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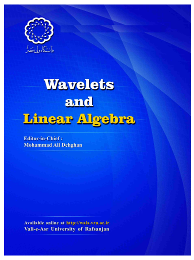

Some classes of interval tensors and their properties

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Abstract

First, we define and investigate some new classes of interval tensors, called interval exceptionally regular tensors ($ER-$tensor) and interval $wP-$tensors which is relevant to interval strictly semi-positive tensors. Also, we show that $ER-$tensor is a wide class of interval tensors, which includes many important structured tensors. Second, some classes of interval matrices are extended to interval tensors, such as interval $R(R_0)$-tensor and column sufficient interval tensor. We discuss their relationships with interval positive semi-definite tensors and some other structured interval tensors. In addition, necessary and sufficient conditions for interval (strictly) copositive and interval E_0-tensors are presented and investigated. Finally, we extend the concept of the column sufficient interval matrix to the column sufficient interval tensor.

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1. Introduction

Tensors (multidimensional arrays) have many similarities with matrices and many related results of matrices such as eigenvalue and eigenvector can be extended to tensors in the multi-linear algebra. Furthermore, structured matrices such as interval matrices can also be extended to interval tensors and these are becoming the focus of recent tensor research [1, 14].

During the last three decades the role of compact intervals as independent objects has continuously increased in numerical analysis when verifying or enclosing solutions to various mathematical problems or when proving that such problems cannot have a solution in a particularly given domain. This was possible by viewing intervals as extensions of real or complex numbers, and by introducing interval matrices and interval tensors. Interval analysis was first introduced by Moore [12, 13]. Since their introduction, interval matrices have been used in some applications such as dynamical systems, mechanics, and engineering [10, 13, 15, 16]. Interval tensors have been treasured for solving multi-linear systems of equations [1], and this has motivated further research [14].

The linear complementarity problem (LCP) appears in many optimization and operations research models such as quadratic programming, bimatrix games, or equilibria in specific economies [3]. Properties of the solution set of interval LCP relate with properties of interval structure matrices, such as $R(R_0)-$matrices and column sufficient matrices [9]. Recently, Song and Qi [18] extended the linear complementarity problem to the tensor complementarity problem, a special class of nonlinear complementarity problems, denoted by TCP. During last several years, the TCP attains much attraction and has been studied extensively with respect to theory, to solution methods and applications. In recent years, various tensors with special structures have been studied (for details, see [2, 5, 18, 19]). Song and Qi [17] studied $P(P_0)-$tensors and $B(B_0)-$tensors and the properties of TCP was studied by Ding, Luo and Qi [5] for $wP-$tensor. Also, concepts such as $R(R_0)$-tensor and exceptionally regular tensor have been introduced and considered the solvability of the TCP [18, 19]. In the theory of TCP, column sufficient tensors were introduced by Chen et al. [2] to the study unique solution of TCP.

As discussed previously with regard to application of TCP, it is appropriate that we extend the concept of interval $R(R_0)-$matrices and interval column sufficient matrices to interval $R(R_0)-$tensors and interval column sufficient tensors. Also, we show that those properties of interval structure matrices still true for interval structure tensors.

In this paper, we make the following contributions. In Section 2, the basic definitions and concepts used in this paper are provided. In Section 3, we define important classes of interval tensors, such as $R(R_0)$-tensor, ER-tensor, wP-tensor, and column sufficient of the interval tensors and show that some results in [9, 19] hold for these classes of interval tensors as well. Also, the relationship between these classes of interval tensors and some other structured tensors is shown.

2. Background and basic concepts

We begin this section with some definitions and statements that are needed for the main results of our work. First, add a comment on the notation that is used. Vectors are written as $\{x, y, \cdots\}$, matrices correspond to $\{A, B, \cdots\}$ and tensors are written as $\{\mathcal{A}, \mathcal{B}, \cdots\}$. Denote $[n] = \{1, 2, \cdots, n\}$.

Let $\mathbb{R}(\mathbb{C})$ be the set of all real (complex) numbers, and $\mathbb{R}^n(\mathbb{C}^n)$ be the set of all dimension n real (complex) vectors. $x \geq 0 \quad (x > 0)$ means $x_i \geq 0 \quad (x_i > 0)$ for all $i \in [n]$. Let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x \geq 0\}$ be the positive cone in $\mathbb{R}^n$. An order m dimension n real tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$, denoted by $\mathcal{A} \in \mathbb{R}^{n_1 \times \cdots \times n_m}$, consists of n^m entries:

$$a_{i_1 i_2 \cdots i_m} \in \mathbb{R}, \quad \forall\ i_j = 1, \cdots, n, \quad j = 1, \cdots, m.$$

If $n_1 = \cdots = n_m = n$, then it is said $\mathcal{A}$ is an m-order n-dimensional cubical tensor or for simplicity just m-order n-dimensional tensor. A vector is a tensor of order 1 and a matrix is a tensor of order 2. A tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{n_1 \times \cdots \times n_m}$ is called nonnegative (positive) if

$$a_{i_1 i_2 \cdots i_m} \geq 0 \ (a_{i_1 i_2 \cdots i_m} > 0), \quad \forall\ i_j = 1, \cdots, n, \quad j = 1, \cdots, m.$$

A tensor $\mathcal{A}$ is said to be symmetric [11] if its entries $a_{i_1 i_2 \cdots i_m}$ are invariant under any permutation of m indices $(a_{i_1 i_2 \cdots i_m})$. For more information about basic definitions and properties of tensors refer to [8, 11]. All the tensors discussed in this paper are real.

For any two tensors, $\mathcal{A} = (a_{i_1 \cdots i_m})$, and $\mathcal{B} = (b_{i_1 \cdots i_m}) \in \mathbb{R}^{n_1 \times \cdots \times n_m}$ of identical order and dimensions, their inner product is defined as

$$\langle \mathcal{A}, \mathcal{B} \rangle = \sum_{i_1 \cdots i_m} a_{i_1 \cdots i_m} b_{i_1 \cdots i_m}.$$

Definition 2.1. If $\mathcal{A} \in \mathbb{R}^{n_1 \times \cdots \times n_m}$ is an m-order tensor and $B \in \mathbb{R}^{J \times n_k}$ is a matrix, then $\mathcal{A} \times_k B$ denotes the mode-k product of $\mathcal{A}$ with B, this product is of size $n_1 \times \cdots \times n_{k-1} \times J \times n_{k+1} \times \cdots \times n_m$ and each element of it is defined as follows

$$(\mathcal{A} \times_k B)_{i_1, \cdots, i_{k-1}, j, i_{k+1}, \cdots, i_m} = \sum_{i_k=1}^{n_k} a_{i_1 \cdots i_m} b_{j, i_k}.$$

If we do the mode-k product of $\mathcal{A}$ and B for all possible $k \in [m]$ as

$$\mathcal{A} \times_1 B \times_2 \cdots \times_m B,$$

and B is reduced to some row vector, say $x^T = (x_1, \cdots, x_n)$, the following frequently used notations are given as below:

$$\mathcal{A}x^m \equiv \mathcal{A} \times_1 x^T \times_2 \cdots \times_m x^T = \sum_{i_1 \cdots i_m = 1}^{n} a_{i_1 \cdots i_m} x_{i_1} \cdots x_{i_m} \in \mathbb{R},$$
$$\mathcal{A}x^{m-1} \equiv \mathcal{A} \times_2 x^T \times_3 \cdots \times_m x^T = \sum_{i_2 \cdots i_m = 1}^{n} a_{i, i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} \in \mathbb{R}^n.$$

