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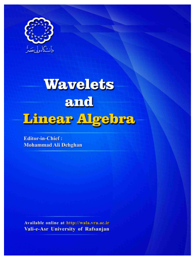

Inverse eigenvalues problem of distance matrices via unit lower triangular matrices

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Abstract

In this paper, for a given set of real numbers such as σ with only one positive number and zero summation, we find a distance matrix in which the given set σ is its spectrum. Finally, we solve special cases of the inverse eigenvalue problem in which the matrix solution is a regular spherical distance matrix.

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1. Introduction

The main foundations of the distance matrix can be traced back to distant years. Schonberg's papers make an important contribution to the distance matrix theory [14, 13, 12].

A matrix $D = (d_{ij}) \in M_{n\times n}$ is said to be an Euclidean distance matrix (EDM), if there are n points $x_1, x_2, \dots, x_n \in \mathbb{R}^r$, such that $d_{ij} = \left\|x_i - x_j\right\|^2$ for all $i, j = 1, 2, \dots, n$, where $\|.\|$ denotes the Euclidean norm. By the definition of EDM the following properties for the matrix D are held:

$$\begin{array}{l}(1)\, D \ \text{ is nonnegative matrix;}\\ (2)\, D \ \text{ is symmetric;}\\ (3)\, D \ \text{ has zero main diagonal and this means that the sum of its}\\ \ \text{eigenvalues is zero.}\end{array} \tag{1.1}$$

Inverse eigenvalues of EDM is an interesting topic in symmetric nonnegative inverse eigenvalue problems (SNIEP). Two main papers regarding SNIEP are [5, 4].

If $x_1, x_2, \dots, x_n \in \mathbb{R}^r$ are the constructive points of EDM D, then

$$X = \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix}^T \in M_{n\times r},$$

is called its coordinate matrix. Since translation and rotation preserve the distance between two points, we deduce that the coordinate matrix associated with an EDM is not unique. The minimum rank of the coordinate matrices associated with an EDM D is called an embedding dimension of D and denoted by $\mathrm{ed}(D)$. If e is a vector with all ones, then D is a distance matrix if and only if D is negative semidefinite on $e^\perp = \{y \in \mathbb{R}^n, y^T e = 0\}$. Therefore an EDM D has at most one positive eigenvalue with algebraic multiplicity $n-1$ on the set $e^\perp$. Considering Property (3) in (1.1), we conclude that D has exactly one positive eigenvalue.

Let S_H be the set of symmetric matrices of order n with zero diagonal and S_C be the set of symmetric matrices B of order n satisfying $Be = 0$. We define the following maps:

$$T : S_H \to S_C \quad \text{and} \quad K : S_C \to S_H,$$

where

$$\begin{array}{l} T(D) = -\frac{1}{2}(I - \frac{ee^T}{n})D(I - \frac{ee^T}{n}),\\ K(B) = \mathrm{diag}(B)e^T + e(\mathrm{diag}(B))^T - 2B. \end{array} \tag{1.2}$$

The linear maps T and K are mutually inverse, and $D \in S_H$ is an EDM of embedding dimension r if and only if $T(D)$ is positive semidefinite of rank r [3].

The distance matrices and their eigenvalues have been studied in several papers such as [3, 15, 7, 2, 11]. In [6] Hayden et al. solved the inverse eigenvalue problem for Euclidean distance matrices of order $n = 3, 4, 5, 6$, and any n for which there exists a Hadamard matrix. They also stated that if there exists a Hadamard matrix of order $n \in \mathbb{N}$, then there are $(n + 1) \times (n + 1)$ and $(n + 2) \times (n + 2)$ distance matrices with eigenvalues satisfying some special conditions for $n \leqslant 16$. Nazari and Mahdinasab [10] solved this problem using orthogonal matrices without employing any Hadamard matrix.

An EDM D is said to be spherical if the construction points of D lie on a hypersphere, otherwise, it is said to be non-spherical. By [15] we know that a distance matrix D of embedding dimension r is spherical if and only if its rank is $r+1$ and D is non-spherical if and only if its rank is $r+2$. A spherical EDM D is called regular if the constructive points of D lie on a hypersphere whose center coincides with the centroid of those points. D is regular spherical if and only if e is the eigenvector of D corresponding to the eigenvalue $\frac{e^TDe}{n}$ [7].

A matrix L is called unit lower triangular if it is a lower triangular with all entries on main diagonal 1. The inverse of L also is an unit lower triangular matrix and is easily computable. Recently, Nazari et al in [9] have solved a special inverse eigenvalue problem of bisymmetric matrices by using unit lower triangular matrices. In this paper for a given set of real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, satisfying $\sum_{i=1}^{n} \lambda_i = 0$ and $\lambda_1 > 0 > \lambda_n \geq \cdots \geq \lambda_2$, we solve the corresponding inverse eigenvalue problem of the distance matrix without using the Hadamard matrices and orthogonal matrices and only using some special unit lower triangular matrices.

2. Construction distance matrix with prescribed eigenvalues

In this section, at first, we solve the inverse eigenvalue problem of distance matrices and then solve some special inverse eigenvalue problem of regular spherical matrices. First, we discuss the following theorem about the inverse of the matrix L.

Theorem 2.1. *Given $\sigma = \{\lambda_1, \lambda_2 \ldots, \lambda_n\}$, with*

1. $\lambda_1 > 0 > \lambda_n \geq \cdots \geq \lambda_2$,
2. $\sum_{i=1}^{n} \lambda_i = 0$,

there exists an $n \times n$ distance matrix that realizes σ.

