Wavelets and Linear Algebra 10(1) (2023) 1- 11

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Wavelets and Linear Algebra

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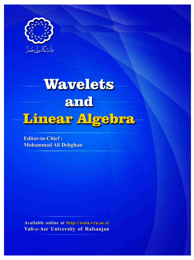

Four new subdivision coronas of two graphs

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Article Info

Article history:
Received 11 December 2021
Accepted 22 February 2023
Available online 5 May 2023
Communicated by Hamid Reza Afshin

Keywords:
Corona, Characteristic polynomial, Laplacian polynomial.

2010 MSC:
05C50.

Abstract

Let G and H be simple graphs and $|V(G)| = n$. The corona of two graphs, denoted by $G \circ H$, is the graph obtained by taking one copy of graph G and n copies of H and joining the i^{th} vertex of G to every vertex of the i^{th} copy of H. Let $S(G)$ be the subdivision of graph G. In this paper we define four new subdivision coronas of two graphs and find the characteristic and Laplacian polynomials of them in case of regularity.

1. Introduction

Let G be a simple graph with the vertex set $\{v_1, \ldots, v_n\}$. The adjacency matrix of G is an $n \times n$ matrix $A(G)$ whose (i, j)-entry is 1 if v_i is adjacent to v_j and 0, otherwise. The characteristic

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http://doi.org/10.22072/wala.2023.544642.1357

polynomial of G, denoted by $f_G(x)$, is the characteristic polynomial of $A(G)$. We will write it simply f_G when there is no confusion. The roots of f_G are called the eigenvalues of G and can be ordered as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Also $L(G) = \Delta(G) - A(G)$ is Laplacian matrix of G, where $\Delta(G)$ is the diagonal matrix which diagonal entries are degree sequences of G. We denoted Laplacian polynomial of G by $f_{L(G)}$ and eigenvalues of $L(G)$ can be ordered as $\mu_n \geq \cdots \geq \mu_1$. We denote the vertices and edges of G_1 by $V(G_1)$ and $E(G_1)$ respectively and the same for G_2. The Kronecker product $A \otimes B$ of two matrices $A(a_{ij})$ and $B(b_{ij})$ of order $m \times n$ and $p \times q$, respectively, is the $mp \times nq$ matrix obtained from A by replacing a_{ij} by $a_{ij}B$. This operation has the properties $(A \otimes B)^T = A^T \otimes B^T$ and $(A \otimes B)(C \otimes D) = (AC \otimes BD)$ whenever AC and BD exist. The latter implies $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ for nonsingular matrices A and B. Moreover, if A and B are $n \times n$ and $p \times p$ matrices respectively, then $\det(A \otimes B) = (\det A)^p.(\det B)^n$. Graph operations are natural techniques for producing new graphs from old ones, and their spectra have been received considerable attention in recent years. The *corona* of G and H, denoted $G \circ H$, is the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and joining the i^{th} vertex of G to every vertex in the i^{th} copy of H. This construction was first introduced by Frucht and Harary in [15] with the goal of constructing a graph whose automorphism group is the wreath product of the automorphism group of their components. Since then a number of papers on graph-theoretic properties of corona have been published. As far as eigenvalues are concerned, the characteristic polynomial and Laplacian polynomial of the corona of any two graphs can be expressed by two graphs [14, 13, 12, 11]. The generalized corona of two graphs also has been defined by Fiuj laali and others in [5]. Some other generalization of various kind of coronas can be found in [4, 3, 2]. The subdivision graph of G, denoted by $S(G)$, is a graph obtained by inserting a new vertex into every edge of G [6]. The characteristic polynomial and Laplacian polynomial neighbourhood corona, subdivision-vertex and subdivision-edge neighbourhood corona of two graphs were completely computed in [10, 9, 8]. The spectra of a graph reveal lots of information on the structural properties of that graph and the study of spectra of graphs has been found applications in variety of fields such as physics, chemistry, computer science, etc(see[7, 16, 6]). In this paper, we define four new subdivision coronas of two graphs and find the characteristic and Laplacian polynomials of them in case of regularity. In Section 2 we give some preliminaries. In Sectin 3 and 4 we find the characteristic and Laplacian polynomial of four coronas respectively.

