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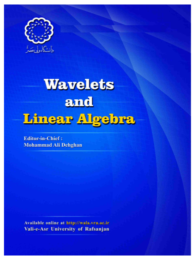

A note on Sonnenschein summability matrices

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Abstract

In this note, we give a simple method for computing the column sums of the Sonnenschein summability matrices.

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1. Introduction

Let $f(z)$ be an analytic function in $|z| < r, r \geq 1$ with $f(1) = 1$. The matrix $S = (f_{n,k})$, where $(f_{n,k})$ are defined by $[f(z)]^n = \sum_{k=0}^{\infty} f_{n,k} z^k$ is called a Sonnenschein matrix [4]. The special choice

$$f(z) = \frac{\alpha + (1-\alpha-\beta) z}{1-\beta z}, \ z \in \mathbb{C} \backslash \{\frac{1}{\beta}\},$$

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where α and β are complex numbers, gives the Karamata matrix $K[\alpha,\beta]$ and its coefficients as a Sonnenschein matrix are given by [3]

$$f_{n,k} = \sum_{v=0}^{k} \binom{n}{v} (1-\alpha-\beta)^v \alpha^{n-v} \binom{n+k-v-1}{k-v} \beta^{k-v}.$$

Recently in [1], the authors have calculated the row and column sums of Karamata matrices in a relatively complicated way. In this note we give a new and simple method for computing the column sums of these matrices which can be also applied to other Sonnenschein matrices.

Start with a general f. It is clear that if $|f(0)| < 1$ then $|f(z)| < 1$ in a neighborhood of $z = 0$. Thus

$$\sum_{n=0}^{\infty} [f(z)]^n = \frac{1}{1-f(z)}.$$

Now the sum we want is the coefficient of z^k of the right-hand side of the above equation. For the case of the Karamata matrices we have $f(0) = \alpha$, so we assume first that $|\alpha| < 1$. Now

$$\frac{1}{1-f(z)} = \frac{1}{1-\alpha} + \frac{(1-\beta)\,z}{(1-\alpha)\,(1-z)} = \frac{1}{1-\alpha} + \frac{1-\beta}{1-\alpha}\sum_{k=1}^{\infty} z^k.$$

The coefficient of z^k is then easily found. Indeed, the sum of the first column is $\frac{1}{1-\alpha}$, and the sum of all other columns are $\frac{1-\beta}{1-\alpha}$.

The point is that this method may apply to other choices of the function f and other Sonnenschein matrices. As another example consider the function

$$h(z) = sin^2(\frac{\pi z}{2}), \quad (z \in \mathbb{C})$$

which is holomorphic function and its coefficients as a Sonnenschein matrix are as follows:

1. If $n = 0$, then $a_{0,k} = \delta_{0k}$, for all k;
2. If $n \neq 0$ and $k = 0$, then $a_{n,k} = \frac{1}{4^n}\binom{2n}{n} + \sum_{r=0}^{n-1} \frac{(-1)^{n+r}}{2^{2n-1}}\binom{2n}{2r}$;
3. If $n \neq 0$ and $k \neq 0$, then $a_{n,k} = \begin{cases} 0 & k \text{ is odd,} \\ \pi^{2k} \sum_{r=0}^{n-1} \frac{(-1)^{n+r+k}(n-r)^{2k}}{2^{2n-1}(2k)!}\binom{2n}{2r} & k \text{ is even.} \end{cases}$

We have $h(0) = 0$, thus

$$\begin{aligned} \frac{1}{1-h(z)} &= \sec^2\left(\frac{\pi z}{2}\right) = \frac{2}{\pi}\frac{d}{dz}\tan\left(\frac{\pi z}{2}\right) \\ &= \frac{2}{\pi}\frac{d}{dz}\left(\sum_{n=0}^{\infty} \frac{(-1)^n 2(2^{2n+2}-1)}{(2n+2)!} B_{2n+2}\, \pi^{2n+1} z^{2n+1}\right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n(8n+4)(2^{2n+2}-1)}{(2n+2)!} B_{2n+2}\, \pi^{2n} z^{2n}, \end{aligned}$$

where B_n is the sequence of Bernoulli numbers (see [2], pp. 274 - 275), defined by

$$B_n = \sum_{k=0}^{n} \frac{1}{k+1} \sum_{r=0}^{k} (-1)^r \binom{k}{r} r^n. \quad (n = 0, 1, 2, ...)$$

Therefore, the sum of odd columns are 0, and the sum of column $2n$ $(n = 0, 1, 2, ...)$ is

$$\frac{(-1)^n\,(8n+4)\,(2^{2n+2}-1)}{(2n+2)!} B_{2n+2}\,\pi^{2n}.$$

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Declaration

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