Wavelets and Linear Algebra 10(2) (2023) 1- 18

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Wavelets and Linear Algebra

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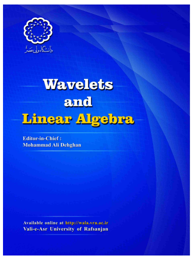

cK-frames and cK-Riesz bases in Hilbert spaces

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Article Info

Article history:
Received 13 March 2022
Accepted 17 June 2023
Available online 22 November 2023
Communicated by Rajab Ali Kamyabi-Gol

Keywords:
cK-frame, cK-Riesz bases, cK-dual, Riesz-type cK-frame.

2010 MSC:
42C15, 42C40, 41A58.

Abstract

In this paper, we prove some new results about cK-frames. Also, we introduce the concept of cK-Riesz basis and we provide a necessary and sufficient condition under which F is a cK-Riesz basis. Finally, for the closed range operator $K \in B(\mathcal{H})$, we prove that under some conditions, $\pi_{R(K)}F$ is a cK-Riesz basis if and only if it has only one dual, where $\pi_{R(K)}$ is the orthogonal projection from $\mathcal{H}$ onto $R(K)$, i.e., the range of K.

1. Introduction

Duffin and Schaeffer introduced the concept of discrete frames in Hilbert spaces in 1952 to study some deep problems in nonharmonic Fourier series [11]. These frames were reintroduced and developed in 1986 by Daubechies et al. [9], and became more popular from then on. A

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http://doi.org/10.22072/wala.2023.550565.1372

sequence $\{f_j\}_{j\in J}$ in a Hilbert space $\mathcal{H}$ is called a frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that for all $f \in \mathcal{H}$,

$$A\|f\|^2 \leqslant \sum_{j\in J} |\langle f, f_j\rangle|^2 \leqslant B\|f\|^2. \tag{1.1}$$

$\{f_j\}_{j\in J}$ is called a Bessel sequence if the right hand side of (1.1) holds for all $f \in \mathcal{H}$.

A particular example of a frame is an orthonormal basis for $\mathcal{H}$ or, more generally, any isomorphic image of an orthonormal basis, i.e., a Riesz basis ([7, 15]). We say that a sequence $\{f_j\}_{j\in J}$ is a Riesz basis for the Hilbert space $\mathcal{H}$ if $\overline{span}\{f_j\}_{j\in J} = \mathcal{H}$ (i.e., $\{f_j\}_{j\in J}$ is complete in $\mathcal{H}$), and there exist constants $A, B > 0$ such that for any finite scalar sequence $\{c_j\}$,

$$A \sum |c_j|^2 \leqslant \Big\| \sum c_j f_j \Big\|^2 \leqslant B \sum |c_j|^2.$$

Continuous frame as a generalization of frame was proposed by G. Kaiser [17] and independently by Ali, Antoine and Gazeau [2] (see also [5], [13]). In mathematical physics these frames are referred to as coherent states [1]. The work done by Rahimi et.al in [21], gives us a more routine study on continuous frames. Also, Gabardo and Han in [13], have provided some basic properties of continuous frames and they have introduced a kind of continuous frame which possesses only one dual, called Riesz-type frame. For more details about continuos frames, the reader can refer to [3, 8, 12, 18, 19].

Throughout this paper, $\mathcal{H}$ is a complex Hilbert space, J is a countable set, (Ω, μ) is a σ-finite measure space with positive measure μ, $B(\mathcal{H})$ is the set of all bounded linear operators on $\mathcal{H}$. If $K \in B(\mathcal{H})$ then $N(K)$ and $R(K)$ are the kernel and the range of the operator K, respectively. $L^2(\Omega)$ represents an infinite-dimentional Hilbert space consisting of all measurable functions $\varphi : \Omega \to \mathbb{C}$ such that

$$\|\varphi\|_2 = \Big(\int_\Omega |\varphi(\omega)|^2 d\mu(\omega) \Big)^{1/2} < \infty.$$

In this space, the inner product is defined by

$$\langle \varphi, \psi \rangle_{L^2} = \int_\Omega \varphi(\omega)\overline{\psi(\omega)} d\mu(\omega),$$

for all $\varphi, \psi \in L^2(\Omega)$. Also, we denote by $L^2(\Omega, \mathcal{H})$ the set of all mappings $F : \Omega \to \mathcal{H}$ such that for all $f \in \mathcal{H}$, the function $\omega \mapsto \langle f, F(\omega)\rangle$ defined almost everywhere on Ω, belongs to $L^2(\Omega)$.

A mapping $F : \Omega \to \mathcal{H}$ is called a continuous frame (or a c-frame) for $\mathcal{H}$, if F is weakly measurable, i.e., for all $f \in \mathcal{H}$, $\omega \mapsto \langle f, F(\omega)\rangle$ is a measurable function on Ω, and there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leqslant \int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega) \leqslant B\|f\|^2, \quad f \in \mathcal{H}. \tag{1.2}$$

If the right hand inequality of (1.2) holds for all $f \in \mathcal{H}$ then we say that $F : \Omega \to \mathcal{H}$ is a c-Bessel mapping. We say that F is a tight c-frame, if there exists a constant $A > 0$ such that $\int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega) = A\|f\|^2$ for all $f \in \mathcal{H}$.

Definition 1.1. [4] A c-Bessel mapping $F \in L^2(\Omega, \mathcal{H})$ is said to be L^2-independent if

$$\int_\Omega \phi(\omega)F(\omega)d\mu(\omega) = 0,$$

for $\phi \in L^2(\Omega)$, implies that $\phi = 0$ almost everywhere.

Proposition 1.2. *([21] Theorem 2.6 and Proposition 2.7). A weakly measurable function $F : \Omega \to \mathcal{H}$ is a c-Bessel mapping for $\mathcal{H}$ if and only if $T_F : L^2(\Omega) \to \mathcal{H}$ weakly defined by*

$$\langle T_F\varphi, h\rangle = \int_\Omega \varphi(\omega)\langle F(\omega), h\rangle d\mu(\omega), \quad h \in \mathcal{H}, \tag{1.3}$$

is well-defined, linear and bounded operator. In this case, its adjoint is given by

$$T_F^* : \mathcal{H} \to L^2(\Omega), \quad (T_F^* h)(\omega) = \langle h, F(\omega)\rangle, \quad \omega \in \Omega. \tag{1.4}$$

The operator T_F is called the pre-frame operator or synthesis operator and T_F^* is called the analysis operator of F.

If F is a c-Bessel mapping for $\mathcal{H}$, then the operator $S_F : \mathcal{H} \to \mathcal{H}$ defined by $S_F = T_F T_F^*$ is a bounded operator. We call S_F the frame operator of F. Thus,

$$\langle S_F f, g\rangle = \int_\Omega \langle f, F(\omega)\rangle\langle F(\omega), g\rangle d\mu(\omega), \quad f, g \in \mathcal{H}. \tag{1.5}$$

The concept of continuous Riesz basis (or c-Riesz basis) was introduced by Arefijamaal et al., in [4]. A c-Bessel mapping $F : \Omega \to \mathcal{H}$ is called μ-complete if

$$\Big\{ \int_\Omega \phi(\omega)F(\omega)d\mu(\omega);\ \phi \in L^2(\Omega)\Big\}$$

is dense in $\mathcal{H}$. It is proved in [4] that a mapping $F : \Omega \to \mathcal{H}$ is c-Bessel if and only if for all $f \in \mathcal{H}$, $\int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega) < \infty$. A mapping $F \in L^2(\Omega, \mathcal{H})$ is called a c-Riesz basis for $\mathcal{H}$ with respect to (Ω, μ) if F is μ-complete and there are two positive constants A and B such that

$$A\Big(\int_{\Omega_1} |\phi(\omega)|^2 d\mu(\omega)\Big)^{1/2} \leqslant \Big\| \int_{\Omega_1} \phi(\omega)F(\omega)d\mu(\omega)\Big\| \leqslant B\Big(\int_{\Omega_1} |\phi(\omega)|^2 d\mu(\omega)\Big)^{1/2},$$

for every $\phi \in L^2(\Omega)$ and for any measurable subset Ω_1 of Ω with $\mu(\Omega_1) < \infty$. The constants A and B are called the c-Riesz basis bounds.