Proof. For $n = 2$, we have $\lambda_1 = -\lambda_2$. In this case letting

$$L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} \lambda_1 & \lambda_1 \\ 0 & \lambda_2 \end{bmatrix},$$

we see that

$$D = L^{-1}AL = \begin{bmatrix} 0 & \lambda_1 \\ -\lambda_2 & 0 \end{bmatrix}.$$

Obviously, D is an EDM realizing the spectrum $\sigma = \{\lambda_1, \lambda_2\}$.

For $n = 3$, we choose the following two matrices

$$A = \begin{bmatrix} \lambda_1 & 2\,\lambda_1 \frac{1}{\sqrt{-2\,\frac{\lambda_1}{\lambda_2}}} & \lambda_1 \frac{1}{\sqrt{-2\,\frac{\lambda_1}{\lambda_2}}} \\ 0 & \lambda_2 & 1/2\,\lambda_2 - 1/2\,\lambda_3 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2\,\sqrt{-2\,\frac{\lambda_1}{\lambda_2}} & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

In this case, we get

$$D = (d_{ij})_{3\times 3} = L^{-1}AL = \begin{bmatrix} 0 & \lambda_1 \frac{1}{\sqrt{-2\frac{\lambda_1}{\lambda_2}}} & \lambda_1 \frac{1}{\sqrt{-2\frac{\lambda_1}{\lambda_2}}} \\ \lambda_1 \frac{1}{\sqrt{-2\frac{\lambda_1}{\lambda_2}}} & 0 & -\lambda_3 \\ \lambda_1 \frac{1}{\sqrt{-2\frac{\lambda_1}{\lambda_2}}} & -\lambda_3 & 0 \end{bmatrix}.$$

It can be easily shown that

$$\begin{aligned} d_{12} + d_{23} &\geq d_{13}, \\ d_{13} + d_{32} &\geq d_{12}, \\ d_{21} + d_{13} &\geq d_{23}. \end{aligned}$$

The first two inequalities are obvious, and we do so to show the third inequality. Since $d_{21} + d_{13} = \sqrt{-2\lambda_1\lambda_2}$ and $\lambda_1 > 0 > \lambda_3 \geq \lambda_2$ then $\sqrt{-2\lambda_1\lambda_2} \geq \sqrt{-2\lambda_1\lambda_3} \geq \sqrt{2}|\lambda_3| \geq -\lambda_3 = d_{23}$, then the above matrix is an EDM and similar to A, which realizes spectrum $\sigma = \{\lambda_1, \lambda_2, \lambda_3\}$.

For $n = 4$ we select the following two matrices

$$A = \begin{bmatrix} \lambda_1 & 1/2\,\lambda_1 - 1/2\,\lambda_2 & 1/2\,\lambda_3 - 1/2\,\lambda_2 & -1/2\,\lambda_4 - 1/2\,\lambda_3 \\ 0 & \lambda_2 & \lambda_2 - \lambda_3 & 1/2\,\lambda_3 - 1/2\,\lambda_2 \\ 0 & 0 & \lambda_3 & 1/2\,\lambda_4 - 1/2\,\lambda_3 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}, \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & -1 & 1 & 1 \end{bmatrix}.$$

Then we see that the matrix

$$D = L^{-1}AL = \begin{bmatrix} 0 & -1/2\,\lambda_3 - 1/2\,\lambda_2 & -1/2\,\lambda_4 - 1/2\,\lambda_2 & -1/2\,\lambda_4 - 1/2\,\lambda_3 \\ -1/2\,\lambda_3 - 1/2\,\lambda_2 & 0 & -1/2\,\lambda_4 - 1/2\,\lambda_3 & -1/2\,\lambda_4 - 1/2\,\lambda_2 \\ -1/2\,\lambda_4 - 1/2\,\lambda_2 & -1/2\,\lambda_4 - 1/2\,\lambda_3 & 0 & -1/2\,\lambda_3 - 1/2\,\lambda_2 \\ -1/2\,\lambda_4 - 1/2\,\lambda_3 & -1/2\,\lambda_4 - 1/2\,\lambda_2 & -1/2\,\lambda_3 - 1/2\,\lambda_2 & 0 \end{bmatrix},$$

is a symmetric and nonnegative matrix with zeroes diagonal and it is to check that $d_{ij} + d_{jk} \geq d_{ik}$ for all $i, j, k = 1, 2, 3, 4$, then D is an EDM and similar to matrix A which realizes the spectrum $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$.

For $n \geq 5$, we consider the unit lower bidiagonal matrix $L = (l_{ij})_{n\times n}$ with the entries

$$\begin{aligned} l_{ii} &= 1, \qquad i = 1, 2, \ldots, n, \\ l_{i+1,i} &= -\sqrt{\frac{\lambda_{i+2}\sum_{k=1}^{i}\lambda_k}{\lambda_{i+1}\left(\sum_{k=1}^{i+1}\lambda_k - \lambda_{i+2}\right)}}, \qquad i = 1, 2, \ldots, n-1. \end{aligned} \tag{2.1}$$

It is worth noting that for $i = n - 1$, it follows from $\sum_{k=1}^{n}\lambda_k = 0$ that

$$l_{n,n-1} = -\sqrt{\frac{\lambda_{n+1} \times (-\lambda_n)}{-\lambda_n \times \lambda_{n+1}}} = -1.$$