We call a number $\lambda \in \mathbb{C}$ an eigenvalue of $\mathcal{A}$ if it and a nonzero vector $x \in \mathbb{C}^n$ are solutions of the following homogeneous polynomial equations:

$$\left(\mathcal{A}x^{m-1}\right)_i = \lambda x_i^{m-1}, \quad \forall i = 1, \cdots, n, \tag{2.1}$$

and call the solution x an eigenvector of $\mathcal{A}$ associated with the eigenvalue λ. If we denote $x^{[m-1]}$ as a vector in $\mathbb{C}^n$ such that its ith component is x_i^{m-1}, then (2.1) can be simply expressed as

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}.$$

The set of all the eigenvalues of $\mathcal{A}$ is called the spectrum of $\mathcal{A}$. The largest modulus of the elements in the spectrum of $\mathcal{A}$ is called the spectral radius of $\mathcal{A}$, denoted as $\rho(\mathcal{A})$.

Let us recall the PerronFrobenius theorem for nonnegative tensors given in [6, 20].

Theorem 2.2. *(The PerronFrobenius theorem for nonnegative tensors) If $\mathcal{A}$ is a nonnegative tensor of order m and dimension n, then $\rho(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$ with a nonnegative eigenvector $x \in \mathbb{R}^n_+$.*

Interval linear algebra is a mathematical field developed from classical linear algebra. The only difference is, that we do not work with real numbers but with real closed intervals $x^I := \left[\underline{x}, \overline{x}\right]$, where $\underline{x} \leq \overline{x}$. Sometimes in applications, we do not know some parameters precisely, that is why we rather use intervals of possible values. When the components of a tensor possess interval uncertainty, we have an interval tensor. In applied problems in which there is minimal information about the nature of the tensor coefficient uncertainty, the tensor is an interval. In [1], Bozorgmanesh et al. introduced interval tensors. An interval tensor is a tensor so that every element is an interval. An m-order n-dimensional cubical interval tensor is denoted by $\mathcal{A}^I := \left[\underline{\mathcal{A}}, \overline{\mathcal{A}}\right]$ where $\underline{\mathcal{A}}$ and $\overline{\mathcal{A}}$ are real tensors, and

$$\mathcal{A}^I(i_1, \cdots, i_m) = \left[\underline{\mathcal{A}}(i_1, \cdots, i_m), \overline{\mathcal{A}}(i_1, \cdots, i_m)\right].$$

We can also present $\mathcal{A}^I$ in the form of $\mathcal{A}^I = \left[\mathcal{A}^c - \mathcal{A}^\Delta, \mathcal{A}^c + \mathcal{A}^\Delta\right]$, where

$$\mathcal{A}^c = \frac{1}{2}(\underline{\mathcal{A}} + \overline{\mathcal{A}}), \quad \mathcal{A}^\Delta = \frac{1}{2}(\overline{\mathcal{A}} - \underline{\mathcal{A}}).$$

Note that by definition, we have $\mathcal{A}^\Delta \geq 0$. The above notations can be also used for interval matrices and vectors. The set of all interval tensors of size $n_1 \times \cdots \times n_m$ is denoted by $\mathbb{IR}^{n_1\times\cdots\times n_m}$.

Example 2.3. Consider the following $2 \times 2 \times 2$ interval tensor,

$$\mathcal{A}^I(:,:,1) = \begin{pmatrix} [2,20] & [14,100] \\ [0,1] & [5,21] \end{pmatrix}, \quad \mathcal{A}^I(:,:,2) = \begin{pmatrix} 3 & [-9,-7] \\ [0,6] & [-3,-2] \end{pmatrix}.$$

Here, the corresponding $\mathcal{A}^c$ and $\mathcal{A}^\Delta$ are as follows,

$$\mathcal{A}^c(:,:,1) = \begin{pmatrix} 11 & 57 \\ 0.5 & 13 \end{pmatrix}, \quad \mathcal{A}^c(:,:,2) = \begin{pmatrix} 3 & -8 \\ 3 & -2.5 \end{pmatrix},$$

$$\mathcal{A}^\Delta(:,:,1) = \begin{pmatrix} 9 & 43 \\ 0.5 & 8 \end{pmatrix}, \quad \mathcal{A}^\Delta(:,:,2) = \begin{pmatrix} 0 & 1 \\ 3 & 0.5 \end{pmatrix}.$$

Before ending this section, we present the following definition and result of [1], which will be used in the sequel.

Definition 2.4. [1, Definition 12] Let $\mathcal{A}^I$ be an n-dimensional cubical interval tensor. For vectors $y^j \in \mathbb{R}^n$, $1 \leq j \leq n$, define

$$\mathcal{A}_{y^1 y^2 \ldots y^m} = \mathcal{A}^c - D_{y^1}(\mathcal{A}^\Delta \times_m D_{y^2} \times_{m-1} D_{y^3} \cdots \times_2 D_{y^m}), \tag{2.2}$$

where D_{y^j} is a diagonal matrix having y^j on its diagonal, that is, $D_{y^j} = diag(y^j)$.

Proposition 2.5. *[1, Proposition 3] Suppose $\mathcal{A}^I$ is defined as before, if for $y^j \in \mathbb{R}^n$, $1 \leq j \leq n$, we have $\left|y^j\right| = e = (1, 1, \cdots, 1)^T$, then we have,*

$$\left(\mathcal{A}_{y^1 \ldots y^m}\right)_{i_1 i_2 \cdots i_m} = \begin{cases} \left(\mathcal{A}^c - \mathcal{A}^\Delta\right)_{i_1 i_2 \cdots i_m} & if \quad y^1_{i_1} \times \cdots \times y^m_{i_m} = 1, \\ \left(\mathcal{A}^c + \mathcal{A}^\Delta\right)_{i_1 i_2 \cdots i_m} & if \quad y^1_{i_1} \times \cdots \times y^m_{i_m} = -1. \end{cases} \tag{2.3}$$

3. Particular interval tensor classes

In this section, some structure of interval matrices are generalized to interval tensors. Whereas some of their proofs are similar to their analogous matrix case (see [9]), others will be proved here.

In the following definition, we review the basic definitions of some structure tensors for the study of some classes of interval tensors.