2. PERELIMINARIES

The subdivision graph of G, denoted by $S(G)$, is a graph obtained by inserting a new vertex into every edge of G. We denote the set of such new vertices by $I(G)$ and obviously $V(S(G)) = V(G) \cup I(G)$. Throughout the paper when we say *V-part* of $V(S(G))$, we refer to $V(G)$ of $V(S(G))$ and when we say *I-part* of $V(S(G))$, we refer to $I(G)$ of $V(S(G))$.

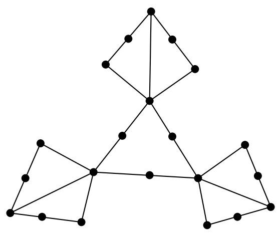
Definition 2.1. *Let G_1 and G_2 be two simple graphs. The subdivision vertex-vertex corona of G_1 and G_2 denoted by $S(G_1) \underset{V_1 \sim V_2}{\circ} S(G_2)$ is obtained by taking a copy of $S(G_1)$ and $|V(G_1)|$ copies of $S(G_2)$, and joining the i^{th} vertex of the V-part of $S(G_1)$ to every vertex of the V-part of the i^{th} copy of $S(G_2)$.*

Definition 2.2. *Let G_1 and G_2 be two simple graphs. The subdivision edge-edge corona of G_1 and G_2 denoted by $S(G_1) \underset{I_1 \sim I_2}{\circ} S(G_2)$ is obtained by taking a copy of $S(G_1)$ and $|I(G_1)|$ copies of $S(G_2)$, and joining the i^{th} vertex of I-part of $S(G_1)$ to every vertex of the I-part of the i^{th} copy of $S(G_2)$.*

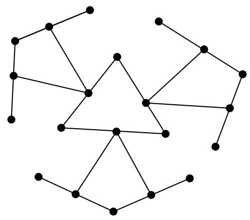
Definition 2.3. *Let G_1 and G_2 be two simple graphs. The subdivision vertex-vertexedge corona of G_1 and G_2 denoted by $S(G_1) \underset{V_1 \sim V_2, I_2}{\circ} S(G_2)$ is obtained by taking a copy of $S(G_1)$ and $|V(G_1)|$ copies of $S(G_2)$, and joining the i^{th} vertex of V-part of $S(G_1)$ to every vertex of the i^{th} copy of $S(G_2)$.*

Definition 2.4. *Let G_1 and G_2 be two simple graphs. The subdivision edge-vertexedge corona of G_1 and G_2 denoted by $S(G_1) \underset{I_1 \sim V_2, I_2}{\circ} S(G_2)$ is obtained by taking a copy of $S(G_1)$ and $|I(G_1)|$ copies of $S(G_2)$, and joining the i^{th} vertex of I-part of $S(G_1)$ to every vertex of the i^{th} copy of $S(G_2)$.*

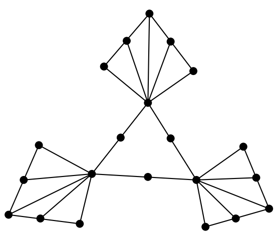
Below you see all possible coronas between K_3 and P_3.