This shows that $l_{n,n-1}$ is independent of λ_{n+1} and in the above relation, there is no need to define λ_{n+1}, and using this relation, λ_{n+1} will be deleted, and in order not to separate the last line of L, we have written it in the same way as before. Let $A = (a_{ij})$ be an upper triangular matrix with $a_{ii} = \lambda_i$, for $i = 1, 2, \ldots, n$, and

$$a_{ij} = \frac{\lambda_i - \lambda_{i+1}}{\sum_{k=1}^{i} \lambda_k - \lambda_{i+1}} \sqrt{\frac{\lambda_{i+1} \prod_{m=i}^{j-1} \left(\sum_{k=1}^{m} \lambda_k\right) \left(\sum_{k=1}^{j} \lambda_k - \lambda_{j+1}\right)}{\lambda_{j+1} \prod_{m=i+1}^{j-1} \left(\sum_{k=1}^{m} \lambda_k - \lambda_{m+1}\right)}}, \quad for \;\; i < j. \tag{2.2}$$

Then we set $D = L^{-1}AL$ that solves the problem. It is not difficult to calculate the inverse of the matrix L, however, in the general cases we propose a method for computing the inverse of L for $n = 5$. For general case it can be computed using induction. For $n = 5$, we have

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\sqrt{\frac{\lambda_3\lambda_1}{\lambda_2(\lambda_1+\lambda_2-\lambda_3)}} & 1 & 0 & 0 & 0 \\ 0 & -\sqrt{\frac{\lambda_4(\lambda_1+\lambda_2)}{\lambda_3(\lambda_1+\lambda_2+\lambda_3-\lambda_4)}} & 1 & 0 & 0 \\ 0 & 0 & -\sqrt{\frac{\lambda_5(\lambda_1+\lambda_2+\lambda_3)}{\lambda_4(\lambda_1+\lambda_2+\lambda_3+\lambda_4-\lambda_5)}} & 1 & 0 \\ 0 & 0 & 0 & -\sqrt{-\frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4}{\lambda_5}} & 1 \end{bmatrix}.$$

The inverse of matrix L is:

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ l_{21}^{-1} & 1 & 0 & 0 & 0 \\ l_{31}^{-1} & l_{32}^{-1} & 1 & 0 & 0 \\ l_{41}^{-1} & l_{42}^{-1} & l_{43}^{-1} & 1 & 0 \\ l_{51}^{-1} & l_{52}^{-1} & l_{53}^{-1} & l_{54}^{-1} & 1 \end{bmatrix},$$

where

$$\begin{aligned}
l_{21}^{-1} &= \sqrt{\frac{\lambda_3\lambda_1}{\lambda_2(\lambda_1+\lambda_2-\lambda_3)}}, \\
l_{31}^{-1} &= \sqrt{\frac{\lambda_3\lambda_1}{\lambda_2(\lambda_1+\lambda_2-\lambda_3)}}, \\
l_{41}^{-1} &= \sqrt{\frac{\lambda_5(\lambda_1+\lambda_2+\lambda_3)}{\lambda_4(\lambda_1+\lambda_2+\lambda_3+\lambda_4-\lambda_5)}} \sqrt{\frac{\lambda_4(\lambda_1+\lambda_2)}{\lambda_3(\lambda_1+\lambda_2+\lambda_3-\lambda_4)}} \sqrt{\frac{\lambda_3\lambda_1}{\lambda_2(\lambda_1+\lambda_2-\lambda_3)}}, \\
l_{51}^{-1} &= \sqrt{\frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4}{\lambda_5}} \sqrt{\frac{\lambda_5(\lambda_1+\lambda_2+\lambda_3)}{\lambda_4(\lambda_1+\lambda_2+\lambda_3+\lambda_4-\lambda_5)}} \sqrt{\frac{\lambda_4(\lambda_1+\lambda_2)}{\lambda_3(\lambda_1+\lambda_2+\lambda_3-\lambda_4)}} \sqrt{\frac{\lambda_3\lambda_1}{\lambda_2(\lambda_1+\lambda_2-\lambda_3)}},
\end{aligned}$$

$$\begin{aligned}
l_{32}^{-1} &= \sqrt{\frac{\lambda_4(\lambda_1+\lambda_2)}{\lambda_3(\lambda_1+\lambda_2+\lambda_3-\lambda_4)}}, \\
l_{42}^{-1} &= \sqrt{\frac{\lambda_5(\lambda_1+\lambda_2+\lambda_3)}{\lambda_4(\lambda_1+\lambda_2+\lambda_3+\lambda_4-\lambda_5)}} \sqrt{\frac{\lambda_4(\lambda_1+\lambda_2)}{\lambda_3(\lambda_1+\lambda_2+\lambda_3-\lambda_4)}}, \\
l_{52}^{-1} &= \sqrt{\frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4}{\lambda_5}} \sqrt{\frac{\lambda_5(\lambda_1+\lambda_2+\lambda_3)}{\lambda_4(\lambda_1+\lambda_2+\lambda_3+\lambda_4-\lambda_5)}} \sqrt{\frac{\lambda_4(\lambda_1+\lambda_2)}{\lambda_3(\lambda_1+\lambda_2+\lambda_3-\lambda_4)}},
\end{aligned}$$

$$l_{43}^{-1} = \sqrt{\frac{\lambda_5(\lambda_1+\lambda_2+\lambda_3)}{\lambda_4(\lambda_1+\lambda_2+\lambda_3+\lambda_4-\lambda_5)}},$$
$$l_{53}^{-1} = \sqrt{\frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4}{\lambda_5}}\sqrt{\frac{\lambda_5(\lambda_1+\lambda_2+\lambda_3)}{\lambda_4(\lambda_1+\lambda_2+\lambda_3+\lambda_4-\lambda_5)}},$$
$$l_{54}^{-1} = \sqrt{-\frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4}{\lambda_5}}.$$

