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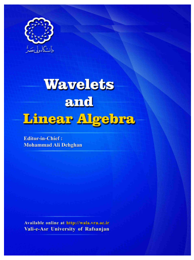

C-spectral norm inequalities between operator matrices and their entries

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Abstract

In this paper, the notion of $C-$spectral norm is introduced for operators; it was defined and studied for matrices before. Here, some $C-$spectral norm inequalities between operator matrices and their operator entries, for 2×2 and $n \times n$ operator matrices, are studied. Also, some $C-$spectral norm equalities between operator matrices are brought.

1. Introduction

Let $\mathbb{B}(\mathbb{H})$ denotes the set of bounded linear operators on the complex Hilbert space $\mathbb{H}$ and $\mathbb{U}(\mathbb{H})$ be the set of bounded unitary operators on the Hilbert space $\mathbb{H}$.

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When $\mathbb{H} = \mathbb{C}^n$, $\mathbb{B}(\mathbb{H})$ is the set of complex $n \times n$ matrices, denoted by $\mathbb{M}_n$. We use $\mathbb{U}_n$ to denote the set of unitary $n \times n$ matrices with complex entries. The $C-$spectral norm of matrices, abbreviated $C-$norm, is defined and denoted by $ııı_C(A) = max\{|tr(CUAV)| : U, V \in \mathbb{U}_n\}$, has been studied by researchers, for example see [2] and [3]. We generalize this concept for the bounded linear operators. Note that The trace concept has been generalized from finite dimensional Hilbert space to infinite dimensional case, with using the notion of trace class operators. By a trace class operator we mean a linear operator for which trace may be defined, such that the trace is a finite number independent of the choice of basis used to compute the trace. If T is in trace class, the trace of T is defined and denoted by $tr(T) = \sum_k \langle Te_k, e_k\rangle$, where $(e_k)_k$ is an arbitrary orthonormal basis of $\mathbb{H}$. The set of all trace class operators is a two sided ideal in the C^*-algebra $\mathbb{B}(\mathbb{H})$. One can see [4] for details of this concept. Here, we let $\mathbb{T}(\mathbb{H})$ be the set of trace class operators. At first, we denote the following set,

$$\mathcal{N}_C(T) = \{tr(CUTV) : U, V \in \mathbb{U}(\mathbb{H})\},$$

where $T \in \mathbb{B}(\mathbb{H})$ and $C \in \mathbb{T}(\mathbb{H})$. Note that since $\mathbb{T}(\mathbb{H})$ is a two sided ideal in $\mathbb{B}(\mathbb{H})$ and $C \in \mathbb{T}(\mathbb{H})$, it follows that $CUTV \in \mathbb{T}(\mathbb{H})$ for all $U, V \in \mathbb{U}(\mathbb{H})$. Now, we use $\mathcal{N}_C(T)$ to define the $C-$spectral norm for operators.

Definition 1.1. Let $T \in \mathbb{B}(\mathbb{H})$ and $C \in \mathbb{T}(\mathbb{H})$. The $C-$spectral norm, abbreviated $C-$norm, of T is defined and denoted by

$$ııı_C(T) = \sup_{\alpha \in \mathcal{N}_C(T)} |\alpha|.$$

When $\mathbb{H}$ has finite dimension, this definition coincides with definition of $C-$ norm of matrices. One can see properties of this concept for matrices in [1]. By an operator matrix, we mean a matrix $A = [T_{ij}]$, where $T_{ij} \in \mathbb{B}(\mathbb{H})$, $i, j = 1, 2, ..., n$. We want to show some inequalities on the $C-$norm of operator matrices. The following properties are easily derived from definition.

Proposition 1.2. *Let $T \in \mathbb{B}(\mathbb{H}), C \in \mathbb{T}(\mathbb{H})$ and $U, V \in \mathbb{U}(\mathbb{H})$. Then, the following statements hold:*

(*i*) *$\mathcal{N}_C(T)$ is compact;*

(*ii*) *$\mathcal{N}_C(T) = \mathcal{N}_T(C)$; Also, $ııı_C(T) = ııı_T(C)$;*

(*iii*) *$\mathcal{N}_{U^*CU}(V^*TV) = \mathcal{N}_C(T)$; Also, $ııı_C(U^*TU) = ııı_C(T)$.*

$C-$norm of matrices is a semi norm; It is a vector norm if and only if the matrix $C \neq 0$; Also, it is a matrix norm if and only if its largest singular value is more than or equal to one [2, Theorem 3.1]. It also has more useful properties. In the following proposition, we list some properties of $C-$norm of matrices that are useful in this article.

