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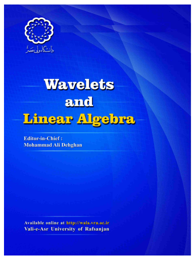

Introduction and Different Properties of Space $c(I)$: Diameter Norm Study

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Abstract

In this paper, the space $c(I)$ is introduced and some of its properties examined. Then with the help of a diameter norm on the space $c_0(I)$, a norm is defined on the space $c(I)$ called as D-norm, which is an extension of the $d-$norm. It is also shown that the D- norm is equivalent to the supremum norm. The extreme points of the unit ball of the spaces $c_0(I)$ and $c(I)$ are also specified. In addition we find some orthogonal vectors in the space $c(I)$.

1. Introduction

The norm $\|\cdot\|_D$ was used by Hagler in [5] to construct a separable Banach space X with non-separable dual such that l_1 does not embed in X and every weakly normalized null sequence

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in X has a subsequence equivalent to the canonical basis of c_0. In paper [1], Bayati considered a natural preorder on $c_0(I)$ and defined a norm on $c_0(I)$, where I is assumed to be an infinite set.

The large volume of researches dealing with extreme points makes it apparent that this area is an important segment of functional analysis. Extreme points have been studied since the early part of the 19th century. In recent years mathematicians have dealt with extreme points in infinite dimensional spaces. These studies have led to useful theorems concerning the isometric and isomorphic properties of Banach spaces. The study of such properties is one of the most active areas in functional analysis. In [4], Gerald characterized the extreme points of the unit ball in some well known Banach spaces, he showed that the set of extreme points of the unit ball of c_0 is empty.

The present study is thus aimed to examine the properties of $c(I)$ by difining D-norm on $c(I)$ which is extension of the $d-$norm. The extreme points of the unit balls of the spaces $c_0(I)$ and $c(I)$ are also specified and, in addition we find some orthogonal vectors in space $c(I)$.

Definition 1.1. [1]. Let I be an infinite set (with a discrete topology) the point $l \in \mathbb{R}$ is called the limit of $f : I \to \mathbb{R}$ and is denoted by

$$\lim_{i \in I} f(i) = l,$$

(or more briefly $\lim f = l$) if for each neighbor V of l there exists a finite set $F \subseteq I$ such that $f(i) \in V$, for all $i \in I \setminus F$.

It can be easily verified that $\lim f$ is unique if it exists. The notation of $c(I)$ is used for the set of all functions $f : I \to \mathbb{R}$ for which $\lim f$ exists. We define a norm on $c(I)$, where I is assumed to be an infinite set, that is equivalent to $\|\cdot\|_\infty$. The set of all bounded functions $f : I \to \mathbb{R}$ is denote by $l^\infty(I)$, thus $c(I) \subseteq l^\infty(I)$. It can be shown that $c(I)$ is a Banach space with the norm $\|f\|_\infty = \sup\{|f(i)| : i \in I\}$. The notation $c_0(I)$ is used for the set of all functions $f : I \to \mathbb{R}$ with $\lim f = 0$. Then $c_0(I)$ is a vector space, thus, it is a subspace of $c(I)$. We define $\mathbf{e} : I \to \mathbb{R}$ by $\mathbf{e}(i) = 1$ for all $i \in I$ and $e_i : I \to \mathbb{R}$ by $e_i(j) = \delta_{ij}$, the Kroneckers delta.

For a non-empty subset C of metric space (X, d) the diameter of C is denoted by $\mathrm{diam}(C)$ and defined as $\mathrm{diam}(C) = \sup\{d(x, y); x, y \in C\}$.
Also, we put

$$c_1(I) = \{f \in c(I) : \exists\, l \in \mathbb{R} \text{ and finite set } F \subset I \text{ such that } f(i) = l \;\forall i \in I \setminus F\}.$$

To simplify notations, $\mathrm{diam}(f)$, $\inf(f)$ and $\sup(f)$ are employed instead of
$\mathrm{diam}(\mathrm{Im}(f))$, $\inf\{f(i) : i \in I\}$, $\sup\{f(i) : i \in I\}$.

Definition 1.2. [2]. Let X be a real Banach space. For any two elements $x, y \in X$, x is said to be orthogonal to y in the sense of Birkhoff-James, written as $x \perp_B y$, if

$$\| x \| \leq \|x + \lambda y\|,$$

for all $\lambda \in \mathbb{R}$.

Definition 1.3. [4]. Let K be a convex set. Then a point $x \in K$ is said to be an extreme point of K if whenever $y, z \in K$ with $x = \alpha y + (1 - \alpha)z$, $0 < \alpha < 1$, then $x = y = z$. The set of extreme points of K will be denoted by $\mathrm{ext}K$.

The unit ball of a space X is $U(X) := \{x \in X : \|x\| \leq 1\}$.