L. Găvruta in [14] introduced a frame with respect to a bounded linear operator K in a Hilbert space $\mathcal{H}$ to study the atomic decomposition systems. These frames are known as K-frames. For $K \in B(\mathcal{H})$, a sequence $\{f_j\}_{j\in J}$ in a Hilbert space $\mathcal{H}$ is called a K-frame for $\mathcal{H}$, if there exist constants $A, B > 0$ such that for each $f \in \mathcal{H}$,

$$A\|K^* f\|^2 \leqslant \sum_{j\in J} |\langle f, f_j\rangle|^2 \leqslant B\|f\|^2.$$

Some properties of K-frames have been studied in [26].

Y. Huang and D. Hua [16], introduced the concept of K-Riesz basis in Hilbert spaces (see also [27]). A sequence $\{f_j\}_{j\in J}$ in $\mathcal{H}$ is called $\overline{R(K)}$-complete, if

$$\{f \in \mathcal{H} : \langle f, f_j\rangle = 0,\ j \in J\} \subset \overline{R(K)}^{\perp} = N(K^*).$$

A sequence $\{f_j\}_{j\in J}$ is a K-Riesz basis for $\mathcal{H}$, if it is $\overline{R(K)}$-complete and there exist constants $A, B > 0$ such that for all finite scalar sequence $\{c_j\}$,

$$A\sum |c_j|^2 \leqslant \Big\| \sum c_j f_j \Big\|^2 \leqslant B\sum |c_j|^2.$$

Rahimlou, et al., introduced the concept of continuous K-frame (or cK-frame) in [22], and they studied some properties of it in [23].

Definition 1.3. [22] Let $K \in B(\mathcal{H})$ and $F : \Omega \to \mathcal{H}$ be weakly measurable. Then the map F is called a cK-frame for $\mathcal{H}$, if there exist constants $A, B > 0$ such that for each $h \in \mathcal{H}$,

$$A\|K^*h\|^2 \leqslant \int_\Omega |\langle h, F(\omega)\rangle|^2 d\mu(\omega) \leqslant B\|h\|^2.$$

A cK-frame F is called a tight cK-frame, whenever there exists $A > 0$ such that for every $h \in \mathcal{H}$,

$$\int_\Omega |\langle h, F(\omega)\rangle|^2 d\mu(\omega) = A\|K^*h\|^2,$$

and it is called a Parseval cK-frame, whenever for every $h \in \mathcal{H}$,

$$\int_\Omega |\langle h, F(\omega)\rangle|^2 d\mu(\omega) = \|K^*h\|^2.$$

Lemma 1.4. *[22] Let $F : \Omega \to \mathcal{H}$ be weakly measurable, and $K \in B(\mathcal{H})$. Then the following assertions are equivalent:*
(i) F is a cK-frame for $\mathcal{H}$;
(ii) F is a c-Bessel mapping for $\mathcal{H}$ and there exists $G \in B(\mathcal{H}, L^2(\Omega))$ such that

$$Kh = \int_\Omega G(h)(\omega)F(\omega)d\mu(\omega), \quad h \in \mathcal{H}. \tag{1.6}$$

Note 1.5. *Since a cK-frame F is a c-Bessel mapping for $\mathcal{H}$, the operators T_F, T_F^* and S_F can be defined for F. If F is a Parseval cK-frame then $S_F = KK^*$.*

Lemma 1.6. *[7] Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and suppose that $U : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded operator with closed range $R(U)$. Then there exists a bounded operator $U^\dagger : \mathcal{H}_2 \to \mathcal{H}_1$ for which*

$$UU^\dagger f = f, \quad f \in R(U).$$

We call $U^\dagger$ the pseudo-inverse of the operator U.

Theorem 1.7. *[22] Let $K \in B(\mathcal{H})$ and $F : \Omega \to \mathcal{H}$ be a cK-frame for $\mathcal{H}$ with bounds A, B. If K is closed range then S_F is invertible on $R(K)$ and for each $h \in R(K)$,*

$$B^{-1}\|h\|^2 \leqslant \langle (S_F\, |_{R(K)})^{-1}h, h\rangle \leqslant A^{-1}\|K^{\dagger}\|^2\|h\|^2,$$

where $K^{\dagger}$ is the pseudo-inverse of K.

Definition 1.8. [22] Let $K \in B(\mathcal{H})$ and let $F : \Omega \to \mathcal{H}$ and $G : \Omega \to \mathcal{H}$ be c-Bessel mappings for $\mathcal{H}$. We say that G is a cK-dual of F, if for all $h, g \in \mathcal{H}$,

$$\langle Kh, g\rangle = \int_{\Omega} \langle h, G(\omega)\rangle\langle F(\omega), g\rangle d\mu(\omega).$$

Definition 1.9. [20] A function $F : \Omega \to \mathcal{H}$ is called Bochner measurable if there exists a sequence of simple functions $\{F_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \|F_n - F\| = 0$, μ-almost everywhere. A Bochner measurable function $F : \Omega \to \mathcal{H}$ is called Bochner integrable if there exists a sequence of integrable simple functions $\{F_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \int_{\Omega} \|F_n(\omega) - F(\omega)\| d\mu(\omega) = 0.$$

In this case, we have

$$\int_E F(\omega)d\mu(\omega) = \lim_{n\to\infty} \int_E F_n(\omega)d\mu(\omega),$$

where E is a countable set.

Proposition 1.10. *[22] Let $K \in B(\mathcal{H})$ be closed range, and $F : \Omega \to \mathcal{H}$ be a cK-frame for $\mathcal{H}$ and Bochner integrable. Then $\Pi = K^*(S_F\, |_{R(K)})^{-1}\pi_{S_F(R(K))}F$ is a cK-dual of $\pi_{R(K)}F$ with bounds B^{-1} and $A^{-1}\|K\|^2\|K^{\dagger}\|^2$, respectively, where A and B are cK-frame bounds for F. Π is called the standard cK-dual of $\pi_{R(K)}F$.*

What follows have been used in the rest of this paper, in which $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces and $B(\mathcal{H}_1, \mathcal{H}_2)$ is the set of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$.

Definition 1.11. [6] We say that $T \in B(\mathcal{H}_1, \mathcal{H}_2)$ is left-invertible (or, respectively, right-invertible) if there exists a bounded linear operator $G : \mathcal{H}_2 \to \mathcal{H}_1$ such that

$$GT = I_{\mathcal{H}_1}, \quad (\text{or } \; TG = I_{\mathcal{H}_2}).$$

We say that such an operator G is a bounded linear left (or right) inverse of T.

Proposition 1.12. *[6] For $T \in B(\mathcal{H}_1, \mathcal{H}_2)$ the following assertions are equivalent:*
(i) T is injective and $R(T)$ is closed in $\mathcal{H}_2$.
(ii) T is left-invertible.