Proposition 1.3. *Let $A, C \in \mathbb{M}_n$. The following statements hold:*

(*i*) *[5, Lemma 1.5] If $a_1 \geq a_2 \geq ... \geq a_n$ and $c_1 \geq c_2 \geq ... \geq c_n$ are singular values of A and C respectively, then the set $\{tr(CUAV) : U, V \in \mathbb{U}_n\}$ is circular disk centered the origin with radius $\Sigma_{j=1}^n a_j c_j$ and so, $ııı_C(A) = \Sigma_{j=1}^n a_j c_j$;*

(*ii*) *[2, Corollary 3.3] Let* $C \neq 0$ *with the largest singular value* c_1. *Then* $\text{n}_C(A^k) \leq (\text{n}_C(A))^k$ *for all* $k = 1, 2, ...,$ *if and only if* $c_1 \geq 1$*;*

(*iii*) *If* C *is Hermitian, then* $\text{n}_C(A) = \text{n}_C(A^*)$*;*

In this paper, some $C-$spectral norm for operator matrices are studied; In Section 2, some lower bounds for the $C-$spectral norm of 2×2 operator matrices according to the $C-$spectral norm of their operator entries are brought. In Section 3, some equalities and inequalities for the $C-$spectral norm of $n \times n$ operator matrices are given.

2. $C-$Norm Inequalities between 2×2 Operator Matrices and their entries

At first, we have a proposition which is useful in other results of this paper.

Proposition 2.1. *Let* $T, S \in \mathbb{B}(\mathbb{H})$, $C_1, C_2 \in \mathbb{T}(\mathbb{H})$ *and* $C' = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}$. *Also, let* $0 \in \mathcal{N}_{C_1}(T) \cap \mathcal{N}_{C_2}(S)$. *Then, the following inequalities hold:*

(*i*) $\text{n}_{C'}(\begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix}) \geq max\{\text{n}_{C_1}(T), \text{n}_{C_2}(S)\}$;

(*ii*) $\text{n}_{C'}(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}) \geq max\{\text{n}_{C_2}(T), \text{n}_{C_1}(S)\}$.

Proof. Let $U = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}$ and $V = \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix}$, where $U_i, V_i \in \mathbb{U}(\mathbb{H})$, $(i = 1, 2)$. Then, we have

$$tr(C'U \begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix} V) = tr(C_1 U_1 T V_1) + tr(C_2 U_2 S V_2).$$

So, one can see

$$\mathcal{N}_{C_1}(T) + \mathcal{N}_{C_2}(S) \subseteq \mathcal{N}_{C'}(\begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix}).$$

Since $0 \in \mathcal{N}_{C_1}(T)$, we have $\mathcal{N}_{C_2}(S) \subseteq \mathcal{N}_{C'} \begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix}$, and so, $\text{n}_{C_2}(S) \leq \text{n}_{C'}(\begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix})$. Therefor, the result in part (*i*) is satisfied. At the same way, the inequality in part (*ii*) is satisfied by using $U = \begin{bmatrix} 0 & U_1 \\ U_2 & 0 \end{bmatrix}$ and $V = \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix}$. □

The inequality mentioned in the above proposition can not be established in the form of equality. The next example shows this, by using Proposition 1.3(*i*).

Example 2.2. Let $T = diag(3, 2, 1), S = diag(5, 4, 3)$ and $C = diag(1, 2, 1)$. Using $U = diag(-1, 1, -1)$ and $V = I_3$, one can see that $0 \in \mathcal{N}_C(T) \cap \mathcal{N}_C(S)$. Clearly, singular values of T, in descending order, are $3 \geq 2 \geq 1$. Also, $5 \geq 4 \geq 3$ are singular values of S, and $2 \geq 1 \geq 1$ are singular values of C. Since T, S and C are diagonal matrices, one can see that singular values of

$T \oplus S$ are $5 \geq 4 \geq 3 \geq 3 \geq 2 \geq 1$ and singular values of $C \oplus C$ are $2 \geq 2 \geq 1 \geq 1 \geq 1 \geq 1$, considering repetition in descending order. So, by Proposition 1.3(i), we have $\text{n}_C(T) = 6 + 2 + 1 = 9$, $\text{n}_C(S) = 10 + 4 + 3 = 17$ and $\text{n}_{C\oplus C}(T \oplus S) = 10 + 8 + 3 + 3 + 2 + 1 = 27$. So, we have $max\{\text{n}_C(S), \text{n}_C(T)\} = 17 < 27 = \text{n}_{C\oplus C}(T \oplus S)$.