Definition 3.1. [5, 17, 18, 19] An m-order n-dimensional tensor $\mathcal{A}$ is said to be
(1) positive definite (PD), if for any nonzero vector $x \in \mathbb{R}^n$, $\mathcal{A}x^m > 0$,
(2) positive semidefinite (PSD), if for any vector $x \in \mathbb{R}^n$, $\mathcal{A}x^m \geq 0$,
(3) P-tensor, if for each nonzero $x \in \mathbb{R}^n$, there exists some index i such that

$$x_i \left(\mathcal{A}x^{m-1}\right)_i > 0, \tag{3.1}$$

(4) P_0-tensor, if for each nonzero $x \in \mathbb{R}^n$, there exists some index i such that

$$x_i \neq 0 \quad and \quad x_i \left(\mathcal{A}x^{m-1}\right)_i \geq 0, \tag{3.2}$$

(5) wP-tensor, if for each nonzero $x \in \mathbb{R}^n$, there exists some index i such that

$$x_i^{m-1} \left(\mathcal{A}x^{m-1}\right)_i > 0, \tag{3.3}$$

(6) wP_0-tensor, if for each nonzero $x \in \mathbb{R}^n$, there exists some index i such that

$$x_i \neq 0 \quad and \quad x_i^{m-1} \left(\mathcal{A}x^{m-1}\right)_i \geq 0, \tag{3.4}$$

(7) semi-positive (SP), if for each $x \in \mathbb{R}^n_+ \setminus \{0\}$, there exists an index $i \in [n]$ such that

$$x_i > 0 \quad and \quad \left(\mathcal{A}x^{m-1}\right)_i \geq 0,$$

(8) strictly semi-positive (SSP), if for each $x \in \mathbb{R}^n_+ \setminus \{0\}$, there exists an index $i \in [n]$ such that

$$x_i > 0 \quad and \quad \left(\mathcal{A}x^{m-1}\right)_i > 0,$$

(9) strictly copositive, if $\mathcal{A}x^m > 0$ for all $x \in \mathbb{R}^n_+ \setminus \{0\}$,
(10) copositive, if $\mathcal{A}x^m \geq 0$ for all $x \in \mathbb{R}^n_+ \setminus \{0\}$,
(11) R-tensor, if there exists no $(x, t) \in (\mathbb{R}^n_+ \setminus \{0\}) \times \mathbb{R}_+$ such that

$$\begin{cases} \left(\mathcal{A}x^{m-1}\right)_i + t = 0, & if \quad x_i > 0, \\ \left(\mathcal{A}x^{m-1}\right)_i + t \geq 0, & if \quad x_i = 0, \end{cases} \tag{3.5}$$

(12) R_0-tensor, if the system (3.5) has no nonzero solution when $t = 0$, i.e., there exists no $x \in \mathbb{R}^n_+ \setminus \{0\}$ such that

$$\begin{cases} \left(\mathcal{A}x^{m-1}\right)_i = 0, & if \quad x_i > 0, \\ \left(\mathcal{A}x^{m-1}\right)_i \geq 0, & if \quad x_i = 0, \end{cases} \tag{3.6}$$

(13) ER-tensor(exceptionally regular tensor), if there exists no $(x, t) \in (\mathbb{R}^n_+ \setminus \{0\}) \times \mathbb{R}_+$ such that

$$\begin{cases} \left(\mathcal{A}x^{m-1}\right)_i + tx_i = 0, & if \quad x_i > 0, \\ \left(\mathcal{A}x^{m-1}\right)_i \geq 0, & if \quad x_i = 0. \end{cases} \tag{3.7}$$

Lemma 3.2. *Clearly, every strictly semi-positive tensor is a semi-positive tensor and every P_0-tensor is certainly semi-positive. When $m = 2$, an ER-tensor reduces to a matrix, and we call it an ER-matrix. It is obvious that every R-tensor is an R_0-tensor, but the converse does not hold. From the definition of strictly semi-positive tensor, we can get that the class of strictly semi-positive tensors is a subset of the class of ER-tensors. Note that the class of ER-tensors is different from the class of R-tensors [19, Examples 3.1, 3.2].*

Now, we recall the following result on the properties of the interval tensor which is important to obtain some theoretical results in this paper.

Corollary 3.3. *[14] Let $\mathcal{A}^I = [\underline{\mathcal{A}}, \overline{\mathcal{A}}]$ be an m-order n-dimensional interval tensor and $x \in \mathbb{R}^n_+$. Then for each $\mathcal{A} \in [\underline{\mathcal{A}}, \overline{\mathcal{A}}]$ and each $i \in \{1, \cdots, n\}$, we have $\left(\mathcal{A}x^{m-1}\right)_i \geq \left(\underline{\mathcal{A}}x^{m-1}\right)_i$ and hence $\mathcal{A}x^{m-1} \geq \underline{\mathcal{A}}x^{m-1}$.*

Definition 3.4. [14] Let $\mathcal{A}^I$ be an m-order n-dimensional interval tensor. $\mathcal{A}^I$ is said to be an interval P, P_0, PD, PSD, SP, SSP, and (strictly) copositive tensor if every $\mathcal{A} \in \mathcal{A}^I$ is a P, P_0, PD, PSD, SP, SSP, and (strictly) copositive tensor.

3.1. Interval $wP(wP_0)$-tensors

Base on interval $P(P_0)$-tensors, we define interval $wP(wP_0)$-tensors, and then we investigate their properties.

Definition 3.5. Let $\mathcal{A}^I$ be an m-order n-dimensional interval tensor. $\mathcal{A}^I$ is called an interval $wP(wP_0)$-tensor if every $\mathcal{A} \in \mathcal{A}^I$ is a $wP(wP_0)$-tensor.

Recently, Rahmati et al. [14, Theorem 3.7] proved the following theorem for the P-tensors. We further present that this result can also be applied to the wP-tensors.

Theorem 3.6. *An m-order n-dimensional interval tensor $\mathcal{A}^I$ is a wP-tensor if and only if for each $z \in Z$, $\mathcal{A}_{z\cdots z}$ is a wP-tensor, where Z denotes the set of all ± 1 vectors, i.e., $Z = \{z \in \mathbb{R}^n : |z| = e\}$.*

Proof. If $\mathcal{A}^I$ is a wP-tensor, then each $\mathcal{A}_{z\cdots z}$ is a wP-tensor, since $\mathcal{A}_{z\cdots z} \in \mathcal{A}^I$ for each $z \in Z$. Conversely, let each $\mathcal{A}_{z\cdots z}$ be a wP-tensor. Take $\mathcal{A} \in \mathcal{A}^I$, $x \neq 0$. Define $z = \text{sgn}(x)$, $|x| = D_z x$, $D_z = diagz$. Since $\mathcal{A}_{z\cdots z}$ is a wP-tensor, there exists an $i \in \{1, \cdots, n\}$ such that $x_i^{m-1}\left(\mathcal{A}_{z\cdots z}x^{m-1}\right)_i > 0$. Then we have

$$
\begin{aligned}
&x_i^{m-1}\left(\mathcal{A}x^{m-1}\right)_i \\
&= x_i^{m-2}\left(\sum_{i_2\cdots i_m} (\mathcal{A}^c)_{ii_2\cdots i_m} x_i x_{i_2}\cdots x_{i_m} + \sum_{i_2\cdots i_m} (\mathcal{A}-\mathcal{A}^c)_{ii_2\cdots i_m} x_i x_{i_2}\cdots x_{i_m}\right) \\
&\geq x_i^{m-2}\left(\sum_{i_2\cdots i_m} (\mathcal{A}^c)_{ii_2\cdots i_m} x_i x_{i_2}\cdots x_{i_m} - \sum_{i_2\cdots i_m} \mathcal{A}^{\Delta}_{ii_2\cdots i_m} |x_i|\left|x_{i_2}\right|\cdots\left|x_{i_m}\right|\right) \\
&= x_i^{m-2}\left(\sum_{i_2\cdots i_m} \left((\mathcal{A}^c)_{ii_2\cdots i_m} - D_z\left(\mathcal{A}^{\Delta}_{ii_2\cdots i_m} \times_m D_z \cdots \times_2 D_z\right)\right) x_i x_{i_2}\cdots x_{i_m}\right) \\
&= x_i^{m-1}\left(\mathcal{A}_{z\cdots z}x^{m-1}\right)_i > 0.
\end{aligned}
$$