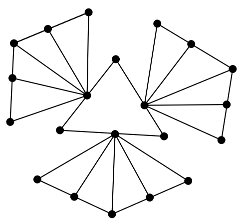
$S(K_3) \underset{V(K_3) \sim V(P_3)}{\circ} S(P_3)$



$S(K_3) \underset{I(K_3) \sim I(P_3)}{\circ} S(P_3)$



$S(K_3) \underset{V(K_3) \sim V(P_3), I(P_3)}{\circ} S(P_3)$



$S(K_3) \underset{I(K_3) \sim V(P_3), I(P_3)}{\circ} S(P_3)$

Let X_1 and X_2 be the incidence matrix [6] of G_1 and G_2 respectively. By a proper labeling, the adjacency matrix of them are as follow

$$A(S(G_1) \underset{V_1\sim V_2}{\circ} S(G_2)) = \begin{array}{c} \\ V(G_1)\\ I(G_1)\\ V(G_2)\\ I(G_2)\end{array} \begin{array}{c} \begin{array}{cccc} V(G_1) & I(G_1) & V(G_2) & I(G_2) \end{array}\\ \begin{bmatrix} 0_{n_1\times n_1} & X_1 & {j_{n_2}}^T\otimes I_{n_1} & 0_{n_1\times m_2}\\ {X_1}^T & 0_{m_1\times m_1} & 0_{m_1\times n_2} & 0_{m_1\times m_2}\\ j_{n_2}\otimes I_{n_1} & 0_{n_2\times m_1} & 0_{n_2\times n_2} & X_2\otimes I_{n_1}\\ 0_{m_2\times n_1} & 0_{m_2\times m_1} & {X_2}^T\otimes I_{n_1} & 0_{m_2\times m_2} \end{bmatrix} \end{array},$$

$$A(S(G_1) \underset{I_1\sim I_2}{\circ} S(G_2)) = \begin{array}{c} \\ V(G_1)\\ I(G_1)\\ V(G_2)\\ I(G_2)\end{array} \begin{array}{c} \begin{array}{cccc} V(G_1) & I(G_1) & V(G_2) & I(G_2) \end{array}\\ \begin{bmatrix} 0_{n_1\times n_1} & X_1 & 0_{n_1\times n_2} & 0_{n_1\times m_2}\\ {X_1}^T & 0_{m_1\times m_1} & {j_{m_2}}^T\otimes I_{m_1} & 0_{m_1\times m_2}\\ 0_{n_2\times n_1} & j_{m_2}\otimes I_{m_1} & 0_{n_2\times n_2} & {X_2}^T\otimes I_{m_1}\\ 0_{m_2\times n_1} & 0_{m_2\times m_1} & X_2\otimes I_{m_1} & 0_{m_2\times m_2} \end{bmatrix} \end{array},$$

$$A(S(G_1) \underset{V_1\sim V_2,I_2}{\circ} S(G_2)) = \begin{array}{c} \\ V(G_1)\\ I(G_1)\\ V(G_2)\\ I(G_2)\end{array} \begin{array}{c} \begin{array}{cccc} V(G_1) & I(G_1) & V(G_2) & I(G_2) \end{array}\\ \begin{bmatrix} 0_{n_1\times n_1} & X_1 & {j_{n_2}}^T\otimes I_{n_1} & {j_{m_2}}^T\otimes I_{n_1}\\ {X_1}^T & 0_{m_1\times m_1} & 0_{m_1\times n_2} & 0_{m_1\times m_2}\\ j_{n_2}\otimes I_{n_1} & 0_{n_2\times m_1} & 0_{n_2\times n_2} & X_2\otimes I_{n_1}\\ j_{m_2}\otimes I_{n_1} & 0_{m_2\times m_1} & {X_2}^T\otimes I_{n_1} & 0_{m_2\times m_2} \end{bmatrix} \end{array},$$

$$A(S(G_1) \underset{I_1\sim V_2,I_2}{\circ} S(G_2)) = \begin{array}{c} \\ V(G_1)\\ I(G_1)\\ V(G_2)\\ I(G_2)\end{array} \begin{array}{c} \begin{array}{cccc} V(G_1) & I(G_1) & V(G_2) & I(G_2) \end{array}\\ \begin{bmatrix} 0_{n_1\times n_1} & X_1 & 0_{n_1\times n_2} & 0_{n_1\times m_2}\\ {X_1}^T & 0_{m_1\times m_1} & {j_{n_2}}^T\otimes I_{m_1} & {j_{m_2}}^T\otimes I_{m_1}\\ 0_{n_2\times n_1} & j_{n_2}\otimes I_{m_1} & 0_{n_2\times n_2} & X_2\otimes I_{m_1}\\ 0_{m_2\times n_1} & j_{m_2}\otimes I_{m_1} & {X_2}^T\otimes I_{m_1} & 0_{m_2\times m_2} \end{bmatrix} \end{array},$$