Theorem 2.3. *Let $Q, R, S, T \in \mathbb{B}(\mathbb{H})$, $C \in \mathbb{T}(\mathbb{H} \oplus \mathbb{H})$ and $\theta \in \mathbb{R}$. Then, the following statements hold:*

(i) $\text{n}_C(\begin{bmatrix} 0 & T \\ e^{i\theta}S & 0 \end{bmatrix}) = \text{n}_C(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix})$, ;

(ii) $\text{n}_C(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}) = \text{n}_C(\begin{bmatrix} 0 & S \\ T & 0 \end{bmatrix})$;

(iii) $\text{n}_C(\begin{bmatrix} Q & R \\ S & T \end{bmatrix}) \geq max\{\text{n}_C(\begin{bmatrix} Q & 0 \\ 0 & T \end{bmatrix}), \text{n}_C(\begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix})\}$.

Proof. Parts (i) and (ii) are satisfied using Proposition 1.2(iii) with unitary matrices $\begin{bmatrix} I & 0 \\ 0 & e^{\frac{i\theta}{2}}I \end{bmatrix}$ and $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$, respectively. To see part (iii), we note that using Proposition 1.2(iii) with $\begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}$, we have

$$\text{n}_C(\begin{bmatrix} Q & R \\ S & T \end{bmatrix}) = \text{n}_C(\begin{bmatrix} Q & -R \\ -S & T \end{bmatrix}). \tag{2.1}$$

Also, Proposition 1.2(iii) with $\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$ shows that

$$\text{n}_C(\begin{bmatrix} Q & R \\ S & T \end{bmatrix}) = \text{n}_C(\begin{bmatrix} -Q & R \\ S & -T \end{bmatrix}). \tag{2.2}$$

Using the true relation $\begin{bmatrix} Q & 0 \\ 0 & T \end{bmatrix} = (1/2)\begin{bmatrix} Q & R \\ S & T \end{bmatrix} + (1/2)\begin{bmatrix} Q & -R \\ -S & T \end{bmatrix}$, and triangle inequality with the relation (2.1), we have

$$\text{n}_C(\begin{bmatrix} Q & 0 \\ 0 & T \end{bmatrix}) \leq \text{n}_C(\begin{bmatrix} Q & R \\ S & T \end{bmatrix}).$$

At the same way, Using $\begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix} = (1/2)\begin{bmatrix} Q & R \\ S & T \end{bmatrix} + (1/2)\begin{bmatrix} -Q & R \\ S & -T \end{bmatrix}$ and relation (2.2), we have

$$\text{n}_C(\begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix}) \leq \text{n}_C(\begin{bmatrix} Q & R \\ S & T \end{bmatrix}).$$

Now, the result in (iii) follows from the last two inequalities. □

In the following theorem, we have $C-$norm lower bounds for special 2×2 block matrices.

Theorem 2.4. *Let $S, T \in \mathbb{B}(\mathbb{H})$, $C_1, C_2 \in \mathbb{T}(\mathbb{H})$ and $C' = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}$. Also, let $0 \in \mathcal{N}_{C_i}(Q)$ for any $Q \in \mathbb{B}(\mathbb{H})$ and $i = 1, 2$. Then, the following statements hold:*

(*i*) $\text{n}_{C'}(\begin{bmatrix} T & S \\ S & T \end{bmatrix}) \geq max\{\text{n}_{C_2}(T - S), \text{n}_{C_1}(T + S)\}$. *In particular, for $i = 1, 2$,*

$$\text{n}_{C'}(\begin{bmatrix} 0 & S \\ S & 0 \end{bmatrix}) \geq \text{n}_{C_i}(S);$$

(*ii*) $\text{n}_{C'}(\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}) \geq \frac{1}{2} max\{\text{n}_{C_1}(T + S), \text{n}_{C_2}(T - S)\}$.

(*iii*) $\text{n}_{C'}(\begin{bmatrix} T & S \\ -S & -T \end{bmatrix}) \geq max\{\text{n}_{C_i}(S), \text{n}_{C_i}(T)\}, (i = 1, 2)$.