Hence $\mathcal{A}$ is a wP-tensor. This proves that $\mathcal{A}^I$ is a wP-tensor. □

Proposition 3.7. *Let $\mathcal{A}^I$ be an m-order n-dimensional interval tensor, $x \in \mathbb{R}^n$ and $z = \text{sgn}(x)$. Then for each $\mathcal{A} \in \mathcal{A}^I$, we have*

$$x_i^{m-1}\left(\mathcal{A}x^{m-1}\right)_i \geq x_i^{m-1}\left(\mathcal{A}_{z\cdots z}x^{m-1}\right)_i, \qquad \forall i \in [n].$$

Proof. This follows from Theorem 3.6. □

The following corollary is a sufficient and necessary condition for an interval tensor to be an interval wP-tensor.

Corollary 3.8. *An m-order n-dimensional interval tensor $\mathcal{A}^I$ is an interval wP-tensor if and only if for each nonzero vector $x \in \mathbb{R}^n$, and $z \in Z$ there exists a positive diagonal matrix D_x and $\mathcal{A}_{z\cdots z}$ such that*

$$\langle D_x x^{[m-1]}, \mathcal{A}_{z\cdots z}x^{m-1}\rangle > 0.$$

Proof. It is similar to the proof of [5, Theorem 2.3]. □

The proof of the following proposition is similar to the proof of [5, Proposition 3.1].

Proposition 3.9. *An interval positive definite tensor is an interval wP-tensor, and an interval positive semidefinite tensor is an interval wP_0-tensor.*

3.2. *Interval multi-linear systems*

In [1], Bozorgmanesh et. al extended the basic linear interval system to interval multi-linear systems.

Let $\mathcal{A}x = b$, $\mathcal{A} \in \mathcal{A}^I$, $b \in b^I$ where $\mathcal{A}^I$ is a real interval tensor and b^I is a real interval vector. This system is called an interval multi-linear system. The solution set of this system is defined as follows,

$$S\left(\mathcal{A}^I, b^I\right) = \{x \in \mathbb{R}^n | \quad \mathcal{A}x = b, \mathcal{A} \in \mathcal{A}^I, b \in b^I\}.$$

A system is called solvable if it has a solution, and feasible if it has a nonnegative solution. A system $\mathcal{A}^I x = b^I$ is said to be weakly solvable (feasible) if some system $\mathcal{A}x = b$ with data $\mathcal{A} \in \mathcal{A}^I$, $b \in b^I$ is solvable (feasible), and it is called strongly solvable (feasible) if each system $\mathcal{A}x = b$ with data $\mathcal{A} \in \mathcal{A}^I$, $b \in b^I$ is solvable (feasible).

To characterization of weakly feasible interval multi-linear systems, we need the following theorem.

Theorem 3.10. *[1] Let* $\mathcal{A}^I = \left[\mathcal{A}^c - \mathcal{A}^\Delta, \mathcal{A}^c + \mathcal{A}^\Delta\right]$ *be a cubical m-order interval tensor and* $b^I = \left[b^c - b^\delta, b^c + b^\delta\right]$ *be an interval vector, then the solution set is*

$$S\left(\mathcal{A}^I, b^I\right) = \{x \in \mathbb{R}^n | \quad |\mathcal{A}^c x^{(m-1)} - b^c| \leq \mathcal{A}^\Delta |x|^{(m-1)} + b^\delta\}.$$

The following theorem is the extension of [7, Theorem 2.13] for the interval tensor case.

Theorem 3.11. *A system* $\mathcal{A}^I x = b^I$ *is weakly feasible if and only if the system*

$$\underline{\mathcal{A}}x^{m-1} \leq \overline{b} \tag{3.8}$$

$$-\overline{\mathcal{A}}x^{m-1} \leq -\underline{b} \tag{3.9}$$

is feasible.

Proof. If $\mathcal{A}^I x = b^I$ is weakly feasible, then it possesses a nonnegative weak solution x that by Theorem 3.10 satisfies

$$|\mathcal{A}^c x^{(m-1)} - b^c| \leq \mathcal{A}^\Delta |x|^{(m-1)} + b^\delta, \tag{3.10}$$

thus

$$-\mathcal{A}^\Delta |x|^{(m-1)} - b^\delta \leq \mathcal{A}^c x^{(m-1)} - b^c \leq \mathcal{A}^\Delta |x|^{(m-1)} + b^\delta, \tag{3.11}$$

which is (3.8), (3.9). Conversely, if (3.8), (3.9) has a nonnegative solution x, then it satisfies (3.10) and (3.11) and by Theorem 3.10 it is a nonnegative weak solution to $\mathcal{A}^I x = b^I$ which means that system is weakly feasible. □

3.3. Interval R_0-tensor, R-tensor and ER-tensor

Interval $R(R_0)$-matrices were defined by Hladk [9], which arose in connection with the linear complementarity problems. In this part, we extend this notion to interval $R(R_0)$-tensors. On the other hand, we introduce the definition of ER-tensor for interval tensors and several of its properties are presented. Furthermore, the equivalent definition of $R_0(R)$-tensor and ER-tensor are obtained.

In the first, the principal subtensors [2] are defined as follows:

Let $\alpha, \beta \subseteq [n]$, $\beta := [n] \setminus \alpha$ be an index sets, then the principal subtensors $\mathcal{A}^{\alpha} = (a^{\alpha}_{i_1\cdots i_m})$ and $\mathcal{A}^{\beta\alpha} = (a^{\beta\alpha}_{i_1\cdots i_m})$ is defined as follows,

$$a^{\alpha}_{i_1\cdots i_m} := a_{i_1\cdots i_m}, \quad i_1\cdots i_m \in \alpha,$$

and

$$a^{\beta\alpha}_{i_1\cdots i_m} := a_{i_1\cdots i_m}, \quad i_1 \in \beta, i_2\cdots i_m \in \alpha.$$

Theorem 3.12. *Let $\mathcal{A}$ be an n-dimensional cubical tensor. The following two statements are equivalent.*
(a) $\mathcal{A}$ is a R_0-tensor.
(b) For each index set $\emptyset \neq \alpha \subseteq [n]$, the system

$$\begin{cases} \mathcal{A}^{\alpha}x^{m-1} = 0, \\ \mathcal{A}^{\beta\alpha}x^{m-1} \geq 0, \\ 0 \neq x \geq 0, \end{cases} \tag{3.12}$$

is infeasible, where $\beta := [n] \setminus \alpha$.