According to the shape of the entries of the matrix L^{-1} in the state $n = 5$, we claim that the entries of this matrix for $i > j$ in the general case are as follows:

$$l_{i,j}^{-1} = \sqrt{\frac{\lambda_{i+1}(\lambda_1+\lambda_2+\cdots+\lambda_{i-1})}{\lambda_i(\lambda_1+\lambda_2+\cdots+\lambda_i-\lambda_{i+1})}}\sqrt{\frac{\lambda_i(\lambda_1+\lambda_2+\cdots+\lambda_{i-2})}{\lambda_{i-1}(\lambda_1+\lambda_2+\cdots+\lambda_{i-1}-\lambda_i)}}\cdots\sqrt{\frac{\lambda_j\left(\lambda_1+\lambda_2+\cdots+\lambda_j\right)}{\lambda_{j+1}\left(\lambda_1+\lambda_2+\cdots+\lambda_{i-1}-\lambda_j\right)}}. \tag{2.3}$$

To prove the relation (2.3), if we assume that the matrix

$$T = (t_{i,j})_{n\times n} = LL^{-1}, \tag{2.4}$$

then we show that $T = I_n$. Because L is a lower triangular matrix, then the matrix L^{-1} also is a lower triangular matrix, and consequently, the matrix T is a lower triangular matrix and it is trivial that all entries of the main diagonal of T are one. Now we show that $t_{ij} = 0$ for $i > j$. Since T is a lower bidiagonal matrix, from (2.4) the following relation can be written

$$t_{i,j} = l_{i+1,i}l_{i,j}^{-1} + l_{i+1,j}^{-1}. \tag{2.5}$$

Now we have

$$l_{i+1,i} = -\sqrt{\frac{\lambda_{i+2}\,(\lambda_1+\lambda_2+\cdots+\lambda_i)}{\lambda_{i+1}\,(\lambda_1+\lambda_2+\cdots+\lambda_i-\lambda_{i+2})}}.$$

According to Equation (2.3), if we write $l_{i+1,j}^{-1}$, we see that the beginning of it has the negative of the member $l_{i+1,i}$ and the rest is the same as in $l_{i,j}^{-1}$, so $t_{ij} = 0$ for $i > j$.

For $n = k$ we write the upper triangular matrix $A = (a_{ij})$ as following

$$a_{ii} = \lambda_i, i = 1, 2, \cdots, k,$$
$$a_{12} = \sqrt{\frac{\lambda_2\lambda_1(\lambda_1+\lambda_2-\lambda_3)}{\lambda_3}},$$
$$a_{13} = \sqrt{\frac{\lambda_2\lambda_1(\lambda_1+\lambda_2)(\lambda_1+\lambda_2+\lambda_3-\lambda_4)}{\lambda_4(\lambda_1+\lambda_2-\lambda_3)}},$$
$$a_{14} = \sqrt{\frac{\lambda_2\lambda_1(\lambda_1+\lambda_2)(\lambda_1+\lambda_2+\lambda_3)(\lambda_1+\lambda_2+\lambda_3+\lambda_4-\lambda_5)}{\lambda_5(\lambda_1+\lambda_2-\lambda_3)(\lambda_1+\lambda_2+\lambda_3-\lambda_4)}},$$
$$\vdots$$
$$a_{1k-1} = \sqrt{\frac{\lambda_2\lambda_1(\lambda_1+\lambda_2)(\lambda_1+\lambda_2+\lambda_3)\cdots(\lambda_1+\lambda_2+\lambda_3+\cdots+\lambda_{k-1}-\lambda_k)}{\lambda_k(\lambda_1+\lambda_2-\lambda_3)(\lambda_1+\lambda_2+\lambda_3-\lambda_4)\cdots(\lambda_1+\lambda_2+\lambda_3+\cdots+\lambda_{k-2}+-\lambda_{k-1})}},$$

$$a_{1k} = \sqrt{\frac{\lambda_2\lambda_1(\lambda_1+\lambda_2)(\lambda_1+\lambda_2+\lambda_3)\cdots(\lambda_1+\lambda_2+\lambda_3+\cdots+\lambda_{k-1})}{(\lambda_1+\lambda_2-\lambda_3)(\lambda_1+\lambda_2+\lambda_3-\lambda_4)\cdots(\lambda_1+\lambda_2+\lambda_3+\cdots+\lambda_{k-1}-\lambda_k)}},$$
$$a_{23} = \frac{\lambda_2-\lambda_3}{\lambda_1+\lambda_2-\lambda_3}\sqrt{\frac{\lambda_3(\lambda_1+\lambda_2)(\lambda_1+\lambda_2+\lambda_3-\lambda_4)}{\lambda_4}},$$
$$a_{24} = \frac{\lambda_2-\lambda_3}{\lambda_1+\lambda_2-\lambda_3}\sqrt{\frac{\lambda_3(\lambda_1+\lambda_2)(\lambda_1+\lambda_2+\lambda_3)(\lambda_1+\lambda_2+\lambda_3+\lambda_4-\lambda_5)}{\lambda_5(\lambda_1+\lambda_2+\lambda_3-\lambda_4)}},$$
$$\vdots$$
$$a_{25} = \frac{\lambda_2-\lambda_3}{\lambda_1+\lambda_2-\lambda_3}\sqrt{\frac{\lambda_3(\lambda_1+\lambda_2)(\lambda_1+\lambda_2+\lambda_3)(\lambda_1+\lambda_2+\lambda_3+\lambda_4)}{(\lambda_1+\lambda_2+\lambda_3-\lambda_4)(\lambda_1+\lambda_2+\lambda_3+\lambda_4-\lambda_5)}},$$