Proof. Let $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ I & -I \end{bmatrix}$. Using Proposition 1.2(*iii*), we have $\text{n}_{C'}(\begin{bmatrix} T & S \\ S & T \end{bmatrix}) = \text{n}_{C'}(\begin{bmatrix} T + S & 0 \\ 0 & T - S \end{bmatrix})$. Now, part (*i*) is derived from Proposition 2.1(*i*). The particular case is derived by $T = 0$. To see part (*ii*), let $W = \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}$ and $U = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. Then,

$$\begin{bmatrix} T & S \\ S & T \end{bmatrix} = W + U^* W U.$$

So, by triangle inequality, we have

$$\begin{aligned} \text{n}_{C'}(\begin{bmatrix} T & S \\ S & T \end{bmatrix}) &\leq \text{n}_{C'}(W) + \text{n}_{C'}(U^* W U) \\ &= 2\text{n}_{C'}(W), \end{aligned}$$

where the equality is derived by Proposition 1.2(*iii*). Now, by part (*i*), we have $max\{\text{n}_{C_1}(T + S), \text{n}_{C_2}(T - S) \leq 2\text{n}_{C'}(W)$ and part (*ii*) is derived.
To see part (*iii*), by Theorem 2.3(*iii*) and Proposition 2.1, we have

$$\text{n}_{C'}(\begin{bmatrix} S & T \\ -T & -S \end{bmatrix}) \geq \text{n}_{C'}(\begin{bmatrix} S & 0 \\ 0 & -S \end{bmatrix}) \geq \text{n}_{C_i}(S), (i = 1, 2),$$

and

$$\text{n}_{C'}(\begin{bmatrix} S & T \\ -T & -S \end{bmatrix}) \geq \text{n}_{C'}(\begin{bmatrix} 0 & T \\ -T & 0 \end{bmatrix}) \geq \text{n}_{C_i}(T), (i = 1, 2).$$

Then,

$$\text{n}_{C'}(\begin{bmatrix} S & T \\ -T & -S \end{bmatrix}) \geq max\{\text{n}_{C_i}(S), \text{n}_{C_i}(T)\}, (i = 1, 2).$$

□

In the next theorem, using Proposition 2.1, we find two $C-$norm lower bounds for 2×2 anti-diagonal operator matrices.

Theorem 2.5. *Let $T, S \in \mathbb{B}(\mathbb{H})$, $C_1, C_2 \in \mathbb{T}(\mathbb{H})$ and $C' = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}$. Also, let $0 \in \mathcal{N}_{C_i}(Q)$ for any $Q \in \mathbb{B}(\mathbb{H})$ and $i = 1, 2$. Then, the following statements are true;*

(*i*) $\text{n}_{C'}(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}) \geq \sqrt[2n]{max\{\text{n}_{C_1}(TS)^n, \text{n}_{C_2}(ST)^n\}}$;

(*ii*) $\text{n}_{C'}(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}) \geq \frac{1}{2}(max\{\text{n}_{C_i}(T-S), \text{n}_{C_i}(T+S)\}), \ \ (i = 1, 2)$.

Proof. Let $W = \begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}$. Then, we have

$$W^{2n} = \begin{bmatrix} (TS)^n & 0 \\ 0 & (ST)^n \end{bmatrix}, \quad (n = 1, 2, \ldots).$$

So, by Proposition 2.1(*i*) and Proposition 1.3(*ii*),

$$\begin{aligned} max(\text{n}_{C_1}(TS)^n, \text{n}_{C_2}(ST)^n) &\leq \text{n}_{C'}(\begin{bmatrix} (TS)^n & 0 \\ 0 & (ST)^n \end{bmatrix}) \\ &= \text{n}_{C'}(W^{2n}) \\ &\leq \text{n}_{C'}(W)^{2n} \\ &= \text{n}_{C'}(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix})^{2n}. \end{aligned}$$

This completes the proof of part (*i*).
We have the following relations by using Proposition 2.1(*ii*), Theorem 2.3(*ii*) and triangle inequality, for $i = 1, 2$,

$$\begin{aligned} \text{n}_{C_i}(T+S) &\leq \text{n}_{C'}(\begin{bmatrix} 0 & T+S \\ T+S & 0 \end{bmatrix}) \\ &= \text{n}_{C'}(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix} + \begin{bmatrix} 0 & S \\ T & 0 \end{bmatrix}) \\ &\leq \text{n}_{C'}(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}) + \text{n}_{C'}(\begin{bmatrix} 0 & S \\ T & 0 \end{bmatrix}) \\ &= \text{n}_{C'}(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}) + \text{n}_{C'}(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}) \\ &= 2\text{n}_{C'}(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}). \end{aligned}$$

So,

$$\frac{\text{n}_{C_i}(T+S)}{2} \leq \text{n}_{C'}(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}), (i = 1, 2). \tag{2.3}$$

Now, in the inequality (2.3) replacing S by $-S$, we have

$$\begin{aligned}\frac{\eta_{C_i}(T-S)}{2} &\leq \eta_{C'}(\begin{bmatrix} 0 & T \\ -S & 0 \end{bmatrix}) \\ &= \eta_{C'}(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}), (i=1,2),\end{aligned}$$

where the last equality derived by Theorem 2.3(i). Now, the inequality in part (ii) follows from the last inequality and inequality (2.3).