Proof. For sufficiency, suppose system (3.12) has solution x for some $\alpha \subseteq [n]$. Let

$$z_i = \begin{cases} x_i & i \in \alpha, \\ 0 & i \in \beta. \end{cases}$$

The vector z is a solution of the system (3.6), which is a contradiction. Conversely, suppose x is a solution of system (3.6). Define $\alpha = \{i : x_i > 0\}$, it follows that x satisfies system (3.12). □

Next, it can be shown that a result similar to Theorem 3.12 trivially holds for interval R, ER-tensor.

Theorem 3.13. *Let $\mathcal{A}$ be an n-dimensional cubical tensor. The following two statements are equivalent.*
(a) $\mathcal{A}$ is a R-tensor.
(b) For each index set $\emptyset \neq \alpha \subseteq [n]$, the system

$$\begin{cases} \mathcal{A}^{\alpha}x^{m-1} + et = 0, \\ \mathcal{A}^{\beta\alpha}x^{m-1} + et \geq 0, \\ 0 \neq x \geq 0, \quad t \geq 0, \end{cases} \tag{3.13}$$

is infeasible, where $\beta := [n] \setminus \alpha$.

Proof. Similar to the proof of Theorem 3.12. □

Theorem 3.14. *Let $\mathcal{A}$ be an n-dimensional cubical tensor. The following two statements are equivalent.*
(a) $\mathcal{A}$ is an ER-tensor.
(b) For each index set $\emptyset \neq \alpha \subseteq [n]$, the system

$$\begin{cases} \mathcal{A}^{\alpha} x^{m-1} + tx = 0, \\ \mathcal{A}^{\beta\alpha} x^{m-1} \geq 0, \\ 0 \neq x \geq 0, \quad t \geq 0, \end{cases} \tag{3.14}$$

is infeasible, where $\beta := [n] \setminus \alpha$.

Proof. Similar to the proof of Theorem 3.12. □

Definition 3.15. Let $\mathcal{A}^I$ be an m-order n-dimensional interval tensor. $\mathcal{A}^I$ is called an interval R_0-tensor, R-tensor, and ER-tensor if every $\mathcal{A} \in \mathcal{A}^I$ is a R_0-tensor, R-tensor and ER-tensor.

The following proposition is proved using an approach similar to its analogous matrix case [9].

Proposition 3.16. *Let $\mathcal{A}^I = \left[\underline{\mathcal{A}}, \overline{\mathcal{A}}\right]$ be an m-order n-dimensional interval tensor. Then $\mathcal{A}^I$ is an interval R_0-tensor if and only if the system*

$$\begin{cases} \underline{\mathcal{A}}^{\alpha} x^{m-1} \leq 0, \\ \overline{\mathcal{A}}^{\alpha} x^{m-1} \geq 0, \\ \underline{\mathcal{A}}^{\beta\alpha} x^{m-1} \geq 0, \\ x > 0, \end{cases} \tag{3.15}$$

is infeasible for each admissible α, β.

Proof. $\mathcal{A}^I$ is not an interval R_0-tensor if and only if there are α, β and $\mathcal{A} \in \mathcal{A}^I$ such that (3.12) is feasible. From Theorem 3.11, system (3.12) is feasible for some $\mathcal{A} \in \mathcal{A}^I$ if and only if system (3.15) is feasible, which the statement follows. □

Proposition 3.17. *Let $\mathcal{A}^I = \left[\underline{\mathcal{A}}, \overline{\mathcal{A}}\right]$ be an m-order n-dimensional interval tensor. Then*
1. *$\mathcal{A}^I$ is an interval R-tensor if and only if the system*

$$\begin{cases} \underline{\mathcal{A}}^{\alpha} x^{m-1} + et \leq 0, \quad \overline{\mathcal{A}}^{\alpha} x^{m-1} + et \geq 0, \\ \underline{\mathcal{A}}^{\beta\alpha} x^{m-1} + et \geq 0, \quad x > 0, t \geq 0, \end{cases}$$

is infeasible for each admissible α, β.
2. *$\mathcal{A}^I$ is an interval ER-tensor if and only if the system*

$$\begin{cases} \underline{\mathcal{A}}^{\alpha} x^{m-1} + tx \leq 0, \quad \overline{\mathcal{A}}^{\alpha} x^{m-1} + tx \geq 0, \\ \underline{\mathcal{A}}^{\beta\alpha} x^{m-1} \geq 0, \quad x > 0, t \geq 0, \end{cases}$$

is infeasible for each admissible α, β.

Proof. Similar to the proof of Proposition 3.16. □

From the definition of interval strictly semi-positive tensors, we can show that this structure interval tensors are a subset of interval ER-tensors. We discuss the relationship between interval ER-tensor and related interval tensors.

Proposition 3.18. *Let $\mathcal{A}^I$ be an m-order n-dimensional interval tensor. If $\mathcal{A}^I$ is an interval strictly semi-positive tensor, then $\mathcal{A}^I$ is an interval ER-tensor.*

Proof. Let $\mathcal{A}^I$ be an interval strictly semi-positive tensor. By Definition 3.4, every $\mathcal{A} \in \mathcal{A}^I$ is an strictly semi-positive tensor, it follows that for any $x \in \mathbb{R}^n_+ \setminus \{0\}$, there exists an index $i \in [n]$ such that

$$x_i > 0 \quad and \quad \left(\mathcal{A}x^{m-1}\right)_i > 0.$$

That is to say, for any $x \in \mathbb{R}^n_+ \setminus \{0\}$ and each $\mathcal{A} \in \mathcal{A}^I$ the system (3.7) has no solution. Therefore, $\mathcal{A}^I$ is an ER-tensor. □

Theorem 3.19. *If $\mathcal{A}^I$ is an m-order n-dimensional interval wP-tensor, then $\mathcal{A}^I$ is an interval ER-tensor.*

Proof. Since $\mathcal{A}^I$ is an interval wP-tensor, it follows from Theorem 3.6 that for any $x \in \mathbb{R}^n \setminus \{0\}$, there exists an index $i \in [n]$ such that $x_i^{m-1}\left(\mathcal{A}_{z\cdots z}x^{m-1}\right)_i > 0$. In particular, for any $x \in \mathbb{R}^n_+ \setminus \{0\}$, there exists an index i_0 such that

$$x_{i_0}^{m-1}\left(\mathcal{A}_{z\cdots z}x^{m-1}\right)_{i_0} > 0.$$

Therefore, $x_{i_0} > 0$. Furthermore, we get $\left(\mathcal{A}_{z\cdots z}x^{m-1}\right)_{i_0} > 0$. That is to say, for any $x \in \mathbb{R}^n_+ \setminus \{0\}$, there exists an index i_0 such that $x_{i_0} > 0$ and $\left(\mathcal{A}_{z\cdots z}x^{m-1}\right)_{i_0} > 0$, by Proposition 3.7, for each $\mathcal{A} \in \mathcal{A}^I$, we have

$$x_i^{m-1}\left(\mathcal{A}x^{m-1}\right)_i \geq x_i^{m-1}\left(\mathcal{A}_{z\cdots z}x^{m-1}\right)_i,$$

which implies that the system (3.7) has no solution. Then $\mathcal{A}^I$ is an ER-tensor. □

Remark 3.20. If $\mathcal{A}^I$ is an m-order n-dimensional interval positive definite tensor, then from Proposition 3.9, $\mathcal{A}^I$ is an interval wP-tensor. It is easy to see that an interval strictly copositive tensor is also an interval strictly semi-positive tensor. Thus, it follows from Proposition 3.18 that the tensors mentioned above (interval positive definite tensor, interval $P-$tensor, interval strictly copositive tensor, interval strictly semi-positive tensor) are all ER-tensors.