where $0_{m\times n}$ is a zero matrix of order $m\times n$, I_n is the identity matrix of order n and j_n is the n-th column vector with all elements equal to 1. We find the charactristic and Laplacian polynomials of these four coronas where graphs G_1 and G_2 are both regular. To continue we need to recall following facts.

Lemma 2.5. *[6] Let G be an r-regular graph with n vertices and m edges. Let A(G) be the adjacency matrix and X(G) be the incidence matrix of G and Ln(G) be its line graph. Then* $XX^T = A(G) + rI_n$ *and* $X^TX = A(Ln(G)) + 2I_m$.

Lemma 2.6. *[7]* ***(Schur complement)*** *Let A be an* $n\times n$ *matrix partitioned as*

$$\begin{pmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix},$$

where A_{11} *and* A_{22} *are square matrices. If* A_{11} *and* A_{22} *are invertible, then*

$$\det\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \det(A_{22})\det(A_{11} - A_{12}{A_{22}}^{-1}A_{21})$$
$$= \det(A_{11})\det(A_{22} - A_{21}{A_{11}}^{-1}A_{12}).$$

Remark 2.7. *[12] Let the sum of the all elements of each row of $A_{m\times m}$ be n. Then following equation holds*

$$j_m{}^T.(xI_m - A)^{-1}.j_m = \tfrac{m}{x-n},$$

where j_m is the column vector which all entries 1. Note that, viewed as a matrix over the field of rational functions $\mathbb{C}(x)$, the characteristic matrix $xI_m - A$ has determinant $\det(xI - A) = f_A(x) \neq 0$, so is invertible. We use this fact over and over in next sections.

3. CHARACTRESTIC POLYNOMIAL OF FOUR CORONAS

Theorem 3.1. *Let G_i be r_i-regular graph with n_i vertices, m_i edges and X_i incidence matrix. If $G' = S(G_1) \underset{V_1\sim V_2}{\circ} S(G_2)$, then we have*

$$f_{G'}(\lambda) = \lambda^{n_1m_2-n_1n_2+m_1-n_1}(f_{G_2}(\lambda^2 - r_2))^{n_1} f_{G_1}(\lambda^2 - r_1 - \tfrac{n_2\lambda^2}{\lambda^2-2r_2}).$$

Proof. By Schur complement, we have

$$f_{G'}(\lambda) = \det(\lambda I_{n_1+m_1+n_1(n_2+m_2)} - A(G'))$$
$$= \det\begin{bmatrix} \lambda I_{n_1} & -X_1 & -j_{n_2}{}^T \otimes I_{n_1} & 0 \\ -X_1{}^T & \lambda I_{m_1} & 0 & 0 \\ -j_{n_2} \otimes I_{n_1} & 0 & \lambda I_{n_1n_2} & -X_2 \otimes I_{n_1} \\ 0 & 0 & -X_2{}^T \otimes I_{n_1} & \lambda I_{n_1m_2} \end{bmatrix}$$

$$= \lambda^{n_1m_2}\det\left(\begin{bmatrix} \lambda I_{n_1} & -X_1 & -j_{n_2}{}^T \otimes I_{n_1} \\ -X_1{}^T & \lambda I_{m_1} & 0 \\ -j_{n_2} \otimes I_{n_1} & 0 & \lambda I_{n_1n_2} \end{bmatrix} - Y\right)$$

where,

$$Y = \begin{pmatrix} 0 \\ 0 \\ -X_2 \otimes I_{n_1} \end{pmatrix}(\lambda I_{n_1n_2})^{-1}\begin{pmatrix} 0 \\ 0 \\ -X_2 \otimes I_{n_1} \end{pmatrix}^T.$$