□

In the following remark, we mention a special case of the above theorem.

Remark 2.6. We have a particular case of part (ii) of the above theorem, when $S = T$, it says that for $i = 1, 2$,

$$\eta_{C_i}(T) \leq \eta_{C'}(\begin{bmatrix} T & T \\ -T & -T \end{bmatrix}).$$

In the following corollary, we have a $C-$norm upper bound for the Cartesian decomposition form of an operator which involved its real part and imaginary part.

Corollary 2.7. *Let $T \in \mathbb{B}(\mathbb{H})$, with the Cartesian decomposition $T = A + iB$ and $C \in \mathbb{T}(\mathbb{H})$ be selfadjoint. Also, let $C' = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}$. Then for each $\theta \in \mathbb{R}$,*

$$\frac{\eta_C(T)}{2} \leq \eta_{C'}(\begin{bmatrix} 0 & A \\ e^{i\theta}B & 0 \end{bmatrix}). \tag{2.4}$$

Proof. At first, we apply Theorem 2.5(ii) to matrix $\begin{bmatrix} 0 & A \\ iB & 0 \end{bmatrix}$. So,

$$\frac{max\{\eta_C(A+iB), \eta_C(A-iB)\}}{2} \leq \eta_{C'}(\begin{bmatrix} 0 & A \\ iB & 0 \end{bmatrix}).$$

Now, using Theorem 2.3(i), we have

$$\frac{max(\eta_C(T), \eta_C(T^*))}{2} \leq \eta_{C'}(\begin{bmatrix} 0 & A \\ e^{i\theta}B & 0 \end{bmatrix}). \tag{2.5}$$

It is easy to see that $\eta_C(T) = \eta_{C^*}(T^*)$; In special case, when C is a self adjoint operator, we have $\eta_C(T) = \eta_C(T^*)$. So, the inequality (2.5) can be written in the form of inequality (2.4). □

At the end of this section, using Theorem 2.5, we find a lower bound for $C-$norm of 2×2 operator matrices.

Theorem 2.8. *Let $Q, R, S, T \in \mathbb{B}(\mathbb{H})$, $C_1, C_2 \in \mathbb{T}(\mathbb{H})$ and $C' = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}$. Then,*

$$\eta_{C'}\begin{bmatrix} Q & R \\ S & T \end{bmatrix} \geq max\left(\eta_{C_1}(Q), \eta_{C_2}(T), \frac{\eta_{C_i}(R+S)}{2}, \frac{\eta_{C_i}(R-S)}{2}\right), (i=1,2).$$

Proof. The following relations satisfied with Theorem 2.3(iii), Proposition 2.1 and Theorem 2.5(ii):

$$\begin{aligned}
\eta_{C'}(\begin{bmatrix} Q & R \\ S & T \end{bmatrix}) &\geq max\left(\eta_{C'}(\begin{bmatrix} Q & 0 \\ 0 & T \end{bmatrix}), \eta_{C'}(\begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix})\right) \\
&\geq max\left(max(\eta_{C_1}(Q), \eta_{C_2}(T)), \eta_{C'}(\begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix})\right) \\
&\geq max\left(max(\eta_{C_1}(Q), \eta_{C_2}(T)), \frac{max(\eta_{C_i}(R+S), \eta_{C_i}(R-S))}{2}\right) \\
&= max\left(\eta_{C_1}(Q), \eta_{C_2}(T), \frac{\eta_{C_i}(R+S)}{2}, \frac{\eta_{C_i}(R-S)}{2}\right), (i = 1, 2).
\end{aligned}$$

Now, the proof is complete. □

3. Some lower bounds and equalities for $C-$norm of operator matrices

At first, we obtain a lower bound for the $C-$norm of anti-diagonal $n \times n$ operator matrices.