Proposition 3.21. *Suppose that the tensor $\mathcal{A}^I$ is an mth-order n-dimensional interval ER-tensor. Then the $\mathcal{A}^I$ is an interval R_0-tensor.*

Proof. Let $\mathcal{A}^I$ be an interval ER-tensor, then any point $(x,t) \in (\mathbb{R}^n_+ \setminus \{0\}) \times \mathbb{R}_+$ is not a solution of the system (3.7) for each $\mathcal{A} \in \mathcal{A}^I$. Then the system (3.7) has no nonzero solution when $t = 0$, that is, the system (3.6) has no solution $x \in \mathbb{R}^n_+ \setminus \{0\}$ for each $\mathcal{A} \in \mathcal{A}^I$. Therefore, $\mathcal{A}^I$ is an interval R_0-tensor. □

The following theorem gives the equivalence of three classes of structured tensors within the semi-positive interval tensors.

Theorem 3.22. *If $\mathcal{A}^I$ is an mth-order n-dimensional interval semi-positive, then the following results are equivalent.*
(i) $\mathcal{A}^I$ is an interval R_0-tensor,
(ii) $\mathcal{A}^I$ is an interval ER-tensor,
(iii) $\mathcal{A}^I$ is an interval R-tensor.

Proof. $(i) \Leftrightarrow (iii)$ It is obvious that every interval R-tensor is an interval R_0-tensor. On the other hand, it follows from [18, Theorem 3.4] that every semi-positive R_0-tensor is a R-tensor. Thus, (i) holds if and only if (iii) holds.
$(i) \Leftrightarrow (ii)$ By Proposition 3.21, every interval ER-tensor is an interval R_0-tensor. Thus, (i) holds if (ii) holds. On the other hand, suppose that $\mathcal{A}^I$ is not an ER-tensor, then for some $\mathcal{A} \in \mathcal{A}^I$ there exists a point $(x, t) \in (\mathbb{R}^n_+ \setminus \{0\}) \times \mathbb{R}_+$ satisfying the system (3.7). Since $\mathcal{A}^I$ is a R_0-tensor, we have $t > 0$. Thus, for some $\mathcal{A} \in \mathcal{A}^I$ we have

$$\left(\mathcal{A}x^{m-1}\right)_i + tx_i = 0, \quad if \quad x_i > 0.$$

This implies that for $x \in \mathbb{R}^n_+ \setminus \{0\}$, we have

$$x_i \left(\mathcal{A}x^{m-1}\right)_i = -tx_i^2 < 0,$$

which contradicts the condition that $\mathcal{A}^I$ is a semi-positive tensor. Therefore, $\mathcal{A}^I$ is an ER-tensor. □

Corollary 3.23. *If $\mathcal{A}^I$ is an m-order n-dimensional interval P_0-tensor, then $\mathcal{A}^I$ is an interval R_0-tensor iff $\mathcal{A}^I$ is an interval ER-tensor iff $\mathcal{A}^I$ is an interval R-tensor.*

Proof. Since every interval P_0-tensor is an interval semi-positive tensor, the results follow from Theorem 3.22. □

We continue this section with some fundamental notions and properties developed in tensor analysis which is important to obtain some theoretical results in this paper.

Definition 3.24. [1, 6, 14] Let $\mathcal{A}$ be an m-order and n-dimensional cubical tensor, then
(1) $\mathcal{A}$ is called an $\mathcal{Z}$-tensor if all of its non-diagonal elements are non-positive. This definition is equivalent to having $\mathcal{A} = s\mathcal{I} - \mathcal{B}$, where $s > 0$, $\mathcal{B}$ is a non-negative tensor and $\mathcal{I}$ identity tensor, denoted by $\mathcal{I} = (I_{i_1 \cdots i_m})$, is the tensor with entries

$$I_{i_1 \cdots i_m} = \begin{cases} 1, & i_1 = \cdots = i_m, \\ 0, & otherwise. \end{cases}$$

(2) $\mathcal{A}$ is called an $\mathcal{M}$-tensor if $\mathcal{A}$ is an $\mathcal{Z}$-tensor and $\mathcal{A} = s\mathcal{I} - \mathcal{B}$, $s \geq \rho(\mathcal{B})$. If $s > \rho(B)$, then $\mathcal{A}$ is called a strong (nonsingular) $\mathcal{M}$-tensor.

(3) A tensor $\mathcal{A}$ is called semimonotone (an E_0-tensor) if for each index set $\emptyset \neq \alpha \subseteq \{1, 2, 3, \cdots, n\}$ the system

$$\mathcal{A}^{\alpha} x^{m-1} < 0, \quad x \geq 0, \tag{3.16}$$

is infeasible.
(4) An interval tensor is called an interval $\mathcal{Z}$-tensor, $\mathcal{M}$-tensor and E_0-tensor if every tensor in it is $\mathcal{Z}$-tensor, $\mathcal{M}$-tensor, and E_0-tensor, respectively.

In the following, we give necessary and sufficient conditions for an interval tensor $\mathcal{A}^I$ to be an interval (strictly) copositive tensor and E_0-tensor.

Proposition 3.25. *Let $\mathcal{A}^c$ be an m-order n-dimensional strong $\mathcal{M}$-tensor. Then*
(1) $\mathcal{A}^I = \left[\underline{\mathcal{A}}, \overline{\mathcal{A}}\right]$ *is an interval copositive if and only if $\underline{\mathcal{A}}$ is an $\mathcal{M}$-tensor;*
(2) $\mathcal{A}^I = \left[\underline{\mathcal{A}}, \overline{\mathcal{A}}\right]$ *is an interval strictly copositive if and only if $\underline{\mathcal{A}}$ is an strong $\mathcal{M}$-tensor.*

Proof. (1) First, suppose that $\mathcal{A}^I$ is an interval copositive. If $\underline{\mathcal{A}}$ is not $\mathcal{M}$-tensor, then we have, $\underline{\mathcal{A}} = s\mathcal{I} - \mathcal{B}$, where $s > 0$ and $\rho(\mathcal{B}) > s$. For the corresponding Perron vector $x \gneqq 0$ we have $\mathcal{B}x^{m-1} = \rho(\mathcal{B})x^{m-1} \gneqq sx^{m-1}$, from which $\underline{\mathcal{A}}x^{m-1} = sx^{m-1} - \mathcal{B}x^{m-1} \lneqq 0$. If $x_i = 0$, then $(\mathcal{B}x^{m-1})_i = 0$ and so $(\underline{\mathcal{A}}x^{m-1})_i = 0$. Similarly, if $x_i > 0$, then $(\mathcal{B}x^{m-1})_i > sx_i$ and so $(\underline{\mathcal{A}}x^{m-1})_i < 0$. Hence $\underline{\mathcal{A}}x^m < 0$, which is a contradiction. Now assume that $\underline{\mathcal{A}}$ is $\mathcal{M}$-tensor, then it is positive semidefinite [6] and so it is copositive. By [14, Theorem 3.7] we conclude $\mathcal{A}^I$ is interval copositive.
(2) Similar argument apply to strict copositivity. □

Corollary 3.26. *Let $\mathcal{A}^c$ be an m-order n-dimensional identity tensor. Then*
(1) $\mathcal{A}^I$ is an interval copositive if and only if $\rho(\mathcal{A}^{\Delta}) \leq 1$;
(2) $\mathcal{A}^I$ is an interval strictly copositive if and only if $\rho(\mathcal{A}^{\Delta}) < 1$.