Lemma 1.13. *[10] Let $T_1 \in B(\mathcal{H}_1, \mathcal{H})$ and $T_2 \in B(\mathcal{H}_2, \mathcal{H})$. The following statements are equivalent:*
(i) $R(T_1) \subset R(T_2)$;
(ii) $T_1T_1^* \leqslant \lambda^2 T_2T_2^*$, for some $\lambda \geqslant 0$; and,
(iii) there exists a bounded operator $T_3 \in B(\mathcal{H}_1, \mathcal{H}_2)$ such that $T_1 = T_2T_3$.

Remark 1.14. In Lemma 1.13, if $T_1 \neq 0$, then $\lambda > 0$.

Lemma 1.15. *[25] Let $T \in B(\mathcal{H}_1, \mathcal{H}_2)$, then*

$$R(T)^{\perp} = N(T^*),\ R(T^*)^{\perp} = N(T),\ \overline{R(T)} = N(T^*)^{\perp},\ \overline{R(T^*)} = N(T)^{\perp}.$$

2. Some properties of cK-frames

In this section, the results shown in [26] for K-frames, are extended to continuous case or cK-frames. Also, inspired by results provided in [13], we show similar consequences for cK-frames. Moreover, we give a necessary and sufficient condition for a mapping F to be a cK-frame. Throughout this section, the orthogonal projection of $\mathcal{H}$ onto a closed subspace $V \subseteq \mathcal{H}$ is denoted by π_V.

Lemma 2.1. *Let $K \in B(\mathcal{H})$ and F be a tight c-frame for $\mathcal{H}$ with bound A. Then KF and K^*F are tight cK-frame and tight cK*-frame for $\mathcal{H}$ with bound A, respectively.*

Proof. For all $f \in \mathcal{H}$, we have

$$\int_{\Omega} |\langle f, KF(\omega)\rangle|^2 d\mu(\omega) = A\|K^* f\|^2,$$

and

$$\int_{\Omega} |\langle f, K^*F(\omega)\rangle|^2 d\mu(\omega) = A\|Kf\|^2.$$

□

Theorem 2.2. *[23] Let $F : \Omega \to \mathcal{H}$ be a c-frame for $\mathcal{H}$ and $K \in B(\mathcal{H})$. Then $KF : \Omega \to \mathcal{H}$ is a cK-frame for $\mathcal{H}$.*

In the following, we provide a trivial example of a cK-frame and a non-trivial example of a Parseval cK-frame.

Example 2.3. From Corollary 8.1.4 in [7], for every $0 \neq g \in L^2(\mathbb{R})$, the family $\{E_b T_a g\}_{a,b\in\mathbb{R}}$ is a c-frame for $L^2(\mathbb{R})$ with respect to $\Omega = \mathbb{R}^2$ and the Lebesgue measure $dbda$. Let $K \in B(L^2(\mathbb{R}))$, then by Theorem 2.2, $\{KE_b T_a g\}_{a,b\in\mathbb{R}}$ is a cK-frame for $L^2(\mathbb{R})$.

Example 2.4. Consider $\mathcal{H} = \mathbb{R}^2$ with the standard basis $\{e_1, e_2\}$, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Put $N_{\mathbb{R}^2} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1\}$ and let $N_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1,\ x \geqslant 0\}$ and $N_2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1,\ x < 0\}$ be partitions of $N_{\mathbb{R}^2}$. Let $\Omega = N_{\mathbb{R}^2}$ and λ be the Lebesgue measure. Define $F : N_{\mathbb{R}^2} \to \mathbb{R}^2$ such that

$$F(\omega) = \begin{cases} \dfrac{1}{\sqrt{\lambda(N_1)}} e_1, & \omega \in N_1, \\[2ex] \dfrac{1}{\sqrt{\lambda(N_2)}} e_2, & \omega \in N_2. \end{cases}$$

It is easy to check that F is a Parseval c-frame for $\mathbb{R}^2$ with respect to $(N_{\mathbb{R}^2}, \lambda)$. Define $K : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$Kf = \langle f, e_1\rangle e_2 + \langle f, e_2\rangle e_1,$$

thus,

$$K^* f = \langle f, e_2\rangle e_1 + \langle f, e_1\rangle e_2.$$

Now, for all $f \in \mathcal{H}$ we have

$$\begin{aligned}
\|K^* f\|^2 &= \|\langle f, e_2\rangle e_1 + \langle f, e_1\rangle e_2\|^2 \\
&= \lambda(N_1)|\langle f, \frac{1}{\sqrt{\lambda(N_1)}} e_1\rangle|^2 + \lambda(N_2)|\langle f, \frac{1}{\sqrt{\lambda(N_2)}} e_2\rangle|^2 \\
&= \int_{N_1} |\langle f, \frac{1}{\sqrt{\lambda(N_1)}} e_1\rangle|^2 d\lambda(\omega) + \int_{N_2} |\langle f, \frac{1}{\sqrt{\lambda(N_2)}} e_2\rangle|^2 d\lambda(\omega) \\
&= \int_{N_{\mathbb{R}^2}} |\langle f, F(\omega)\rangle|^2 d\lambda(\omega).
\end{aligned}$$

This shows that F is a Parseval cK-frame for $\mathbb{R}^2$.

Proposition 2.5. *Let $T \in B(\mathcal{H})$ and F be a cK-frame for $\mathcal{H}$ with bounds A and B. Then TF is a cTK-frame for $\mathcal{H}$.*

Proof. We have

$$\int_\Omega |\langle f, TF(\omega)\rangle|^2 d\mu(\omega) = \int_\Omega |\langle T^* f, F(\omega)\rangle|^2 d\mu(\omega) \leqslant B\|T\|^2.\|f\|^2, \ \ f \in \mathcal{H}.$$

Also,

$$A\|(TK)^* f\|^2 = A\|K^* T^* f\|^2 \leqslant \int_\Omega |\langle T^* f, F(\omega)\rangle|^2 d\mu(\omega) = \int_\Omega |\langle f, TF(\omega)\rangle|^2 d\mu(\omega).$$

So, TF is a cTK-frame for $\mathcal{H}$ with bounds A and $B\|T\|^2$. □

Proposition 2.6. *Suppose that $F : \Omega \to \mathcal{H}$ is a cK-frame for $\mathcal{H}$ with respect to (Ω, μ). If $F : \Omega \to \mathcal{H}$ is L^2-independent, then the mapping $G \in B(\mathcal{H}, L^2(\Omega))$ satisfying (1.6) is unique.*

Proof. Assume that there exists $D \in B(\mathcal{H}, L^2(\Omega))$ such that

$$Kh = \int_\Omega D(h)(\omega)F(\omega)d\mu(\omega), \ \ h \in \mathcal{H}.$$

Then we have

$$\int_\Omega (G(h) - D(h))(\omega)F(\omega)d\mu(\omega) = 0, \ \ h \in \mathcal{H}.$$

Since F is L^2-independent, $G(h) - D(h) = 0$ almost everywhere. Since $h \in \mathcal{H}$ is arbitary, so we get $G = D$. Hence the mapping $G \in B(\mathcal{H}, L^2(\Omega))$ satisfying (1.6) is unique. □

Theorems 2.7 and 2.9 are similar to Proposition 2.6 and corollary 2.7 in [13].

Theorem 2.7. *Let $\{e_j\}_{j\in J}$ be an orthonormal basis for $\mathcal{H}$ and let $K \in B(\mathcal{H})$ be an invertible operator. Then the following are equivalent:*
(i) F is a Parseval cK-frame for $\mathcal{H}$,
(ii) there exists an orthonormal set $\{\psi_j\}$ in $L^2(\Omega)$ having the property that $\sum_{j\in J}|\psi_j(\omega)|^2 < \infty$ for a.e. $\omega \in \Omega$ such that $F(\omega) = \sum_{j\in J}\psi_j(\omega)Ke_j$ holds for a.e. $\omega \in \Omega$.