Theorem 3.1. *Let* $T = \begin{bmatrix} 0 & 0 & ... & 0 & T_1 \\ 0 & 0 & ... & T_2 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ T_n & 0 & ... & 0 & 0 \end{bmatrix}$*, where* $T_i \in \mathbb{B}(\mathbb{H})$*,* $(i = 1, 2, ..., n)$*, and* $C' = \begin{bmatrix} C_1 & 0 & ... & 0 & 0 \\ 0 & C_2 & ... & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & ... & 0 & C_n \end{bmatrix}$*, where* $C_i \in \mathbb{T}(\mathbb{H})$*, for* $i = 1, \ldots, n$*. Then,*

$$\eta_{C'}(T) \geq \frac{1}{\sqrt{2}} max_{1\leq i\leq n}\{|\sqrt{\eta_{C_i}(T_iT_{n-i+1} + T_{n-i+1}T_i)}|, |\sqrt{\eta_{C_i}(T_iT_{n-i+1} - T_{n-i+1}T_i)}|\}.$$

Proof. At fist, notice that

$$T^2 + (U^*TU)^2 = (T_1T_n + T_nT_1 \oplus T_2T_{n-1} + T_{n-1}T_2 \oplus \cdots \oplus T_nT_1 + T_1T_n),$$

where the unitary matrix $U = \begin{bmatrix} 0 & 0 & ... & 0 & I \\ 0 & 0 & ... & I & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ I & 0 & ... & 0 & 0 \end{bmatrix}$. Then, we have

$$\begin{aligned}
max\{\eta_{C_i}(T_iT_{n-i+1} + T_{n-i+1}T_i), 1 \leq i \leq n\} &\leq \eta_{C'}(T^2 + (U^*TU)^2) \\
&\leq \eta_{C'}(T^2) + \eta_{C'}(U^*T^2U) \\
&= 2\eta_{C'}(T^2), (i = 1, \ldots, n),
\end{aligned}$$

where the first above inequality derived by using Proposition 2.1 and the second one is by triangle inequality; The equality derived by Proposition 1.2(*iii*). At the same way, by calculating $T^2 - (U^*TU)^2$ and the same argument as above, we can see that

$$max\{\eta_{C_i}(T_iT_{n-i+1} - T_{n-i+1}T_i), 1 \leq i \leq n\} \leq 2\eta_{C'}(T^2), (i = 1, \dots, n).$$

Therefore, we conclude that

$$\eta_{C'}(T^2) \geq \frac{1}{2} max_{1\leq i\leq n}\{\eta_{C_i}(T_iT_{n-i+1} + T_{n-i+1}T_i), \eta_{C_i}(T_iT_{n-i+1} - T_{n-i+1}T_i)\}, (i = 1, \dots, n).$$

Using Proposition 1.3(*ii*), we have $\eta_{C'}(T^2) \leq \eta_{C'}(T)^2$; So, taking square root, we have the result. □

In the following theorem, by using unitary operator matrices, we find some $C-$norm equalities of operator matrices with the roots of unity as coefficients.

Theorem 3.2. *Let $T_i \in \mathbb{B}(\mathbb{H})$, $(i = 1, 2, ..., n)$ and $1, \xi, \xi^2, ..., \xi^{n-1}$ are the $n-$th roots of unity. Also, let $C \in \mathbb{T}(\mathbb{H})$ and* $T = \begin{bmatrix} 0 & & T_1 \\ & T_2 & \\ \vdots & & \\ T_n & & 0 \end{bmatrix}$. *Then,*

$$\begin{aligned}
\eta_C(T) &= \eta_C(\begin{bmatrix} 0 & & & \xi^{n-1}T_n \\ & & \xi^{2n-3}T_{n-1} & \\ & \xi^{3n-5}T_{n-2} & & \\ \vdots & & & \\ \xi^{(n-1)^2}T_1 & & & 0 \end{bmatrix}) \\
&= \eta_C(\begin{bmatrix} 0 & & & \xi T_n \\ & & \xi^3T_{n-1} & \\ & \xi^5T_{n-2} & & \\ \vdots & & & \\ \xi^{2n-1}T_1 & & & 0 \end{bmatrix}) \\
&= \eta_C(\begin{bmatrix} 0 & \cdots & 0 & \xi^3T_{n-1} & 0 & 0 \\ \vdots & \vdots & \xi^5T_{n-2} & & & \\ 0 & \vdots & \vdots & \vdots & & \vdots \\ \xi^{2n-3}T_2 & & & & \vdots & 0 \\ 0 & & & & 0 & \xi^{n-1}T_1 \\ 0 & & \cdots & 0 & \xi^{n+1}T_n & 0 \end{bmatrix}).
\end{aligned}$$