Proof. By the proposition above, $\mathcal{A}^c = \mathcal{I}$ is an strong $\mathcal{M}$-tensor. Further, $\mathcal{I} - \mathcal{A}^{\Delta}$ is an $\mathcal{M}$-tensor if and only if $\rho(\mathcal{A}^{\Delta}) \leq 1$.
(2) Similar argument apply to strict copositivity. □

Proposition 3.27. *Let $\mathcal{A}^c$ be an m-order n-dimensional $\mathcal{M}$-tensor. Then $\mathcal{A}^I = \left[\underline{\mathcal{A}}, \overline{\mathcal{A}}\right]$ is E_0-tensor if and only if $\underline{\mathcal{A}}$ is an $\mathcal{M}$-tensor.*

Proof. First, suppose that $\underline{\mathcal{A}}$ is an $\mathcal{M}$-tensor. If $\mathcal{A}^I$ is not E_0-tensor, then there is $\mathcal{A} \in \mathcal{A}^I$ such that (3.16) is not established. Since $\underline{\mathcal{A}}$ is an $\mathcal{M}$-tensor, therefore $\underline{\mathcal{A}}^{\alpha}$ is $\mathcal{M}$-tensor. Hence we can write it as $\underline{\mathcal{A}}^{\alpha} = s\mathcal{I} - \mathcal{B}$, where $s > 0$ and $\mathcal{B}$ is a non-negative tensor so that $s \geqslant \rho(\mathcal{B})$. Assume that $x > 0$. From (3.16) we have $\mathcal{B}x^{m-1} > sx^{m-1}$, from which $\rho(\mathcal{B}) > s$ and is a contradiction. Now suppose $\mathcal{A}^I$ is E_0-tensor. If $\underline{\mathcal{A}}$ is not an $\mathcal{M}$-tensor. That is, $\underline{\mathcal{A}} = s\mathcal{I} - \mathcal{B}$, where $s > 0$ and $\rho(\mathcal{B}) > s$. Let $x > 0$ be the Perron vector corresponding to $\mathcal{B}$, so that $\mathcal{B}x^{m-1} = \rho(\mathcal{B})x^{m-1} \geqslant sx^{m-1}$. Then $\underline{\mathcal{A}}x^{m-1} = sx^{m-1} - \mathcal{B}x^{m-1} \leqslant 0$. Define $\alpha = \{i : x_i > 0\}$ (x_α denotes the restriction of a vector x to the entries indexed by α). Since $(\mathcal{B}x^{m-1})_i > (sx^{m-1})_i$ for all $i \in [n]$, we get $(\underline{\mathcal{A}}^{\alpha} x^{m-1})_{\alpha} < 0$ for $x_\alpha > 0$ which is a contradiction. □

Corollary 3.28. *Let $\mathcal{A}^c$ be an m-order n-dimensional identity tensor. Then $\mathcal{A}^I$ is interval E_0-tensor if and only if $\rho(\Delta) \leq 1$.*

3.4. Interval column sufficient tensor

In this subsection, we define an interval column sufficient tensor, which is a generalization of an interval column sufficient matrix.

The column sufficient tensor have been defined and investigated by Chen, Qi and Song [2].

Definition 3.29. [2] An m-order n-dimensional tensor $\mathcal{A}$ is called a column sufficient tensor, if for all $x \in \mathbb{R}^n$ satisfies

$$x_i\left(\mathcal{A}x^{m-1}\right)_i \le 0, \quad \forall i \in [n] \Longrightarrow x_i\left(\mathcal{A}x^{m-1}\right)_i = 0, \quad \forall i \in [n]. \tag{3.17}$$

Notice that when $m = 2$, this definition reduces to the notion of column sufficient matrix [4].

Let $\mathcal{A}$ be a m-order n-dimentional tensor. For each pair of disjoint index sets $I, J \subseteq \{1, 2, 3, \cdots, n\}$ whose union is nonempty, we define $\mathcal{A}^{IJ} = (a^{IJ}_{i_1,i_2,\cdots,i_m})$,

$$a^{IJ}_{i_1,i_2,\cdots,i_m} = \begin{cases} a_{i_1,i_2,\cdots,i_m} & if \quad i_2, i_3, \cdots, i_m \in I, \\ a_{i_1,i_2,\cdots,i_m} & if \quad i_2, i_3, \cdots, i_m \in J, \\ 0 & if \quad i_2, i_3, \cdots, i_m \in K, \\ -a_{i_1,i_2,\cdots,i_m} & otherwise, \end{cases}$$

where $K = \{1, 2, \cdots, n\} \setminus (I \cup J)$.

Theorem 3.30. *Let $\mathcal{A}$ be a m-order n-dimentional tensor. The following two statements are equivalent.*
(a) $\mathcal{A}$ is column sufficient.
(b) For each pair of disjoint sets $I, J \subseteq \{1, 2, \cdots, n\}$, whose union is nonempty, the system

$$\begin{cases} 0 \neq \mathcal{A}^{IJ}x^{m-1} \le 0, \\ x_i > 0 \quad for \quad i \in I \cup J, \\ x_i = 0 \quad for \quad i \in K, \end{cases} \tag{3.18}$$

is infeasible.

Proof. It is similar to the proof of [3, Proposition 3.5.9]. □

Definition 3.31. An m-order n-dimensional interval tensor $[\underline{\mathcal{A}}, \overline{\mathcal{A}}]$ is said to be an interval column sufficient tensor if each $\mathcal{A} \in [\underline{\mathcal{A}}, \overline{\mathcal{A}}]$ is a column sufficient tensor.

In the following results, some necessary and sufficient conditions for interval column sufficient tensors are obtained, which is a generalization of [9].