Proof. Assume that F is a Parseval cK-frame for $\mathcal{H}$. Let T_F^* be the analysis operator for F defined by (1.4), and write $\psi_j = \overline{\phi_j}$, where $\phi_j = T_F^*(K^{-1})^*e_j$. For all $i, j \in J$, by Note 1.5 we have

$$\begin{aligned}\langle\psi_j, \psi_i\rangle = \langle\phi_i, \phi_j\rangle = \langle T_F^*(K^{-1})^*e_i, T_F^*(K^{-1})^*e_j\rangle &= \langle T_F T_F^*(K^{-1})^*e_i, (K^{-1})^*e_j\rangle \\ &= \langle K^{-1}S_F(K^{-1})^*e_i, e_j\rangle \\ &= \langle K^{-1}KK^*(K^{-1})^*e_i, e_j\rangle \\ &= \langle e_i, e_j\rangle.\end{aligned}$$

Therefore, $\{\psi_j\}_{j\in J}$ is an orthonormal set. Also, we have

$$\begin{aligned}\sum_{j\in J}|\psi_j(\omega)|^2 = \sum_{j\in J}|\overline{\phi_j(\omega)}|^2 = \sum_{j\in J}|\overline{(T_F^*(K^{-1})^*e_j)(\omega)}|^2 &\\ = \sum_{j\in J}|\langle F(\omega), (K^{-1})^*e_j\rangle|^2 &\\ = \sum_{j\in J}|\langle K^{-1}F(\omega), e_j\rangle|^2 &\\ = \|K^{-1}F(\omega)\|^2 &\\ \leqslant \|K^{-1}\|^2.\|F(\omega)\|^2 < \infty, &\end{aligned}$$

and

$$\begin{aligned}\sum_{j\in J}\psi_j(\omega)e_j = \sum_{j\in J}\overline{\phi_j(\omega)}e_j = \sum_{j\in J}\overline{(T_F^*(K^{-1})^*e_j)(\omega)}e_j &\\ = \sum_{j\in J}\overline{\langle(K^{-1})^*e_j, F(\omega)\rangle}e_j &\\ = \sum_{j\in J}\langle K^{-1}F(\omega), e_j\rangle e_j &\\ = K^{-1}F(\omega). &\end{aligned}$$

So, $F(\omega) = \sum_{j\in J}\psi_j(\omega)Ke_j$.

Conversly if (ii) holds then for any $f \in \mathcal{H}$, we have

$$\begin{aligned}\int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega) = \int_\Omega \Big|\Big\langle f, \sum_{j\in J}\psi_j(\omega)Ke_j\Big\rangle\Big|^2 d\mu(\omega) &= \int_\Omega \Big|\sum_{j\in J}\langle f, Ke_j\rangle\overline{\psi_j}(\omega)\Big|^2 d\mu(\omega)\\ &= \Big\|\sum_{j\in J}\langle f, Ke_j\rangle\overline{\psi_j}\Big\|_2^2\\ &= \sum_{j\in J}|\langle K^* f, e_j\rangle|^2\\ &= \|K^* f\|^2.\end{aligned}$$

□

Corollary 2.8. *Theorem 2.7 holds also for a unitary operator $K \in B(\mathcal{H})$. It is enough to put $\phi_j = T_F^* K e_j$.*

Theorem 2.9. *Let $0 \neq K \in B(\mathcal{H})$ be an invertible operator. Then the following are equivalent:*
(i) F is a cK-frame for $\mathcal{H}$;
(ii) $F(\omega) = \sum_{j\in J}\psi_j(\omega)Ke_j$ for some orthonormal basis $\{e_j\}_{j\in J}$ of $\mathcal{H}$ and some family $\{\psi_j\}_{j\in J}$ in $L^2(\Omega)$ with the properties that $\{\psi_j\}_{j\in J}$ is a Riesz basis for $\overline{span}\{\psi_j\}_{j\in J}$ and that $\sum_{j\in J}|\psi_j(\omega)|^2 < \infty$ for a.e. $\omega \in \Omega$.
(iii) $F(\omega) = \sum_{j\in J}\psi_j(\omega)g_j$ for some K-Riesz basis $\{g_j\}_{j\in J}$ of $\mathcal{H}$ and some family $\{\psi_j\}_{j\in J}$ in $L^2(\Omega)$ with the proprties that $\{\psi_j\}_{j\in J}$ is a Riesz basis for $\overline{span}\{\psi_j\}_{j\in J}$ and that $\sum_{j\in J}|\psi_j(\omega)|^2 < \infty$ for a.e. $\omega \in \Omega$.

Proof. $(i) \Rightarrow (ii)$. Let F be a cK-frame for $\mathcal{H}$ with bounds A and B. Let $\{e_j\}_{j\in J}$ be an orthonormal basis for $\mathcal{H}$ and put $\psi_j = \overline{\phi_j}$ where $\phi_j = T_F^*(K^{-1})^* e_j$. Then for any finite scalar sequence $\{c_j\}$ we have

$$\begin{aligned}\Big\|\sum c_j\psi_j\Big\|^2 = \Big\|\sum c_j\overline{\phi_j}\Big\|^2 = \Big\|\overline{\sum \overline{c_j}\phi_j}\Big\|^2 &= \Big\|\overline{T_F^*\Big(\sum \overline{c_j}(K^{-1})^* e_j\Big)}\Big\|_2^2\\ &= \int_\Omega \Big|\overline{\Big\langle \sum \overline{c_j}(K^{-1})^* e_j, F(\omega)\Big\rangle}\Big|^2 d\mu(\omega)\\ &= \int_\Omega \Big|\Big\langle F(\omega), \sum \overline{c_j}(K^{-1})^* e_j\Big\rangle\Big|^2 d\mu(\omega).\end{aligned}$$

Since F is a cK-frame, we have

$$A\Big\|K^*\Big(\sum \overline{c_j}(K^{-1})^* e_j\Big)\Big\|^2 \leqslant \Big\|\sum c_j\psi_j\Big\|^2 \leqslant B\Big\|\sum \overline{c_j}(K^{-1})^* e_j\Big\|^2.$$

Then,

$$\begin{aligned}A\sum |c_j|^2 = A\Big\|\sum \overline{c_j}e_j\Big\|^2 = A\Big\|K^*\Big(\sum \overline{c_j}(K^{-1})^* e_j\Big)\Big\|^2 &\leqslant \Big\|\sum c_j\psi_j\Big\|^2\\ &\leqslant B\Big\|\sum \overline{c_j}(K^{-1})^* e_j\Big\|^2\\ &\leqslant B\|K^{-1}\|^2\sum |c_j|^2.\end{aligned}$$

Therefore, $\{\psi_j\}_{j\in J}$ is a Riesz basis for $\overline{span}\{\psi_j\}_{j\in J}$ with bounds A and $B\|K^{-1}\|^2$. Moreover, $F(\omega) = \sum_{j\in J}\psi_j(\omega)Ke_j$ and $\sum_{j\in J}|\psi_j(\omega)|^2 < \infty$ for a.e. $\omega \in \Omega$.
$(ii) \Rightarrow (i)$. Since $\{\psi_j\}_{j\in J}$ is a Riesz basis for $\overline{span}\{\psi_j\}_{j\in J}$, there exist $0 < A \leqslant B < \infty$ such that

$$A\sum|c_j|^2 \leqslant \left\|\sum c_j\psi_j\right\|^2 \leqslant B\sum|c_j|^2, \quad \{c_j\}_{j\in J}\in l^2(J).$$