Proof. Let $U_1 = \begin{bmatrix} 0 & & & \xi^{2n-2}I \\ & & \xi^{2n-3}I & \\ & \vdots & & \\ \xi^{n-1}I & & & \end{bmatrix}$, $U_2 = \begin{bmatrix} 0 & & & & I \\ & & & \xi I & \\ & & \xi^2 I & & \\ & \vdots & & & \\ \xi^{n-1}I & & & & \end{bmatrix}$ and

$$U_3 = \begin{bmatrix} 0 & \cdots & & 0 & I & 0 \\ & & & \xi I & & \\ \vdots & \vdots & \xi^2 I & & & \\ 0 & \vdots & \vdots & & & \vdots \\ \xi^{n-2}I & & & & & 0 \\ 0 & \cdots & & & 0 & \xi^{n-1}I \end{bmatrix}.$$

To complete the proof, we use Proposition 1.2(*iii*) to have $\mathfrak{n}_C(T) = \mathfrak{n}_C(U_1TU_1^*) = \mathfrak{n}_C(U_2TU_2^*) = \mathfrak{n}_C(U_3TU_3^*)$. □

The above theorem along with Theorem 2.3(*i*) gives the following result.

Corollary 3.3. *Let $X_1, X_2, X_3 \in \mathbb{B}(\mathbb{H})$ and $1, \xi, \xi^2$ be the third roots of unity. Also, let $C \in \mathbb{T}(\mathbb{H} \oplus \mathbb{H} \oplus \mathbb{H})$ is self adjoint. Then*

$$\mathfrak{n}_C(\begin{bmatrix} X_2 & \xi^2 X_1 & \xi X_3 \\ \xi X_3 & X_2 & \xi^2 X_1 \\ \xi^2 X_1 & \xi X_3 & X_2 \end{bmatrix}) \leq 3\mathfrak{n}_C(\begin{bmatrix} 0 & 0 & X_3 \\ 0 & X_2 & 0 \\ X_1 & 0 & 0 \end{bmatrix}).$$

Proof. We have the following relations; At first, we use triangle inequality; Then, we use the generalization of Theorem 2.3(*i*) and Theorem 3.2. For second equality, we use the unitary operator matrix $U = \begin{bmatrix} \xi^2 I & 0 & 0 \\ 0 & 0 & I \\ 0 & \xi I & 0 \end{bmatrix}$ and Proposition 1.2(*iii*). To find third equality, notice that C is self adjoint and $\mathfrak{n}_C(T) = \mathfrak{n}_C(T^*)$, for an operator T.

$$
\begin{aligned}
\mathfrak{n}_C(\begin{bmatrix} X_2 & \xi^2 X_1 & \xi X_3 \\ \xi X_3 & X_2 & \xi^2 X_1 \\ \xi^2 X_1 & \xi X_3 & X_2 \end{bmatrix}) &\leq \mathfrak{n}_C(\begin{bmatrix} 0 & 0 & \xi X_3 \\ 0 & X_2 & 0 \\ \xi^2 X_1 & 0 & 0 \end{bmatrix}) + \mathfrak{n}_C(\begin{bmatrix} X_2 & 0 & 0 \\ 0 & 0 & \xi^2 X_1 \\ 0 & \xi X_3 & 0 \end{bmatrix}) \\
&+ \mathfrak{n}_C(\begin{bmatrix} 0 & \xi^2 X_1 & 0 \\ \xi X_3 & 0 & 0 \\ 0 & 0 & X_2 \end{bmatrix}) \\
&= 2\mathfrak{n}_C(\begin{bmatrix} 0 & 0 & X_3 \\ 0 & X_2 & 0 \\ X_1 & 0 & 0 \end{bmatrix}) + \mathfrak{n}_C(\begin{bmatrix} 0 & \xi^2 X_1 & 0 \\ \xi X_3 & 0 & 0 \\ 0 & 0 & X_2 \end{bmatrix}) \\
&= 2\mathfrak{n}_C(\begin{bmatrix} 0 & 0 & X_3 \\ 0 & X_2 & 0 \\ X_1 & 0 & 0 \end{bmatrix}) + \mathfrak{n}_C(\begin{bmatrix} 0 & 0 & X_1 \\ 0 & X_2 & 0 \\ X_3 & 0 & 0 \end{bmatrix}) \\
&= 2\mathfrak{n}_C(\begin{bmatrix} 0 & 0 & X_3 \\ 0 & X_2 & 0 \\ X_1 & 0 & 0 \end{bmatrix}) + \mathfrak{n}_C(\begin{bmatrix} 0 & 0 & X_3^* \\ 0 & X_2^* & 0 \\ X_1^* & 0 & 0 \end{bmatrix}) \\
&= 2\mathfrak{n}_C(\begin{bmatrix} 0 & 0 & X_3 \\ 0 & X_2 & 0 \\ X_1 & 0 & 0 \end{bmatrix}) + \mathfrak{n}_C(\begin{bmatrix} 0 & 0 & X_3 \\ 0 & X_2 & 0 \\ X_1 & 0 & 0 \end{bmatrix}^*) \\
&= 3\mathfrak{n}_C(\begin{bmatrix} 0 & 0 & X_3 \\ 0 & X_2 & 0 \\ X_1 & 0 & 0 \end{bmatrix}).
\end{aligned}
$$

□

In the following Theorem, we use the unitary similarity in Proposition 1.2(*iii*) to show another $C-$norm equalities.

