} = \frac{1}{A(D_3)} \int_{D_3} y\, \mathrm{d}x\mathrm{d}y = \frac{1}{4}(3a + \gamma).$$

On the other hand, we have $\beta_2 = \frac{a-\gamma}{2a}$ and $\beta_3 = \frac{a+\gamma}{2a}$. Now, according to Corollary 3.12, we put

$$\begin{aligned}\bar{x} &= \beta_2 X_{D_2} + \beta_3 (Y_{D_3} - \gamma) - 1 \\ &= \frac{a-\gamma}{2a}\frac{3a-\gamma}{4} + \frac{a+\gamma}{2a}\Big(\frac{3a+\gamma}{4} - \gamma\Big) - 1 \\ &= -\frac{1}{4a}\gamma^2 - \frac{1}{2}\gamma + \frac{3}{4}a - 1\end{aligned}$$

and $\bar{y} = \bar{x} + \gamma$. By hypothesis, we have $|\gamma| \leq \sqrt{a^2 - 4a}$. By a simple calculation, this implies that $(\bar{x}, \bar{y}) \in D$. But $x_M^D = y_M^D = 2a$, so it follows from Corollary 3.12 that $((\bar{x}, \bar{y}), \bar{\alpha}) \in Q(D)$ for any $\bar{\alpha} \geq \min\{2a - \bar{x}, 2a - \bar{y}\}$ (a simple calculation shows that $((\bar{x}, \bar{y}), \bar{\alpha}) \in Q(D)$ for any $\bar{\alpha} \geq \frac{1}{4a}\gamma^2 + \frac{3}{4}a + 1$). It follows from Remark 3.2 that the following inequality holds for each sub-topical function f:

$$f\Big(-\frac{1}{4a}\gamma^2 - \frac{1}{2}\gamma + \frac{3}{4}a - 1 + \bar{\alpha}, -\frac{1}{4a}\gamma^2 + \frac{1}{2}\gamma + \frac{3}{4}a - 1 + \bar{\alpha}\Big) - \bar{\alpha} + 1 \leq \frac{1}{a^2}\int_D f(x,y)\,\mathrm{d}x\mathrm{d}y$$

for each $\bar{\alpha} \geq \frac{1}{4a}\gamma^2 + \frac{3}{4}a + 1$.

Example 4.2. Now we consider the set $D \subset \mathbb{R}^2$ as a solid half-disk with radius a. In other words, D in polar coordinates has the following form:

$$D = \Big\{(r, \theta) \mid 0 \leq r \leq a, \ -\frac{3\pi}{4} \leq \theta \leq \frac{\pi}{4}\Big\}.$$

We assume that $a \geq \frac{3\sqrt{2}\pi}{3\pi-4}$. Consider the line $R = \{(x,y) \in \mathbb{R}^2 \mid y = x\}$ (so $D \subset R^- = \{(x,y) \in \mathbb{R}^2 \mid y \leq x\}$). We are looking for a point $((\bar{x}, \bar{y}), \bar{\alpha}) \in Q(D)$ that $(\bar{x}, \bar{y}) \in D \cap R$ and $int(D_1) = \emptyset$. According to Corollary 3.10, we must calculate Y_D. We have $A(D) = \frac{\pi}{2}a^2$ and

$$Y_D = \frac{1}{A(D)}\int_D y\,\mathrm{d}x\mathrm{d}y = -\frac{2\sqrt{2}}{3\pi}a.$$

Therefore $(\bar{x}, \bar{y}) = \big(-\frac{2\sqrt{2}}{3\pi}a - 1, -\frac{2\sqrt{2}}{3\pi}a - 1\big)$. Since $a \geq \frac{3\sqrt{2}\pi}{3\pi-4}$, we conclude that $(\bar{x}, \bar{y}) \in D$. But, $y_M^D = \frac{a}{\sqrt{2}}$. It follows from Corollary 3.10 that $((\bar{x}, \bar{y}), \bar{\alpha}) \in Q(D)$ for any $\bar{\alpha} \geq (\frac{1}{\sqrt{2}} + \frac{2\sqrt{2}}{3\pi})a + 1$. It follows from Remark 3.2 that the following inequality holds for each sub-topical function f:

$$f\Big(-\frac{2\sqrt{2}}{3\pi}a - 1 + \bar{\alpha}, -\frac{2\sqrt{2}}{3\pi}a - 1 + \bar{\alpha}\Big) - \bar{\alpha} + 1 \leq \frac{2}{\pi a^2}\int_D f(x,y)\,\mathrm{d}x\mathrm{d}y$$

for each $\bar{\alpha} \geq (\frac{1}{\sqrt{2}} + \frac{2\sqrt{2}}{3\pi})a + 1$.

Example 4.3. Let $a > 3$. We will now consider the square in $\mathbb{R}^2$ formed by the points $(0,0)$, $(0,a)$, $(a,0)$ and (a,a) as vertices, which we denoted as D.

Consider the line $R = \{(x,y) \in \mathbb{R}^2 \mid y = x\}$. We are looking for a point $((\bar{x}, \bar{y}), \bar{\alpha}) \in Q(D)$ that $(\bar{x}, \bar{y}) \in D \cap R$ and $int(D_1) \neq \emptyset$. Let $\bar{x} \in [0, \frac{a}{3} - 1)$ be arbitrary. So $(\bar{x}, \bar{x}) \in R \cap D$. Now let

$\bar{\alpha}$ be such that $int(D_1) \neq \emptyset$. We have $A(D) = a^2$, $A(D_1) = (a - (\bar{x} + \bar{\alpha}))^2$ and $A(D_2) = A(D_3) = \frac{1}{2}(a^2 - (a - (\bar{x} + \bar{\alpha})))^2$. So we get

$$X_{D_2} = \frac{1}{A(D_2)} \int_{D_2} x \, \mathrm{d}x\mathrm{d}y = \frac{1}{3} \frac{3a(\bar{x} + \bar{\alpha}) - 2(\bar{x} + \bar{\alpha})^2}{2a - (\bar{x} + \bar{\alpha})}$$

and

$$Y_{D_3} = \frac{1}{A(D_3)} \int_{D_3} y \, \mathrm{d}x\mathrm{d}y = \frac{1}{3} \frac{3a(\bar{x} + \bar{\alpha}) - 2(\bar{x} + \bar{\alpha})^2}{2a - (\bar{x} + \bar{\alpha})}.$$

According to Theorem 3.9, we must have $(\beta_2 + \beta_3)\bar{x} = \beta_1\bar{\alpha} + \beta_2 X_{D_2} + \beta_3 Y_{D_3} - 1$. Substituting the quantities in this equation and simplifying, gives $(\bar{\alpha} + (\bar{x} - a))^3 = -a^3 + 3a^2(\bar{x} + 1)$. Therefore

$$\bar{\alpha} = a - \bar{x} + \sqrt[3]{-a^3 + 3a^2(\bar{x} + 1)}.$$

Note that since $\bar{x} \in [0, \frac{a}{3} - 1)$, we get $0 < \bar{x} + \bar{\alpha} < a$. This implies that $int(D_1) \neq \emptyset$. Summarizing, we have $\left((\bar{x}, \bar{x}), a - \bar{x} + \sqrt[3]{-a^3 + 3a^2(\bar{x} + 1)}\right) \in Q(D)$ for each $\bar{x} \in [0, \frac{a}{3} - 1)$.
On the other hand, the minimal point of the set D is $(0, 0)$. So (3.3) implies the following inequality:

$$\int_D f(x) \, \mathrm{d}x \leq a^2 f(0,0) + \frac{2}{3}a^3.$$

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