$$a_{34} = \frac{\lambda_3-\lambda_4}{\lambda_1+\lambda_2+\lambda_3-\lambda_4} \sqrt{\frac{\lambda_4(\lambda_1+\lambda_2+\lambda_3)(\lambda_1+\lambda_2+\lambda_3+\lambda_4-\lambda_5)}{\lambda_5}},$$
$$a_{35} = \frac{\lambda_3-\lambda_4}{\lambda_1+\lambda_2+\lambda_3-\lambda_4} \sqrt{\frac{\lambda_4(\lambda_1+\lambda_2+\lambda_3)(\lambda_1+\lambda_2+\lambda_3+\lambda_4)}{\lambda_1+\lambda_2+\lambda_3+\lambda_4-\lambda_5}},$$
$$a_{45} = \frac{(\lambda_4-\lambda_5)\sqrt{\lambda_5(\lambda_1+\lambda_2+\lambda_3+\lambda_4)}}{\lambda_1+\lambda_2+\lambda_3+\lambda_4-\lambda_5}.$$

In this case, for the matrix $D = L^{-1}AL = (d_{ij})$ we have

$$d_{ii} = 0, i = 1, 2, \cdots, 5,$$
$$d_{12} = d_{21} = \sqrt{\frac{\lambda_3\lambda_2\lambda_1}{(\lambda_1+\lambda_2-\lambda_3)}},$$
$$d_{13} = d_{31} = \sqrt{\frac{\lambda_4\lambda_2\lambda_1(\lambda_1+\lambda_2)}{(\lambda_1+\lambda_2-\lambda_3)(\lambda_1+\lambda_2+\lambda_3-\lambda_4)}},$$
$$d_{14} = d_{41} = \sqrt{\frac{\lambda_5\lambda_2\lambda_1(\lambda_1+\lambda_2)(\lambda_1+\lambda_2+\lambda_3)}{(\lambda_1+\lambda_2-\lambda_3)(\lambda_1+\lambda_2+\lambda_3-\lambda_4)(\lambda_1+\lambda_2+\lambda_3+\lambda_4-\lambda_5)}},$$
$$\vdots$$
$$d_{1,k-1} = d_{k-1,1} = \sqrt{\frac{\lambda_k\lambda_2\lambda_1(\lambda_1+\lambda_2)(\lambda_1+\lambda_2+\lambda_3)\cdots(\lambda_1+\lambda_2+\lambda_3+\cdots+\lambda_{k-1})}{(\lambda_1+\lambda_2-\lambda_3)(\lambda_1+\lambda_2+\lambda_3-\lambda_4)(\lambda_1+\lambda_2+\lambda_3+\cdots+\lambda_{k-1}-\lambda_k)}},$$
$$d_{1k} = d_{k1} = \sqrt{-\frac{\lambda_2\lambda_1(\lambda_1+\lambda_2)(\lambda_1+\lambda_2+\lambda_3)\cdots(\lambda_1+\lambda_2+\lambda_3+\lambda_{k-1})}{(\lambda_1+\lambda_2-\lambda_3)(\lambda_1+\lambda_2+\lambda_3-\lambda_4)\cdots(\lambda_1+\lambda_2+\lambda_3+\cdots+\lambda_{k-1}-\lambda_k)}},$$
$$d_{23} = d_{32} = -\lambda_3\sqrt{\frac{\lambda_4(\lambda_1+\lambda_2)}{\lambda_3(\lambda_1+\lambda_2+\lambda_3-\lambda_4)}},$$
$$d_{24} = d_{42} = -\lambda_3\sqrt{\frac{\lambda_5(\lambda_1+\lambda_2)(\lambda_1+\lambda_2+\lambda_3)}{(\lambda_1+\lambda_2+\lambda_3+\lambda_4-\lambda_5)\lambda_3(\lambda_1+\lambda_2+\lambda_3-\lambda_4)}},$$
$$d_{25} = d_{52} = -\lambda_3\sqrt{-\frac{(\lambda_1+\lambda_2+\lambda_3+\lambda_4)(\lambda_1+\lambda_2+\lambda_3)(\lambda_1+\lambda_2)}{(\lambda_1+\lambda_2+\lambda_3+\lambda_4-\lambda_5)\lambda_3(\lambda_1+\lambda_2+\lambda_3-\lambda_4)}},$$
$$\vdots$$
$$d_{34} = d_{43} = -\lambda_4\sqrt{\frac{\lambda_5(\lambda_1+\lambda_2+\lambda_3)}{\lambda_4(\lambda_1+\lambda_2+\lambda_3+\lambda_4-\lambda_5)}},$$
$$d_{35} = d_{53} = -\lambda_4\sqrt{-\frac{(\lambda_1+\lambda_2+\lambda_3+\lambda_4)(\lambda_1+\lambda_2+\lambda_3)}{\lambda_4(\lambda_1+\lambda_2+\lambda_3+\lambda_4-\lambda_5)}},$$
$$\vdots$$
$$d_{45} = d_{54} = -\lambda_5\sqrt{-\frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4}{\lambda_5}},$$
$$\vdots$$

Now we show that the matrix D is EDM. To do this, we prove that the entries of all rows (as well as columns, because of the symmetry of D) form a decreasing sequence. At first, we show that