Theorem 3.32. *Let $\mathcal{A}^I = \left[\underline{\mathcal{A}}, \overline{\mathcal{A}}\right]$ be an m-order n-dimensional interval tensor. Then $\mathcal{A}^I$ is column sufficient if and only if the system*

$$\begin{cases} 0 \neq \mathcal{B}^{IJ}x^{m-1} \le 0, \\ x_i > 0 \quad for \quad i \in I \cup J, \\ x_i = 0 \quad for \quad i \in K, \end{cases} \tag{3.19}$$

has no solution for each sets $I, J \subseteq \{1, 2, \cdots, n\}$ where $I \cup J \neq \emptyset$, $K = \{1, 2, \cdots, n\} \setminus (I \cup J)$ and

$$\mathcal{B}^{IJ}_{i_1,i_2,\cdots,i_m} = \begin{cases} \underline{\mathcal{A}}_{i_1,i_2,\cdots,i_m} & if \quad i_2, \cdots, i_m \in I \quad or \quad i_2, \cdots, i_m \in J, \\ 0 & if \quad i_2, \cdots, i_m \in K, \\ -\overline{\mathcal{A}}_{i_1,i_2,\cdots,i_m} & otherwise. \end{cases}$$

Proof. The necessity of the condition is obviously a direct consequence of Theorem 3.30. To establish its sufficiency, assume that $\mathcal{A}^I$ is not column sufficient, then by Theorem 3.30, for some $I, J \subseteq \{1, 2, \cdots, n\}$ where $I \cup J \neq \emptyset$, $K = \{1, 2, \cdots, n\} \setminus (I \cup J)$ the system 3.18 has a solution x^*. Since $\mathcal{B}^{IJ} \leq \mathcal{A}^{IJ}$ and x^* is nonnegative vector, we can get

$$\mathcal{B}^{IJ} {x^*}^{m-1} \leq \mathcal{A}^{IJ} {x^*}^{m-1} \leq 0.$$

This shows that x^* is a solution to (3.19), which is a contradiction. □

Let S_m is the symmetric group of order m and define T as the product of n factors of S_m. That is,

$$T = \{(t_1, \cdots, t_n) \mid t_i \in S_m, \forall i \in \{1, \cdots, n\}\}.$$

Rahmati and Tawhid [14], defined a set of tensors

$$\varepsilon := \{\mathcal{A}_t \in \mathbb{R}^{n_1 \cdots n_m} \mid t \in T\},$$

by

$$(\mathcal{A}_t)_{i_1 \cdots i_m} = \begin{cases} \underline{\mathcal{A}}_{i_1 \cdots i_m}, & sgn(t_{i_1} \cdots t_{i_m}) = 1, \\ \overline{\mathcal{A}}_{i_1 \cdots i_m}, & sgn(t_{i_1} \cdots t_{i_m}) = -1, \end{cases}$$

where $t_{i_1} \cdots t_{i_m}$ is the product (composition) of permutations $t_{i_1}, \cdots, t_{i_m}$. For a vector $x \in \mathbb{R}^n$, we define $sgn(x) := (t_1, \cdots, t_n) \in T$ by

$$t_i = \begin{cases} id, & x_i \geq 0, \\ (1 \quad 2), & x_i < 0, \end{cases}$$

where id and $(1 \quad 2)$ are considered as permutations in S_m and $sgn(id) = 1$ and $sgn((1 \quad 2)) = -1$. Therefore, $|x_i| = sgn(t_i)x_i$.

Theorem 3.33. *Let $\mathcal{A}^I = [\underline{\mathcal{A}}, \overline{\mathcal{A}}]$ be an m-order n-dimensional interval tensor. Then $\mathcal{A}^I$ is an interval column sufficient tensor if and only if each $\mathcal{A}_t \in \varepsilon$ is column sufficient tensor.*

Proof. The necessity of the condition is obviously, because, for every $t \in T$, $\mathcal{A}_t \in [\underline{\mathcal{A}}, \overline{\mathcal{A}}]$. To prove sufficiency, let $\mathcal{A} \in \mathcal{A}^I$ and $x \in \mathbb{R}^n$ with $x_i(\mathcal{A}x^{m-1})_i \leq 0$, $\forall i \in [n]$. Set $t = sign(x)$, by [14, Proposition 3.2], we know that $x_i(\mathcal{A}x^{m-1})_i \geq x_i(\mathcal{A}_t x^{m-1})_i$ for all $i \in [n]$, combining this with the fact that each $\mathcal{A}_t$ is column sufficient, we obtain $x_i(\mathcal{A}x^{m-1})_i = x_i(\mathcal{A}_t x^{m-1})_i = 0$ for all $i \in [n]$, which implies that $\mathcal{A}$ is a column sufficent, so is $\mathcal{A}^I$. □

Corollary 3.34. *Let $\mathcal{A}^I = [\underline{\mathcal{A}}, \overline{\mathcal{A}}]$ be an m-order n-dimensional interval tensor. Then $\mathcal{A}^I$ is column sufficient in $\mathbb{R}^n_+$, if $\underline{\mathcal{A}}$ is column sufficient in $\mathbb{R}^n_+$.*

Proof. Let $x \in \mathbb{R}^n_+$ and $t = sign(x) = (id, \cdots, id)$, we have $\mathcal{A}_t = \underline{\mathcal{A}}$. Therefore, from Theorem 3.33 , $\mathcal{A}^I$ is column sufficient in $\mathbb{R}^n_+$. □

Theorem 3.35. *Let $\mathcal{A}^I = \left[\underline{\mathcal{A}}, \overline{\mathcal{A}}\right]$ be an m-order n-dimensional interval tensor. Then $\mathcal{A}^I$ is interval column sufficient tensor if and only if tensors of the form $\mathcal{A}_{zz\cdots z} := \mathcal{A}^C - D_z(\mathcal{A}^\Delta \times_m D_z \times_{m-1} \cdots D_z \times_2 D_z)$, are column sufficient for each $z \in \{\pm 1\}^n$.*

Proof. First, suppose that $\mathcal{A}^I$ is column sufficient, then $\mathcal{A}_{zz\cdots z}$ is column sufficient, since $\mathcal{A}_{zz\cdots z} \in \mathcal{A}^I$. Now assume that $\mathcal{A}_{zz\cdots z}$ are column sufficient for each $z \in \{\pm 1\}^n$. If $\mathcal{A}^I$ is not column sufficient, then (3.19) has a solution for certain I and J. We define z by

$$z_i = \begin{cases} 1 & if \quad i \in I, \\ -1 & if \quad otherwise. \end{cases}$$

Since $(\mathcal{A}^{IJ}_{zz\cdots z})_{i_1, i_2, \cdots, i_m} = \mathcal{B}^{IJ}_{i_1, i_2, \cdots, i_m}$, by Theorem 3.30, $\mathcal{A}_{zz\cdots z}$ is not column sufficient, which is a contradiction. □

Theorem 3.36. *Any symmetric interval positive semidefinite tensor is an interval column sufficient tensor.*

Proof. Let $\mathcal{A}^I$ be a symmetric interval positive semidefinite tensor. By the definition of symmetric interval tensors and [14, Theorem 3.7], $\mathcal{A}_z$ is a symmetric positive semidefinite tensor, and by [2, Theorem 1], A_z is a column sufficient tensor. Then, Theorem 3.35 shows that $\mathcal{A}^I$ is an interval column sufficient tensor. □

4. Conclusion

Interval tensors, like interval matrices, can be a useful theoretic and numerical instrument for getting better results and analysis. In this survey, we extended some classes of interval matrices to classes of interval tensors, and derive various characterizations, and properties of them. We established theoretic properties for several interval tensor. It is shown that our definitions also has connections to the interval positive semi-definite tensors.

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