After computing Y and replacing in the determinant, we obtain

$$\lambda^{n_1m_2}\det\begin{bmatrix} \lambda I_{n_1} & -X_1 & -{j_{n_2}}^T\otimes I_{n_1} \\ -{X_1}^T & \lambda I_{m_1} & 0 \\ -j_{n_2}\otimes I_{n_1} & 0 & \lambda I_{n_1n_2}-\frac{1}{\lambda}(X_2{X_2}^T)\otimes I_{n_1} \end{bmatrix}.$$

Since G_2 is regular, by Lemma 2.5 we obtain

$$f_{G'}(\lambda)=\lambda^{n_1m_2}\det\begin{bmatrix} \lambda I_{n_1} & -X_1 & -{j_{n_2}}^T\otimes I_{n_1} \\ -{X_1}^T & \lambda I_{m_1} & 0 \\ -j_{n_2}\otimes I_{n_1} & 0 & \frac{1}{\lambda}((\lambda^2-r_2)I_{n_2}-A(G_2))\otimes I_{n_1} \end{bmatrix}.$$

Again by using Schur complement, it follows that

$$f_{G'}(\lambda)=\lambda^{n_1m_2-n_1n_2}\Big(f_{G_2}(\lambda^2-r_2)\Big)^{n_1}\det\left(\begin{bmatrix} \lambda I_{n_1} & -X \\ -{X_1}^T & \lambda I_{m_1} \end{bmatrix}-S\right)$$

where,

$$S=\begin{pmatrix} -{j_{n_2}}^T\otimes I_{n_1} \\ 0 \end{pmatrix}\left(\frac{1}{\lambda}\left(\left(\lambda^2-r_2\right)I_{n_2}-A(G_2\right)\otimes I_{n_1}\right)^{-1}\begin{pmatrix} -j_{n_2}\otimes I_{n_1} & 0 \end{pmatrix}.$$

By Remark 2.7 it can be easily seen that

$$S=\begin{pmatrix} \frac{\lambda n_2}{\lambda^2-2r_2}I_{n_1} & 0 \\ 0 & 0 \end{pmatrix}.$$

So, we obtain

$$f_{G'}(\lambda)=\lambda^{n_1m_2-n_1n_2}\Big(f_{G_2}(\lambda^2-r_2)\Big)^{n_1}\det\begin{bmatrix} (\lambda-\frac{\lambda n_2}{\lambda^2-2r_2})I_{n_1} & -X_1 \\ -{X_1}^T & \lambda I_{m_1} \end{bmatrix}.$$

By Schur complement and Lemma 2.5, it follows that

$$f_{G'}(\lambda)=\lambda^{n_1m_2-n_1n_2+m_1}\Big(f_{G_2}(\lambda^2-r_2)\Big)^{n_1}\det((\lambda-\frac{\lambda n_2}{\lambda^2-2r_2})I_{n_1}-\frac{1}{\lambda}(X_1{X_1}^T))$$

$$=\lambda^{n_1m_2-n_1n_2+m_1}\Big(f_{G_2}(\lambda^2-r_2)\Big)^{n_1}\det((\lambda-\frac{r_1}{\lambda}-\frac{\lambda n_2}{\lambda^2-2r_2})I_{n_1}-\frac{1}{\lambda}A(G_1))$$

$$=\lambda^{n_1m_2-n_1n_2+m_1}\Big(f_{G_2}(\lambda^2-r_2)\Big)^{n_1}f_{G_1}(\lambda^2-r_1-\frac{\lambda n_2}{\lambda^2-2r_2}).$$

□

By the same argument in the proof of Theorem 3.1, we can prove following theorems.