Then, for all $f \in \mathcal{H}$ we have

$$\begin{aligned} A\|K^*f\|^2 = A\sum_{j\in J}|\langle e_j, K^*f\rangle|^2 &\leqslant \left\|\sum_{j\in J}\langle e_j, K^*f\rangle\psi_j\right\|_2^2 \\ &\leqslant B\sum_{j\in J}|\langle e_j, K^*f\rangle|^2 \\ &= B\|K^*f\|^2 \leqslant B\|K\|^2.\|f\|^2, \end{aligned}$$

and

$$\begin{aligned} \int_\Omega|\langle f, F(\omega)\rangle|^2 d\mu(\omega) = \int_\Omega|\langle F(\omega), f\rangle|^2 d\mu(\omega) &= \int_\Omega\left|\left\langle\sum_{j\in J}\psi_j(\omega)Ke_j, f\right\rangle\right|^2 d\mu(\omega) \\ &= \int_\Omega\left|\sum_{j\in J}\psi_j(\omega)\langle Ke_j, f\rangle\right|^2 d\mu(\omega) \\ &= \left\|\sum_{j\in J}\langle e_j, K^*f\rangle\psi_j\right\|_2^2. \end{aligned}$$

Therefore,

$$A\|K^*f\|^2 \leqslant \left\|\sum_{j\in J}\langle e_j, K^*f\rangle\psi_j\right\|_2^2 = \int_\Omega|\langle f, F(\omega)\rangle|^2 d\mu(\omega) \leqslant B\|K\|^2.\|f\|^2,$$

so, F is a cK-frame for $\mathcal{H}$ with bounds A and $B\|K\|^2$.
$(ii) \Rightarrow (iii)$. Put $\psi_j = \overline{\phi_j}$ where $\phi_j = T_F^*(K^{-1})^*e_j$. According to $(i) \Rightarrow (ii)$, $\{\psi_j\}_{j\in J}$ is a Riesz basis for $\overline{span}\{\psi_j\}_{j\in J}$ and $\sum_{j\in J}|\psi_j(\omega)|^2 < \infty$ for a.e. $\omega \in \Omega$. Also, $F(\omega) = \sum_{j\in J}\psi_j(\omega)g_j$ where $g_j = Ke_j$ for all $j \in J$ and $\{g_j\}_{j\in J}$ is a K-Riesz basis for $\mathcal{H}$ with bounds $\|K^{-1}\|^{-2}$ and $\|K\|^2$, recpectively.
$(iii) \Rightarrow (i)$. $\{g_j\}_{j\in J}$ is a K-Riesz basis for $\mathcal{H}$, so it is a K-frame ([24]), then there exist $C, D > 0$ such that

$$C\|K^*f\|^2 \leqslant \sum_{j\in J}|\langle f, g_j\rangle|^2 \leqslant D\|f\|^2.$$

On the other hand, let $A, B > 0$ be the bounds of Riesz basis $\{\psi_j\}_{j\in J}$. Then,

$$A\sum|c_j|^2 \leqslant \left\|\sum c_j\psi_j\right\|^2 \leqslant B\sum|c_j|^2, \quad \{c_j\}_{j\in J}\in l^2(J).$$

For all $f \in \mathcal{H}$, we have

$$\begin{aligned}\int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega) = \int_\Omega |\langle F(\omega), f\rangle|^2 d\mu(\omega) &= \int_\Omega \Big|\Big\langle \sum_{j\in J} \psi_j(\omega) g_j, f\Big\rangle\Big|^2 d\mu(\omega)\\ &= \int_\Omega \Big|\sum_{j\in J} \psi_j(\omega)\langle g_j, f\rangle\Big|^2 d\mu(\omega)\\ &= \Big\|\sum_{j\in J}\langle g_j, f\rangle\psi_j\Big\|_2^2.\end{aligned}$$

Moreover,

$$AC\|K^* f\|^2 \leqslant A\sum_{j\in J} |\langle g_j, f\rangle|^2 \leqslant \Big\|\sum_{j\in J}\langle g_j, f\rangle\psi_j\Big\|_2^2 \leqslant B\sum_{j\in J} |\langle g_j, f\rangle|^2 \leqslant BD\|f\|^2,$$

for all $f \in \mathcal{H}$. Therefore,

$$AC\|K^* f\|^2 \leqslant \int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega) \leqslant BD\|f\|^2.$$

□

Corollary 2.10. *Theorem 2.9 also holds for any unitary operator $K \in B(\mathcal{H})$.*

Proposition 2.11. *Let $0 \neq K \in B(\mathcal{H})$ be an invertible operator and let $F(\omega) = \sum_{j\in J} \psi_j(\omega) g_j$ in which $\{g_j\}_{j\in J}$ is a K-frame for $\mathcal{H}$ and $\{\psi_j\}_{j\in J}$ is an orthonormal set in $L^2(\Omega)$ with the property that $\sum_{j\in J} |\psi_j(\omega)|^2 < \infty$, for a.e. $\omega \in \Omega$ then F is a cK-frame for $\mathcal{H}$.*

Proof. For all $f \in \mathcal{H}$ we have

$$\begin{aligned}\int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega) &= \int_\Omega |\langle F(\omega), f\rangle|^2 d\mu(\omega)\\ &= \int_\Omega \Big|\Big\langle \sum_{j\in J} \psi_j(\omega) g_j, f\Big\rangle\Big|^2 d\mu(\omega)\\ &= \int_\Omega \Big|\sum_{j\in J} \psi_j(\omega)\langle g_j, f\rangle\Big|^2 d\mu(\omega)\\ &= \Big\|\sum_{j\in J}\langle g_j, f\rangle\psi_j\Big\|_2^2\\ &= \sum_{j\in J} |\langle g_j, f\rangle|^2.\end{aligned}$$

Then there exist $A, B > 0$ such that

$$A\|K^* f\|^2 \leqslant \sum_{j\in J} |\langle g_j, f\rangle|^2 = \int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega) \leqslant B\|f\|^2.$$

□

Theorem 2.12. *Let $0 \neq K \in B(\mathcal{H})$ and $\{\psi_i\}_{i\in I}$ be an orthonormal basis for $L^2(\Omega)$ and $F : \Omega \to \mathcal{H}$ be weakly measurable. Then F is a cK-frame for $\mathcal{H}$ if and only if there exists a bounded linear operator $\Theta : L^2(\Omega) \to \mathcal{H}$ such that*

$$\langle x, \Theta\psi_i\rangle = \Big\langle \langle x, F(\cdot)\rangle, \psi_i \Big\rangle, \;\; x \in \mathcal{H}, \;\; i \in I,$$

and $R(K) \subset R(\Theta)$.