Theorem 3.4. *Let* $T_i \in \mathbb{B}(\mathbb{H})$, $(i = 1, 2, ..., n)$ *where* $n \geq 3$ *and* $C \in \mathbb{T}(\underbrace{\mathbb{H} \oplus \cdots \oplus \mathbb{H}}_{n-times})$. *Also, let*

$$
T = \begin{bmatrix} 0 & & T_1 \\ & T_2 & \\ \vdots & & \\ T_n & & 0 \end{bmatrix} \text{ and } S = \begin{bmatrix} 0 & & T_2 \\ & T_3 & \\ \vdots & & \\ T_{n-1} & & 0 \end{bmatrix}. \text{ Then,}
$$

$$
\begin{aligned}
\mathfrak{n}_C(T) &= \mathfrak{n}_C(\begin{bmatrix} 0 & 0 & T_1 \\ 0 & S^t & 0 \\ T_n & 0 & 0 \end{bmatrix}) = \mathfrak{n}_C(\begin{bmatrix} 0 & 0 & T_n \\ 0 & S & 0 \\ T_1 & 0 & 0 \end{bmatrix}) = \mathfrak{n}_C(\begin{bmatrix} S & 0 & 0 \\ 0 & 0 & T_1 \\ 0 & T_n & 0 \end{bmatrix}) \\
&= \mathfrak{n}_C(\begin{bmatrix} S^t & 0 & 0 \\ 0 & 0 & T_n \\ 0 & T_1 & 0 \end{bmatrix}) = \mathfrak{n}_C(\begin{bmatrix} 0 & T_n & 0 \\ T_1 & 0 & 0 \\ 0 & 0 & S \end{bmatrix}) = \mathfrak{n}_C(\begin{bmatrix} 0 & T_n & 0 \\ T_1 & 0 & 0 \\ 0 & 0 & S^t \end{bmatrix}).
\end{aligned}
$$

At the end of this section, we have the following corollary derived from the last theorem.

Corollary 3.5. *Let* $X_1, X_2, X_3 \in \mathbb{B}(\mathbb{H})$ *and* $C \in \mathbb{T}(\mathbb{H})$. *Then*

$$\eta_C(\begin{bmatrix} X_1 & X_2 & X_3 \\ X_3 & X_1 & X_2 \\ X_2 & X_3 & X_1 \end{bmatrix}) \leq 3\eta_C(\begin{bmatrix} 0 & 0 & X_3 \\ 0 & X_1 & 0 \\ X_2 & 0 & 0 \end{bmatrix}).$$

Proof. By using triangle inequality, we have

$$\begin{aligned} \eta_C(\begin{bmatrix} X_1 & X_2 & X_3 \\ X_3 & X_1 & X_2 \\ X_2 & X_3 & X_1 \end{bmatrix}) &\leq \eta_C(\begin{bmatrix} 0 & 0 & X_3 \\ 0 & X_1 & 0 \\ X_2 & 0 & 0 \end{bmatrix}) + \eta_C(\begin{bmatrix} 0 & X_2 & 0 \\ X_3 & 0 & 0 \\ 0 & 0 & X_1 \end{bmatrix}) + \eta_C(\begin{bmatrix} X_1 & 0 & 0 \\ 0 & 0 & X_2 \\ 0 & X_3 & 0 \end{bmatrix}) \\ &= \eta_C(\begin{bmatrix} 0 & 0 & X_3 \\ 0 & X_1 & 0 \\ X_2 & 0 & 0 \end{bmatrix}) + \eta_C(\begin{bmatrix} 0 & 0 & X_2 \\ 0 & X_3 & 0 \\ X_1 & 0 & 0 \end{bmatrix}) + \eta_C(\begin{bmatrix} 0 & 0 & X_2 \\ 0 & X_1 & 0 \\ X_3 & 0 & 0 \end{bmatrix}) \\ &= 3\eta_C(\begin{bmatrix} 0 & 0 & X_3 \\ 0 & X_1 & 0 \\ X_2 & 0 & 0 \end{bmatrix}), \end{aligned}$$

where equalities derived by using two steps of unitary similarity invariant property of $C-$norm. □

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