Theorem 3.2. *Let G_i be r_i-regular graph with n_i vertices, m_i edges and X_i incidence matrix. If $G' = S(G_1) \underset{I_1\sim I_2}{\circ} S(G_2)$, we have*

$$f_{G'}(\lambda) = \lambda^{n_2m_1-m_1m_2+n_1-m_1}(f_{Ln(G_2)}(\lambda^2-2))^{m_1} f_{Ln(G_1)}(\lambda^2-2-\tfrac{m_2\lambda^2}{\lambda^2-2r_2+1}).$$

Theorem 3.3. *Let G_i be r_i-regular graph with n_i vertices, m_i edges and X_i incidence matrix. If $G' = S(G_1) \underset{V_1\sim V_2,I_2}{\circ} S(G_2)$, we have*

$$f_{G'}(\lambda) = \lambda^{n_1m_2-n_1n_2+m_1-n_1}(f_{G_2}(\lambda^2-r_2))^{n_1} f_{G_1}(\lambda^2-r_1-m_2-\tfrac{n_2(\lambda+r_2)^2}{\lambda^2-2r_2}).$$

Theorem 3.4. *Let G_i be r_i-regular graph with n_i vertices, m_i edges and X_i incidence matrix. If $G' = S(G_1) \underset{I_1\sim V_2,I_2}{\circ} S(G_2)$, we have*

$$f_{G'}(\lambda) = \lambda^{m_1m_2-m_1n_2+n_1-m_1}(f_{G_2}(\lambda^2-r_2))^{m_1} f_{Ln(G_1)}(\lambda^2-2\lambda-m_2-\tfrac{n_2(\lambda+r_2)^2}{\lambda^2-2r_2}).$$

4. LAPLACIAN POLYNOMIAL OF FOUR CORONAS

Theorem 4.1. *Let G_i be r_i-regular graph with n_i vertices, m_i edges and X_i incidence matrix. If $G' = S(G_1) \underset{V_1\sim V_2}{\circ} S(G_2)$ we have*

$$\begin{aligned} f_{L(G')}(\lambda) &= \lambda^{n_1m_2}(\lambda-2)^{m_1-n_1n_2-n_1}\Big(f_{G_2}(\lambda^2-r_2)\Big)^{n_1} \\ &\quad .f_{G_1}((\lambda-r_1-n_2)(\lambda-2)-\frac{(\lambda-2)^2n_2}{(\lambda-2)(\lambda-r_2-1)-2r_2}-r_1). \end{aligned}$$

.

Proof. By Schur complement, we have

$$f_{L(G')}(\lambda) = \det(\lambda I_{n_1+m_1+n_1(n_2+m_2)} - \Delta(G') + A(G'))$$

$$= \det\begin{bmatrix} (\lambda - r_1 - n_2)I_{n_1} & X_1 & {j_{n_2}}^T \otimes I_{n_1} & 0 \\ {X_1}^T & (\lambda-2)I_{m_1} & 0 & 0 \\ j_{n_2} \otimes I_{n_1} & 0 & (\lambda - r_2 - 1)I_{n_1 n_2} & X_2 \otimes I_{n_1} \\ 0 & 0 & {X_2}^T \otimes I_{n_1} & (\lambda-2)I_{n_1 m_2} \end{bmatrix}$$

$$= (\lambda-2)^{n_1 m_2} \det(\begin{bmatrix} (\lambda - r_1 - n_2)I_{n_1} & X_1 & {j_{n_2}}^T \otimes I_{n_1} \\ {X_1}^T & (\lambda-2)I_{m_1} & 0 \\ j_{n_2} \otimes I_{n_1} & 0 & (\lambda - r_2 - 1)I_{n_1 n_2} \end{bmatrix}$$

$$- \begin{pmatrix} 0 \\ 0 \\ X_2 \otimes I_{n_1} \end{pmatrix} ((\lambda-2)I_{n_1 m_2})^{-1} \begin{pmatrix} 0 \\ 0 \\ {X_2}^T \otimes I_{n_1} \end{pmatrix}^T)$$

$$= (\lambda-2)^{n_1 m_2} \det \begin{bmatrix} (\lambda - r_1 - n_2)I_{n_1} & X_1 & {j_{n_2}}^T \otimes I_{n_1} \\ {X_1}^T & (\lambda-2)I_{m_1} & 0 \\ j_{n_2} \otimes I_{n_1} & 0 & (\lambda - r_2 - 1)I_{n_1 n_2} - \frac{1}{\lambda-2}(X_2 {X_2}^T) \otimes I_{n_1} \end{bmatrix}.$$