Proof. First, we suppose that F is a cK-frame for $\mathcal{H}$. Then, there exist $A, B > 0$ such that

$$A\|K^* f\|^2 \leqslant \int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega) \leqslant B\|f\|^2, \;\; f \in \mathcal{H}. \tag{2.1}$$

Let the mapping $\Phi : \mathcal{H} \to L^2(\Omega)$ be defined by $\Phi f = \langle f, F(.)\rangle$, for all $f \in \mathcal{H}$. By (2.1), we have

$$\|\Phi f\|_2^2 = \int_\Omega |\Phi f(\omega)|^2 d\mu(\omega) = \int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega) \leqslant B\|f\|^2, \;\; f \in \mathcal{H}.$$

Hence, Φ is a bounded linear operator. Also, by (2.1) we have $A\|K^* f\|^2 \leqslant \|\Phi f\|_2^2$, for all $f \in \mathcal{H}$. Then, $AKK^* \leqslant \Phi^*\Phi$, so by Lemma 1.13, $R(K) \subset R(\Theta)$, where $\Theta = \Phi^*$. Also, for all $x \in \mathcal{H}$, we have

$$\langle x, \Theta\psi_i\rangle = \langle x, \Phi^*\psi_i\rangle = \langle \Phi x, \psi_i\rangle = \Big\langle \langle x, F(\cdot)\rangle, \psi_i \Big\rangle, \;\; i \in I.$$

Conversely, for all $f \in \mathcal{H}$ and for all $i \in I$ we have

$$\langle \Theta^* f, \psi_i\rangle = \langle f, \Theta\psi_i\rangle = \Big\langle \langle f, F(\cdot)\rangle, \psi_i \Big\rangle.$$

So, $\Theta^* f = \langle f, F(\cdot)\rangle$, for all $f \in \mathcal{H}$. F is a c-Bessel mapping, since

$$\int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega) = \|\Theta^* f\|^2 \leqslant \|\Theta\|^2.\|f\|^2.$$

Since, $R(K) \subset R(\Theta)$, Lemma 1.13 and Remark 1.14 imply that there exists $\lambda > 0$ such that $KK^* \leqslant \lambda\Theta\Theta^*$. Then

$$\frac{1}{\lambda}\|K^* f\|^2 \leqslant \|\Theta^* f\|^2 = \int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega), \;\; f \in \mathcal{H}.$$

Therefore, F is a cK-frame for $\mathcal{H}$ with bounds $\frac{1}{\lambda}$ and $\|\Theta\|^2$, respectively. □

3. cK-Riesz bases

In this section, considering the method provided in [4], we introduce the concept of continuous K-Riesz basis or cK-Riesz bases. We show that every cK-Riesz basis is a cK-frame, but a cK-frame can not be a cK-Riesz basis unless some certain conditions satisfied. Moreover, we give the definition of Riesz-type cK-frames and we show that when a Riesz-type cK-frame can be a cK-Riesz basis.

Definition 3.1. Let $K \in B(\mathcal{H})$ and put $M = \left\{ \int_\Omega \phi(\omega)F(\omega)d\mu(\omega)\, ;\, \phi \in L^2(\Omega) \right\}$. A c-Bessel mapping $F : \Omega \to \mathcal{H}$ is called $\overline{R(K)}$-μ-complete, if

$$\overline{R(K)} \subset \overline{M} \tag{3.1}$$

Proposition 3.2. *Let $F \in L^2(\Omega, \mathcal{H})$. Then the following are equivalent:*
(i) F is $\overline{R(K)}$-μ-complete,
(ii) if $f \in \mathcal{H}$ so that $\langle f, F(\omega)\rangle = 0$ for almost all $\omega \in \Omega$, then $f \in N(K^*)$.

Proof. $(i) \Rightarrow (ii)$. Assume that $f \in \mathcal{H}$ and $\langle f, F(\omega)\rangle = 0$, for almost all $\omega \in \Omega$. Then

$$\Big\langle f, \int_\Omega \phi(\omega)F(\omega)d\mu(\omega)\Big\rangle = \int_\Omega \overline{\phi(\omega)}\langle f, F(\omega)\rangle d\mu(\omega) = 0,$$

for all $\phi \in L^2(\Omega)$. Thus $f \in M^\perp$. By (3.1) we have $M^\perp \subset \overline{R(K)}^\perp$ and Lemma 1.15 implies that $f \in N(K^*)$.
$(ii) \Rightarrow (i)$. Let $f \in \mathcal{H}$ be such that $\left\langle f, \int_\Omega \phi(\omega)F(\omega)d\mu(\omega)\right\rangle = 0$ for all $\phi \in L^2(\Omega)$. Taking $\phi = \langle f, F(\cdot)\rangle \in L^2(\Omega)$, we get

$$0 = \Big\langle \int_\Omega \phi(\omega)F(\omega)d\mu(\omega), f\Big\rangle = \int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega),$$

which shows that $\langle f, F(\omega)\rangle = 0$ for almost all $\omega \in \Omega$. Hence $f \in N(K^*)$. Thus,

$$M^\perp \subset \overline{R(K)}^\perp,$$

therefore, $\overline{R(K)} \subset \overline{M}$. □

Definition 3.3. A mapping $F \in L^2(\Omega, \mathcal{H})$ is called a cK-Riesz basis for $\mathcal{H}$ with respect to (Ω, μ) if F is $\overline{R(K)}$-μ-complete and there are two positive numbers A and B such that

$$A\Big(\int_{\Omega_1} |\phi(\omega)|^2 d\mu(\omega)\Big)^{1/2} \leqslant \Big\| \int_{\Omega_1} \phi(\omega)F(\omega)d\mu(\omega)\Big\| \leqslant B\Big(\int_{\Omega_1} |\phi(\omega)|^2 d\mu(\omega)\Big)^{1/2}, \tag{3.2}$$

for every $\phi \in L^2(\Omega)$ and for any measurable subset Ω_1 of Ω with $\mu(\Omega_1) < \infty$. The integral is taken in the weak sense and the constants A and B are called cK-Riesz basis bounds.

Remark 3.4. Let F be a cK-Riesz basis for $\mathcal{H}$. Since (Ω, μ) is σ-finite,

$$A\Big(\int_{\Omega} |\phi(\omega)|^2 d\mu(\omega)\Big)^{1/2} \leqslant \Big\| \int_{\Omega} \phi(\omega)F(\omega)d\mu(\omega)\Big\| \leqslant B\Big(\int_{\Omega} |\phi(\omega)|^2 d\mu(\omega)\Big)^{1/2}.$$

It means that (3.2) holds for whole Ω.

Theorem 3.5. *Let $0 \neq K \in B(\mathcal{H})$ and $\{\psi_i\}_{i\in I}$ be an orthonormal basis for $L^2(\Omega)$ and $F \in L^2(\Omega, \mathcal{H})$. If F is a cK-Riesz basis for $\mathcal{H}$, then there exists a bounded left-invertible operator $\Gamma_F : L^2(\Omega) \to \mathcal{H}$ such that*

$$\langle x, \Gamma_F \psi_i\rangle = \Big\langle \langle x, F(\cdot)\rangle, \psi_i\Big\rangle, \ \ x \in \mathcal{H}, \ \ i \in I,$$

and $R(K) \subset R(\Gamma_F)$.