$d_{12} \geq d_{13}$. Since $\lambda_4 \geq \lambda_3$ and $\lambda_1 + \lambda_2 + \lambda_3 > 0$, we have

$$\begin{aligned}
\lambda_4(\lambda_1+\lambda_2+\lambda_3) &\geq \lambda_3(\lambda_1+\lambda_2+\lambda_3),\\
\Rightarrow \lambda_4(\lambda_1+\lambda_2) &\geq \lambda_3(\lambda_1+\lambda_2+\lambda_3-\lambda_4),\\
\lambda_4\lambda_2\lambda_1(\lambda_1+\lambda_2) &\leq \lambda_3\lambda_2\lambda_1(\lambda_1+\lambda_2+\lambda_3-\lambda_4),\\
\Rightarrow \frac{\lambda_4\lambda_2\lambda_1(\lambda_1+\lambda_2)}{(\lambda_1+\lambda_2-\lambda_3)(\lambda_1+\lambda_2+\lambda_3-\lambda_4)} &\leq \frac{\lambda_3\lambda_2\lambda_1}{(\lambda_1+\lambda_2-\lambda_3)},\\
\Rightarrow d_{12} &\geq d_{13}.
\end{aligned}$$

For the other two consecutive elements of each row of the matrix, we can similarly show that the above inequality holds, and then all rows of this matrix form a decreasing order. Now we show that the matrix D is EDM. Let d_{ij} and d_{jk} be two entries of matrix D. Then it is easy to see that $d_{ij} + d_{jk} \geq d_{ik}$, because if $j < k$ then $d_{ij} \geq d_{ik}$ and if $j > k$, then we have only a conjecture that $d_{ij}^2 + d_{jk}^2 \geq d_{ik}^2$ (numerical examples, even with dimensions greater than 100, confirm this point), so the triangular inequality holds for every three entries of a matrix D as d_{ij}, d_{jk} and d_{ik}. Therefore D is EDM. □

Theorem 2.2. *Given $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ with the following conditions*

1. $\lambda_1 > 0 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$, *and* $\sum_{i=1}^{n} \lambda_i = 0$,
2. $\lambda_i = -\frac{\lambda_1}{n-1}, \ \ i = 2, 3, \ldots, n$

the following regular spherical distance matrix is realized spectrum σ

$$D = \begin{bmatrix} 0 & \frac{\lambda_1}{n-1} & \frac{\lambda_1}{n-1} & \cdots & \frac{\lambda_1}{n-1} & \frac{\lambda_1}{n-1} \\ \frac{\lambda_1}{n-1} & 0 & \frac{\lambda_1}{n-1} & \cdots & \frac{\lambda_1}{n-1} & \frac{\lambda_1}{n-1} \\ \frac{\lambda_1}{n-1} & \frac{\lambda_1}{n-1} & 0 & \cdots & \frac{\lambda_1}{n-1} & \frac{\lambda_1}{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{\lambda_1}{n-1} & \frac{\lambda_1}{n-1} & \frac{\lambda_1}{n-1} & \cdots & \frac{\lambda_1}{n-1} & 0 \end{bmatrix}. \tag{2.6}$$

Proof. We set the matrices A and L as the following:

$$A = \begin{bmatrix} -\frac{\lambda_1}{n-1} & 0 & 0 & \cdots & 0 & \frac{\lambda_i}{n-1} \\ 0 & -\frac{\lambda_1}{n-1} & 0 & \cdots & 0 & 2\frac{\lambda_1}{n-1} \\ 0 & 0 & -\frac{\lambda_1}{n-1} & \cdots & 0 & 3\frac{\lambda_1}{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{\lambda_1}{n-1} & (n-1)\frac{\lambda_1}{n-1} \\ 0 & 0 & 0 & \cdots & 0 & \frac{\lambda_1}{n-1} \end{bmatrix},$$

$$L = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}.$$

It is easy to see that we have $D = L^{-1}AL$. □

Remark 2.3. The matrix D in (2.6) is a regular spherical matrix.

Proof. Since the matrix of eigenvectors of matrix D as following:

$$E = \begin{bmatrix} 1 & -1 & -1 & -1 & \cdots & -1 & -1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 1 & 0 \\ \vdots & & & \vdots & & & \vdots \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

We see that

$$\frac{e^T De}{n} = \frac{1}{n}(1, 1, \ldots, 1)\begin{pmatrix} \sum_1^{n-1} \frac{\lambda_1}{n-1} \\ \vdots \\ \sum_1^{n-1} \frac{\lambda_1}{n-1} \end{pmatrix} = \frac{1}{n} n(n-1)\frac{\lambda_1}{n-1} = \lambda_1,$$

then $\frac{e^T De}{n}$ is an eigenvalue of D corresponding to eigenvector e, consequently by [7] D is a regular spherical matrix. □

3. Numerical Examples

In this section, we provide some numerical examples.

Example 3.1. Consider $\sigma = \{9, -4, -3, -2\}$. Then by Theorem (2.1) for n=4 we present a distance matrix that σ is its spectrum. We solve this problem with the two methods given in Theorem 2.1.