Since G_2 is regular, by Lemma 2.5 it follows that

$$f_{L(G')}(\lambda) = (\lambda-2)^{n_1 m_2}$$

$$. \det \begin{bmatrix} (\lambda - r_1 - n_2)I_{n_1} & X_1 & {j_{n_2}}^T \otimes I_{n_1} \\ {X_1}^T & (\lambda-2)I_{m_1} & 0 \\ j_{n_2} \otimes I_{n_1} & 0 & \frac{1}{\lambda-2}(((\lambda - r_2 - 1)(\lambda-2) - r_2)I_{n_2} - A(G_2)) \otimes I_{n_1} \end{bmatrix}.$$

Again by using Schur complement, it can be seen that

$$f_{G'}(\lambda) = (\lambda-2)^{n_1(m_2-n_2)} (f_{G_2}(((\lambda - r_2 - 1)(\lambda-2) - r_2)))^{n_1}$$

$$. \det\left(\begin{bmatrix} (\lambda - r_1 - n_2)I_{n_1} & -X_1 \\ -{X_1}^T & (\lambda-2)I_{m_1} \end{bmatrix} - \begin{pmatrix} -{j_{n_2}}^T \otimes I_{n_1} \\ 0 \end{pmatrix} S^{-1} \begin{pmatrix} -j_{n_2} \otimes I_{n_1} & 0 \end{pmatrix} \right)$$

$$= (\lambda-2)^{n_1(m_2-n_2)} (f_{G_2}(((\lambda - r_2 - 1)(\lambda-2) - r_2)))^{n_1}$$

$$\det\left(\begin{bmatrix} (\lambda - r_1 - n_2)I_{n_1} & -X_1 \\ -{X_1}^T & (\lambda-2)I_{m_1} \end{bmatrix} - \begin{bmatrix} (-{j_{n_2}}^T \otimes I_{n_1}) S^{-1} (-j_{n_2} \otimes I_{n_1}) & 0 \\ 0 & 0 \end{bmatrix} \right),$$

where $S = \frac{1}{\lambda-2}(((\lambda - r_2 - 1)(\lambda - 2) - r_2)I_{n_2} - A(G_2)) \otimes I_{n_1}$. By Remark 2.7 we have

$$(-{j_{n_2}}^T \otimes I_{n_1})S^{-1}(-j_{n_2} \otimes I_{n_1}) = \frac{(\lambda-2)n_2}{(\lambda-2)(\lambda-r_2-1)-2r_2} \otimes I_{n_1}$$
$$= \frac{(\lambda-2)n_2}{(\lambda-2)(\lambda-r_2-1)-2r_2} I_{n_1}.$$

Thus we obtain

$$f_{G'}(\lambda) = (\lambda-2)^{n_1(m_2-n_2)}(f_{G_2}(((\lambda-r_2-1)(\lambda-2)-r_2)))^{n_1}$$

$$.\det\begin{bmatrix} (\lambda - r_1 - n_2 - \frac{(\lambda-2)n_2}{(\lambda-2)(\lambda-r_2-1)-2r_2})I_{n_1} & X_1 \\ {X_1}^T & (\lambda-2)I_{m_1} \end{bmatrix}.$$