Proof. Suppose that $F \in L^2(\Omega, \mathcal{H})$ is a cK-Riesz basis for $\mathcal{H}$. Then, it is $\overline{R(K)}$-μ-complete and there exist $A, B > 0$ such that for every $\phi \in L^2(\Omega)$,

$$A\Big(\int_\Omega |\phi(\omega)|^2 d\mu(\omega)\Big)^{1/2} \leqslant \Big\| \int_\Omega \phi(\omega)F(\omega)d\mu(\omega)\Big\| \leqslant B\Big(\int_\Omega |\phi(\omega)|^2 d\mu(\omega)\Big)^{1/2}. \tag{3.3}$$

Define $\Gamma_F : L^2(\Omega) \to \mathcal{H}$ by

$$\Gamma_F(\varphi) = \int_\Omega \varphi(\omega)F(\omega)d\mu(\omega).$$

$F \in L^2(\Omega, \mathcal{H})$ implies that F is a c-Bessel mapping, let D be its Bessel bound. Then by (3.3), we have

$$\begin{aligned}\|\Gamma_F(\varphi)\| = \sup_{h\in\mathcal{H},\|h\|=1} |\langle \Gamma_F\varphi, h\rangle| &= \sup_{h\in\mathcal{H},\|h\|=1} \Big|\Big\langle \int_\Omega \varphi(\omega)F(\omega)d\mu(\omega), h\Big\rangle\Big| \\ &\leqslant \sup_{h\in\mathcal{H},\|h\|=1} \Big(\int_\Omega |\varphi(\omega)|^2 d\mu(\omega)\Big)^{1/2}\Big(\int_\Omega |\langle h, F(\omega)|^2 d\mu(\omega)\Big)^{1/2} \\ &\leqslant D^{1/2}\|\varphi\|_2.\end{aligned}$$

Then, Γ_F is well-defined and bounded. Also, (3.3) implies that

$$A\|\varphi\|_2 \leqslant \|\Gamma_F(\varphi)\| \leqslant B\|\varphi\|_2, \quad \varphi \in L^2(\Omega). \tag{3.4}$$

By (3.4) we conclude that Γ_F is injective and $R(\Gamma_F)$ is a closed subspace of $\mathcal{H}$. Therefore, by Proposition 1.5, Γ_F is left-invertible. We have

$$\Gamma_F^* : \mathcal{H} \to L^2(\Omega), \quad (\Gamma_F^* f)(\omega) = \langle f, F(\omega)\rangle. \tag{3.5}$$

Now, if $f \in \mathcal{H}$ and $\Gamma_F^*(f) = 0$, then $\langle f, F(\omega)\rangle = 0$, for all $\omega \in \Omega$. Thus by Proposition 3.2, $f \in N(K^*)$. It means that $N(\Gamma_F^*) \subset N(K^*)$, so $N(K^*)^\perp \subset N(\Gamma_F^*)^\perp$. By Lemma 1.13, $R(K) \subset \overline{R(K)} \subset \overline{R(\Gamma_F)} = R(\Gamma_F)$. Moreover, for all $x \in \mathcal{H}$ and for all $i \in I$ we have

$$\Big\langle \langle x, F(\cdot)\rangle, \psi_i\Big\rangle = \int_\Omega \langle x, F(\omega)\rangle\overline{\psi_i(\omega)}d\mu(\omega) = \Big\langle x, \int_\Omega \psi_i(\omega)F(\omega)d\mu(\omega)\Big\rangle = \langle x, \Gamma_F(\psi_i)\rangle$$

□

Proposition 3.6. *Let $0 \neq K \in B(\mathcal{H})$. If $F \in L^2(\Omega, \mathcal{H})$ is a cK-Riesz basis for $\mathcal{H}$, then it is a cK-frame for $\mathcal{H}$.*

Proof. Let $F \in L^2(\Omega, \mathcal{H})$ be a cK-Riesz basis for $\mathcal{H}$, then it is a c-Bessel mapping with bound D and by Theorem 3.5, there exists a bounded left-invertible operator $\Gamma_F \in B(L^2(\Omega), \mathcal{H})$ such that $R(K) \subset R(\Gamma_F)$ and $T_F^* f = \langle f, F(\cdot)\rangle$. By Lemma 1.13, there exists $\lambda > 0$ such that $KK^* \leqslant \lambda^2\Gamma_F\Gamma_F^*$. Then,

$$\|K^* f\|^2 \leqslant \lambda^2\|\Gamma_F^* f\|^2 = \lambda^2\|\{\langle f, F(\cdot)\rangle\}\|^2 = \lambda^2\int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega), \quad f \in \mathcal{H}.$$

Therefore, F is a cK-frame for $\mathcal{H}$ with bounds $\dfrac{1}{\lambda^2}$ and D. □

Theorem 3.7. *Let $F : \Omega \to \mathcal{H}$ be a cK-frame for $\mathcal{H}$ with respect to (Ω, μ). Then F is a cK-Riesz basis for $\mathcal{H}$ if and only if $R(T_F)$ is closed and F is L^2-independent.*

Proof. First let $F : \Omega \to \mathcal{H}$ be a cK-Riesz basis for $\mathcal{H}$ with respect to (Ω, μ), then it is $\overline{R(K)}$-μ-complete and there exist $A, B > 0$ such that for every $\phi \in L^2(\Omega)$,

$$A\Big(\int_\Omega |\phi(\omega)|^2 d\mu(\omega)\Big)^{1/2} \leqslant \Big\|\int_\Omega \phi(\omega)F(\omega)d\mu(\omega)\Big\| \leqslant B\Big(\int_\Omega |\phi(\omega)|^2 d\mu(\omega)\Big)^{1/2}. \tag{3.6}$$

Let $\phi \in L^2(\Omega)$ and $\int_\Omega \phi(\omega)F(\omega)d\mu(\omega) = 0$, then by (3.6), $\phi = 0$ almost everywhere. Thus, F is L^2-independent. Also from (3.6) we have

$$A\|\phi\|_2 \leqslant \|T_F(\phi)\| \leqslant B\|\phi\|_2,$$

so we conclude that $R(T_F)$ is closed.
Conversely, let $R(T_F)$ be closed and F be L^2-independent. Since F is a cK-frame, then there exist $A, B > 0$ such that for all $f \in \mathcal{H}$,

$$A\|K^* f\|^2 \leqslant \int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega) \leqslant B\|f\|^2.$$

Thus if $\langle f, F(\omega)\rangle = 0$ for almost all $\omega \in \Omega$, then $A\|K^* f\|^2 = 0$, so $f \in N(K^*)$. Therefore, F is $\overline{R(K)}$-μ-complete. Also, for all $\phi \in L^2(\Omega)$ we have

$$\begin{aligned}\Big\|\int_\Omega \phi(\omega)F(\omega)d\mu(\omega)\Big\| &= \sup_{\|f\|=1} \Big|\Big\langle \int_\Omega \phi(\omega)F(\omega)d\mu(\omega), f\Big\rangle\Big| \\ &\leqslant \sup_{\|f\|=1} \Big(\int_\Omega |\phi(\omega)\langle F(\omega), f\rangle| d\mu(\omega)\Big) \\ &\leqslant \sup_{\|f\|=1} \Big(\int_\Omega |\phi(\omega)|^2 d\mu(\omega)\Big)^{1/2}.\Big(\int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega)\Big)^{1/2} \\ &\leqslant B^{1/2}\Big(\int_\Omega |\phi(\omega)|^2 d\mu(\omega)\Big)^{1/2}.\end{aligned}$$

Since F is L^2-independent, $N(T_F) = \{0\}$ and since $R(T_F)$ is closed, by Proposition 1.12, there exists a bounded operator $(T_F)_l^{-1} : \mathcal{H} \to L^2(\Omega)$ such that for any $\phi \in L^2(\Omega)$, $(T_F)_l^{-1} T_F(\phi) = \phi$. Then

$$\int_\Omega |\phi(\omega)|^2 d\mu(\omega) = \|\phi\|_2^2 = \|(T_F)_l^{-1} T_F(\phi)\|^2 \leqslant \|(T_F)_l^{-1}\|^2 \|T_F(\phi)\|^2 = \|(T_F)_l^{-1}\|^2 \Big\|\int_\Omega \phi(\omega)F(\omega)d\mu(\omega)\Big\|^2,$$

for all $\phi \in L^2(\Omega)$. Thus

$$\|(T_F)_l^{-1}\|^{-1}\Big(\int_\Omega |\phi(\omega)|^2 d\mu(\omega)\Big)^{1/2} \leqslant \Big\|\int_\Omega \phi(\omega)F(\omega)d\mu(\omega)\Big\|, \quad \phi \in L^2(\Omega).$$

Therefore, F is a cK-Riesz basis with bounds $\|(T_F)_l^{-1}\|^{-1}$ and $B^{1/2}$, respectively. □

Definition 3.8. Let $K \in B(\mathcal{H})$ be closed range. We say that $F : \Omega \to \mathcal{H}$ is a Riesz-type cK-frame, if $\pi_{R(K)}F$ has only one dual.