The first method: Let

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & -1 & 1 & 1 \end{bmatrix},$$

and

$$A := \begin{bmatrix} 9 & 13/2 & 1/2 & 5/2 \\ 0 & -4 & -1 & 1/2 \\ 0 & 0 & -3 & 1/2 \\ 0 & 0 & 0 & -2 \end{bmatrix},$$

then

$$D = L^{-1}AL = \begin{bmatrix} 0 & 7/2 & 3 & 5/2 \\ 7/2 & 0 & 5/2 & 3 \\ 3 & 5/2 & 0 & 7/2 \\ 5/2 & 3 & 7/2 & 0 \end{bmatrix}.$$

The second method: Let

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3/8\sqrt{6} & 1 & 0 & 0 \\ 0 & -1/6\sqrt{30} & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix},$$

$$A = \begin{bmatrix} 9 & 4\sqrt{6} & 3\sqrt{5} & 3/2\sqrt{5} \\ 0 & -4 & -1/8\sqrt{30} & -1/16\sqrt{30} \\ 0 & 0 & -3 & -1/2 \\ 0 & 0 & 0 & -2 \end{bmatrix},$$

then

$$D = L^{-1}AL = \begin{bmatrix} 0 & 3/2\sqrt{6} & 3/2\sqrt{5} & 3/2\sqrt{5} \\ 3/2\sqrt{6} & 0 & 1/2\sqrt{6}\sqrt{5} & 1/2\sqrt{6}\sqrt{5} \\ 3/2\sqrt{5} & 1/2\sqrt{6}\sqrt{5} & 0 & 2 \\ 3/2\sqrt{5} & 1/2\sqrt{6}\sqrt{5} & 2 & 0 \end{bmatrix}.$$

Example 3.2. Consider $\sigma = \{25, -11, -5, -5, -\frac{7}{2}, -\frac{1}{2}\}$. Then

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{5\sqrt{1045}}{209} & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1/5\sqrt{21} & 1 & 0 & 0 \\ 0 & 0 & 0 & -2/7\sqrt{7} & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix},$$

$$L^{-1}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ \frac{5\sqrt{1045}}{209} & 1 & 0 & 0 & 0 & 0\\ \frac{5\sqrt{1045}}{209} & 1 & 1 & 0 & 0 & 0\\ \frac{\sqrt{21}\sqrt{1045}}{209} & 1/5\sqrt{21} & 1/5\sqrt{21} & 1 & 0 & 0\\ \frac{2\sqrt{7}\sqrt{21}\sqrt{1045}}{1463} & \frac{2\sqrt{7}\sqrt{21}}{35} & \frac{2\sqrt{7}\sqrt{21}}{35} & 2/7\sqrt{7} & 1 & 0\\ \frac{2\sqrt{7}\sqrt{21}\sqrt{1045}}{1463} & \frac{2\sqrt{7}\sqrt{21}}{35} & \frac{2\sqrt{7}\sqrt{21}}{35} & 2/7\sqrt{7} & 1 & 1 \end{bmatrix},$$

$$A=\begin{bmatrix} 25 & \sqrt{1045} & \frac{14\sqrt{1045}}{19} & \frac{15\sqrt{21945}}{133} & \frac{4\sqrt{3135}}{19} & 2/19\sqrt{3135}\\ 0 & -11 & -\frac{84}{19} & -\frac{90\sqrt{21}}{133} & -\frac{24\sqrt{3}}{19} & -\frac{12\sqrt{3}}{19}\\ 0 & 0 & -5 & 0 & 0 & 0\\ 0 & 0 & 0 & -5 & -2/5\sqrt{7} & -1/5\sqrt{7}\\ 0 & 0 & 0 & 0 & -7/2 & -3/2\\ 0 & 0 & 0 & 0 & 0 & -1/2 \end{bmatrix},$$

$$D=L^{-1}AL=\begin{bmatrix} 0 & \frac{5\sqrt{1045}}{19} & \frac{5\sqrt{1045}}{19} & 1/19\sqrt{3135}\sqrt{7} & 2/19\sqrt{3135} & 2/19\sqrt{3135}\\ \frac{5\sqrt{1045}}{19} & 0 & 5 & \sqrt{3}\sqrt{7} & 2\sqrt{3} & 2\sqrt{3}\\ \frac{5\sqrt{1045}}{19} & 5 & 0 & \sqrt{3}\sqrt{7} & 2\sqrt{3} & 2\sqrt{3}\\ 1/19\sqrt{21}\sqrt{1045} & \sqrt{21} & \sqrt{21} & 0 & \sqrt{7} & \sqrt{7}\\ 2/19\sqrt{3}\sqrt{1045} & 2\sqrt{3} & 2\sqrt{3} & \sqrt{7} & 0 & 1/2\\ 2/19\sqrt{3}\sqrt{1045} & 2\sqrt{3} & 2\sqrt{3} & \sqrt{7} & 1/2 & 0 \end{bmatrix}.$$

By (1.2) we find the matrix $T(D)=-\frac{1}{2}(I-\frac{ee^T}{n})D(I-\frac{ee^T}{n})$ as

$$T(D)=\begin{bmatrix} 4.143843 & -1.074683 & -1.074683 & -0.9499423 & -0.522267 & -0.522267\\ -1.074683 & 2.213754 & -0.2862465 & -0.3078960 & -0.2724638 & -0.2724638\\ -1.074683 & -0.2862465 & 2.213754 & -0.3078960 & -0.2724638 & -0.2724638\\ -0.9499433 & -0.3078956 & -0.3078956 & 1.753032 & -0.0936488 & -0.0936488\\ -0.5222664 & -0.2724642 & -0.2724642 & -0.0936491 & 0.7054213 & 0.4554213\\ -0.5222664 & -0.2724642 & -0.2724642 & -0.0936491 & 0.4554213 & 0.7054213 \end{bmatrix}.$$

The eigenvalues of $T(D)$ are

$$\begin{bmatrix} 5.0962034515219839627 + 0.0\,i \\ -3.41 \times 10^{-21} + 0.0\,i \\ 1.5587362413318417489 + 0.0\,i \\ 2.3302837993164271684 + 0.0\,i \\ 2.5000000000000000003 + 0.0\,i \\ 0.25000000000000000011 + 0.0\,i \end{bmatrix},$$

and we see that all the eigenvalues of T are nonnegative. Hence, the matrix T is positive semidefinite and by [3] the matrix D is EDM.