By Schur complement and Lemma 2.5, it follows that

$$f_{L(G')}(\lambda) = (\lambda-2)^{n_1(m_2-n_2)+m_1}(f_{G_2}(((\lambda-r_2-1)(\lambda-2)-r_2)))^{n_1}$$

$$.\det((\lambda - r_1 - n_2 - \frac{(\lambda-2)n_2}{(\lambda-2)(\lambda-r_2-1)-2r_2})I_{n_1} - \frac{1}{\lambda-2}(X_1{X_1}^T))$$

$$= (\lambda-2)^{n_1(m_2-n_2)+m_1}(f_{G_2}(((\lambda-r_2-1)(\lambda-2)-r_2)))^{n_1}$$

$$.\det((\lambda - r_1 - n_2 - \frac{(\lambda-2)n_2}{(\lambda-2)(\lambda-r_2-1)-2r_2} - \frac{r_1}{\lambda-2})I_{n_1} - \frac{1}{\lambda-2}A(G_1))$$

$$= (\lambda-2)^{n_1(m_2-n_2)+m_1}\left(f_{G_2}(\lambda^2-r_2)\right)^{n_1}$$

$$.f_{G_1}((\lambda-r_1-n_2)(\lambda-2) - \frac{(\lambda-2)^2 n_2}{(\lambda-2)(\lambda-r_2-1)-2r_2} - r_1).$$

□

By the same argument in the proof of Theorem 4.1, we can prove following theorems.

Theorem 4.2. *Let G_i be r_i-regular graph with n_i vertices, m_i edges and X_i incidence matrix. If $G' = S(G_1) \underset{I_1 \sim I_2}{\circ} S(G_2)$ Then we have*

$$f_{L(G')}(\lambda) = (\lambda-r_2)^{m_1(n_2-m_2)}(f_{Ln(G_2)}((\lambda-3)(\lambda-r_2)-2))^{n_1}$$
$$.f_{Ln(G_1)}((\lambda-r_1)(\lambda-m_2-2) - \frac{(\lambda-r_2)(\lambda-r_1)}{(\lambda-r_2)(\lambda-3)-2r_2} - 2).$$

.

Theorem 4.3. *Let G_i be r_i-regular graph with n_i vertices, m_i edges and X_i incidence matrix. If $G' = S(G_1) \underset{V_1\sim V_2, I_2}{\circ} S(G_2)$ we have*

$$f_{L(G')}(\lambda) = (\lambda-3)^{n_1(m_2-n_2)}(\lambda-2)^{m_1-n_1}(f_{G_2}((\lambda-3)(\lambda-r_1-1)-r_2))^{n_1}$$
$$.f_{G_1}([\lambda - r_1 - m_2 - n_2 - \frac{1}{\lambda-3}(m_2 + \frac{n_2(\lambda-r_2-3)^2}{(\lambda-3)(\lambda-r_1-1)-2r_2})](\lambda-2)-r_2).$$

Theorem 4.4. *Let G_i be r_i-regular graph with n_i vertices, m_i edges and X_i incidence matrix. If $G' = S(G_1) \underset{I_1\sim V_2, I_2}{\circ} S(G_2)$ we have*

$$f_{L(G')}(\lambda) = (\lambda-3)^{m_1(m_2-n_2)}(\lambda-r_1)^{n_1-m_1}(f_{G_2}(\lambda(\lambda-3)-r_2))^{m_1}$$
$$.f_{Ln(G_1)}([\lambda - m_2 - n_2 - 2 - \frac{1}{\lambda-3}(m_2 + \frac{n_2(\lambda-r_2-3)^2}{\lambda(\lambda-3)-2r_2})](\lambda-r_1)-2).$$

5. Conclusion

In this paper, we introduced four new subdivision coronas of graphs and determined the characteristic and Lapalacian polynomials of them. We can obtain the number of spanning trees and the Kirchhoff index of the new variants of corona of graphs by using theorems 4.1 to 4.4. It seems that such generalization of the corona corresponds to many molecular bonds, thus by obtaining a variety of spectra of such coronas, some of the indices associated with those bonds would be calculated. The determination of the characteristic and Lapalacian polynomials of these four new subdivision coranas in case of irregularity are further research problems.

6. Acknowledgments

The authors wish to sincerely thank the referees for several useful comments.

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