Theorem 3.9. *Let $K \in B(\mathcal{H})$ be closed range and F be a cK-frame and Bochner integrable. Then F is a Riesz-type cK-frame if and only if $R(T^*_{\pi_{R(K)}F}) = L^2(\Omega)$.*

Proof. Let Π be the standard cK-dual of $\pi_{R(K)}F$ and assume that $R(T^*_{\pi_{R(K)}F}) \neq L^2(\Omega)$. We can then choose $h \in (R(T^*_{\pi_{R(K)}F}))^{\perp}$ such that $\|h\| = 1$. Define $D(\omega) = h(\omega)h$ for each $\omega \in \Omega$. Then, for each $f \in L^2(\Omega)$, we have

$$\int_\Omega |\langle f, D(\omega)\rangle|^2 d\mu(\omega) = \int_\Omega |\langle f, h\rangle|^2 |h(\omega)|^2 d\mu(\omega) = |\langle f, h\rangle|^2 \leqslant \|f\|^2.$$

Thus D is c-Bessel. Let $V : L^2(\Omega) \to \mathcal{H}$ be a bounded linear operator such that $Vh \neq 0$. For all $f \in \mathcal{H}$, we have

$$\int_\Omega |\langle f, VD(\omega)\rangle|^2 d\mu(\omega) = \int_\Omega |\langle V^* f, D(\omega)\rangle|^2 d\mu(\omega) \leqslant \|V\|^2 \|f\|^2,$$

so VD is a c-Bessel mapping as well. By Proposition 1.10,

$$\begin{aligned}\int_\Omega |\langle f, (\Pi(\omega) + VD(\omega)\rangle|^2 d\mu(\omega) &\leqslant 2\int_\Omega |\langle f, \Pi(\omega)\rangle|^2 d\mu(\omega) + 2\int_\Omega |\langle f, VD(\omega)\rangle|^2 d\mu(\omega) \\ &\leqslant 2A^{-1}\|K\|^2\|K^\dagger\|^2\|f\|^2 + 2\|V\|^2\|f\|^2, \quad f \in \mathcal{H},\end{aligned}$$

hence, $\Pi + VD$ is a c-Bessel mapping. Note that for each $x \in \mathcal{H}$,

$$\langle x, VD(\omega)\rangle = \langle V^* x, D(\omega)\rangle = \langle V^* x, h\rangle \overline{h(\omega)}.$$

Since h is orthogonal to $R(T^*_{\pi_{R(K)}F})$, it follows that

$$\begin{aligned}\int_\Omega \langle x, VD(\omega)\rangle \langle \pi_{R(K)}F(\omega), y\rangle d\mu(\omega) &= \langle V^* x, h\rangle \int_\Omega \overline{h(\omega)} \langle \pi_{R(K)}F(\omega), y\rangle d\mu(\omega) \\ &= \langle V^* x, h\rangle \Big\langle \langle \pi_{R(K)}F(\cdot), y\rangle, h\Big\rangle \\ &= 0,\end{aligned}$$

for all $x, y \in \mathcal{H}$. Thus , for all $x, y \in \mathcal{H}$ we have

$$\begin{aligned}\int_\Omega \langle x, \Pi(\omega) + VD(\omega)\rangle \langle \pi_{R(K)}F(\omega), y\rangle d\mu(\omega) &= \int_\Omega \langle x, \Pi(\omega)\rangle \langle \pi_{R(K)}F(\omega), y\rangle d\mu(\omega) \\ &\quad + \int_\Omega \langle x, VD(\omega)\rangle \langle \pi_{R(K)}F(\omega), y\rangle d\mu(\omega) \\ &= \int_\Omega \langle x, \Pi(\omega)\rangle \langle \pi_{R(K)}F(\omega), y\rangle d\mu(\omega) \\ &= \langle Kx, y\rangle.\end{aligned}$$

Therefore $\Pi + VD$ is a dual of $\pi_{R(K)}F$ which is different from the standard dual. Then F is not a Riesz-type cK-frame.
Conversely, assume that $\pi_{R(K)}F$ admits two different duals G_1 and G_2 and consider $G_3 = G_1 - G_2 \neq 0$. Then, for all $x, y \in \mathcal{H}$,

$$\int_\Omega \langle x, G_3(\omega)\rangle\langle \pi_{R(K)}F(\omega), y\rangle d\mu(\omega) = \int_\Omega \langle x, G_1(\omega) - G_2(\omega)\rangle\langle \pi_{R(K)}F(\omega), y\rangle d\mu(\omega) = 0.$$

Thus $R(T^*_{G_3}) \perp R(T^*_{\pi_{R(K)}F})$, which implies that $(R(T^*_{\pi_{R(K)}F}))^\perp \neq 0$, so $R(T^*_{\pi_{R(K)}F}) \neq L^2(\Omega)$. □

Theorem 3.10. *Let $K \in B(\mathcal{H})$ be closed range and let $F : \Omega \to \mathcal{H}$ be a cK-frame and Bochner intagrable. Then $\pi_{R(K)}F$ is a cK-Riesz basis if and only if it has only one dual.*

Proof. Let $\pi_{R(K)}F$ be a cK-Riesz basis and F_1 and F_2 are duals of $\pi_{R(K)}F$. Then for all $f, g \in \mathcal{H}$ we have

$$\int_\Omega \langle f, F_1(\omega) - F_2(\omega)\rangle\langle \pi_{R(K)}F(\omega), g\rangle d\mu(\omega) = 0,$$

then

$$A\Big(\int_\Omega |\langle f, F_1(\omega) - F_2(\omega)\rangle|^2 d\mu(\omega)\Big)^{1/2} \leqslant \Big\| \int_\Omega \langle f, F_1(\omega) - F_2(\omega)\rangle \pi_{R(K)}F(\omega) d\mu(\omega)\Big\| = 0,$$

which shows that, for all $f \in \mathcal{H}$

$$\langle f, F_1(\omega)\rangle = \langle f, F_2(\omega)\rangle \quad (\mu - \text{almost all } \ \omega \in \Omega).$$

Hence $F_1 = F_2$.
For the converse, let F be a Riesz-type cK-frame. If $\phi \in L^2(\Omega)$ be such that

$$\int_\Omega \phi(\omega)\pi_{R(K)}F(\omega) d\mu(\omega) = 0,$$

thus for all $h \in \mathcal{H}$,

$$\int_\Omega \phi(\omega)\langle \pi_{R(K)}F(\omega), h\rangle d\mu(\omega) = 0.$$

Then by Theorem 3.9,

$$\phi \in (R(T^*_{\pi_{R(K)}F})^\perp = \{0\}.$$

Hence $\phi = 0$ almost everywhere and $\pi_{R(K)}F$ is L^2-independent. Also, by Theorem 3.9, $R(T^*_{\pi_{R(K)}F})$ is closed, then $R(T_{\pi_{R(K)}F})$ is closed. Now Theorem 3.7 implies that $\pi_{R(K)}F$ is a cK-Riesz basis. □

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