Example 3.3. Let $n = 10$ and $\sigma = \{42, -10, -9, -6, -5, -4, -3, -2, -2, -1\}$. By Theorem 3.4 we find an EDM matrix with σ being its spectrum. To present the matrices we use 4 decimal places. The matrices L (along with its inverse) and the matrix A are given as follows

$$L = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ -0.9599 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & -0.8576 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & -0.9332 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & -0.9220 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & -0.9045 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.8731 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -1.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.8660 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -1.0 & 1.0 \end{bmatrix},$$

$$L^{-1} = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.9599 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.8236 & 0.8576 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.7685 & 0.8003 & 0.9332 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.7088 & 0.7380 & 0.8605 & 0.9220 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.6412 & 0.6677 & 0.7785 & 0.8339 & 0.9045 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.5593 & 0.5826 & 0.6793 & 0.7278 & 0.7896 & 0.8731 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.5593 & 0.5826 & 0.6793 & 0.7278 & 0.7896 & 0.8731 & 1.0 & 1.0 & 0.0 & 0.0 \\ 0.4847 & 0.5049 & 0.5887 & 0.6306 & 0.6838 & 0.7560 & 0.8660 & 0.8660 & 1.0 & 0.0 \\ 0.4847 & 0.5049 & 0.5887 & 0.6306 & 0.6838 & 0.7560 & 0.8660 & 0.8660 & 1.0 & 1.0 \end{bmatrix},$$

$$A = \begin{bmatrix} 42.0 & 43.74 & 39.80 & 33.82 & 28.34 & 23.50 & 19.58 & 13.98 & 9.692 & 4.846 \\ 0.0 & -10.0 & -0.9099 & -0.7731 & -0.6479 & -0.5374 & -0.4477 & -0.3198 & -0.2215 & -0.1108 \\ 0.0 & 0.0 & -9.0 & -2.549 & -2.137 & -1.771 & -1.476 & -1.054 & -0.7304 & -0.3652 \\ 0.0 & 0.0 & 0.0 & -6.0 & -0.8382 & -0.6951 & -0.5789 & -0.4136 & -0.2865 & -0.1433 \\ 0.0 & 0.0 & 0.0 & 0.0 & -5.0 & -0.8292 & -0.6908 & -0.4935 & -0.3419 & -0.1709 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -4.0 & -0.8332 & -0.5953 & -0.4122 & -0.2062 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -3.0 & -0.7143 & -0.4948 & -0.2475 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -2.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -2.0 & -0.5000 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -1.0 \end{bmatrix},$$

then

$$D = L^{-1}AL = \begin{bmatrix} 0.0 & 9.602 & 8.235 & 7.687 & 7.087 & 6.410 & 5.595 & 5.595 & 4.846 & 4.846 \\ 9.602 & 0.0 & 7.719 & 7.205 & 6.643 & 6.009 & 5.245 & 5.245 & 4.542 & 4.542 \\ 8.235 & 7.719 & 0.0 & 5.600 & 5.163 & 4.670 & 4.077 & 4.077 & 3.530 & 3.530 \\ 7.687 & 7.205 & 5.600 & 0.0 & 4.610 & 4.170 & 3.640 & 3.640 & 3.152 & 3.152 \\ 7.087 & 6.643 & 5.163 & 4.610 & 0.0 & 3.618 & 3.158 & 3.158 & 2.735 & 2.735 \\ 6.410 & 6.009 & 4.670 & 4.170 & 3.618 & 0.0 & 2.619 & 2.619 & 2.268 & 2.268 \\ 5.595 & 5.245 & 4.077 & 3.640 & 3.158 & 2.619 & 0.0 & 2.0 & 1.732 & 1.732 \\ 5.595 & 5.245 & 4.077 & 3.640 & 3.158 & 2.619 & 2.0 & 0.0 & 1.732 & 1.732 \\ 4.846 & 4.542 & 3.530 & 3.152 & 2.735 & 2.268 & 1.732 & 1.732 & 0.0 & 1.0 \\ 4.846 & 4.542 & 3.530 & 3.152 & 2.735 & 2.268 & 1.732 & 1.732 & 1.0 & 0.0 \end{bmatrix}.$$

Example 3.4. Consider $\sigma = \{8, -2, -2, -2, -2\}$. Then by Theorem (2.2), we find a regular spherical matrix with eigenvalues σ. Let

$$A = \begin{bmatrix} -2 & 0 & 0 & 0 & 2 \\ 0 & -2 & 0 & 0 & 4 \\ 0 & 0 & -2 & 0 & 6 \\ 0 & 0 & 0 & -2 & 8 \\ 0 & 0 & 0 & 0 & 8 \end{bmatrix}.$$

Then we have by matrix L which is defined in Theorem (2.2)

$$D = L^{-1}AL = \begin{bmatrix} 0 & 2 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 & 2 \\ 2 & 2 & 0 & 2 & 2 \\ 2 & 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 2 & 0 \end{bmatrix}.$$

Remark 3.5. The matrix D in example (3.4) is EDM matrix.

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