Lemma 1.4. *[4]. Let K be a convex subset of a vector space X and $x \in K$. $x \notin extK$ if and only if there exist $v, w \in K$, with $x = (\frac{1}{2})(v + w)$ and $x \neq v$ or $x \neq w$*

Lemma 1.5. *[4]. If $x \in extU(X)$, then $\|x\| = 1$.*

Definition 1.6. [1]. The $d-$norm on $c_0(I)$ is defined as $\|f\|_d := \text{diam}(f)$, for all $f \in c_0(I)$.

It can be easily verified that the $d-$norm is a norm on $c_0(I)$, if I is an infinite set. Moreover, since $\|\cdot\|_\infty \leq \|\cdot\|_d \leq 2\|\cdot\|_\infty$, two norms $\|\cdot\|_d$ and $\|\cdot\|_\infty$, are equivalent. Also, notice that, $\|f\|_d := \sup(f) - \inf(f)$.

Example 1.7. Suppose $f : [0, 1] \to \mathbb{R}$ and

$$f(i) = \begin{cases} i & \text{if } i \in \{1, \frac{1}{2}, \ldots\} \\ 0 & \text{otherwise} \end{cases},$$

then $\lim f = 0$, since for each $\epsilon > 0$, there exists $M \in \mathbb{N}$ such that $\epsilon > \frac{1}{M}$. Consider $F = \{1, \frac{1}{2}, \ldots, \frac{1}{M}\}$, then for all $i \in [0, 1] \setminus F$, we wil have

$$|f(i)| \leq i < \frac{1}{M} < \epsilon.$$

Example 1.8. Suppose $g : \mathbb{R}^2 \to \mathbb{R}$ and

$$g(i, j) = \begin{cases} 1 + \frac{1}{i} + \frac{1}{j} & (i, j) \in \mathbb{N} \times \mathbb{N} \\ 1 & \text{otherwise} \end{cases},$$

this function has no limit on $\mathbb{R}^2$, since, given $m \in \mathbb{N}$, if we put $\epsilon = \dfrac{1}{m}$ and

$$F = \{(m, 1), (m, 2), \ldots\},$$

then F is an infinite set and for each finite set $H \subset \mathbb{R}^2$, $\mathbb{R}^2 \setminus H$, contains infinitely many points of F and $\|g(i, j) - 1\| = \dfrac{1}{i} + \dfrac{1}{j} \geq \dfrac{1}{m}$ for all $(i, j) \in F$.

2. Main results

In this section, some theorems are proposed to represent the properties of members of $c(I)$. The extreme points of the unit balls of the spaces $c_0(I)$ and $c(I)$ are also specified. In addition we find some orthogonal vectors in $c(I)$. Throughout this section, consider I as an infinite set (with a discrete topology) and $\mathbb{R}$ as the set of the real numbers.

Theorem 2.1. *Suppose $f : I \longrightarrow \mathbb{R}$. Then the limit of f on I exists if and only if for each $\epsilon > 0$ there exists a finite subset $F \subset I$ such that $diam f(I \setminus F) \leq \epsilon$.*

Proof. Let $\lim f = l$, then for each $\epsilon > 0$, there exists a finite subset $F \subset I$ such that for each $i \in I \setminus F$, we have

$$|f(i) - l| < \frac{\epsilon}{2}, \tag{2.1}$$

let $i, j \in I \setminus F$. Then by (2.1)

$$|f(i) - f(j)| = |f(i) - l + l - f(j)| \leq |f(i) - l| + |f(j) - l| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which concludes that $\mathrm{diam} f(I \setminus F) \leq \epsilon$. Conversely, suppose for each $\epsilon > 0$, there exists a finite subset $F \subset I$ such that $\mathrm{diam} f(I \setminus F) \leq \epsilon$. Clearly, the function f is bounded on I. Let $(i_n)_{n\in\mathbb{N}} \subseteq I$ be an infinite sequence. Then $\{f(i_n)\}$ is a real bounded sequence, so there exists a subsequence of $\{f(i_n)\}_{n\in\mathbb{N}}$ which converges to l. Without loss of generality, it can be assumed that $\lim_{n\to\infty} f(i_n) = l$. Then for each $\epsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n > N$,

$$|f(i_n) - l| < \frac{\epsilon}{2}. \tag{2.2}$$

On the other hand for $\epsilon > 0$, there exists a finite subset $H \subset I$ such that $\mathrm{diam} f(I \setminus H) \leq \frac{\epsilon}{2}$ then for $i, j \in I \setminus H$, we have

$$|f(i) - f(j)| \leq \mathrm{diam} f(I \setminus F) \leq \frac{\epsilon}{2}. \tag{2.3}$$

For each $\epsilon > 0$, we consider $G = H$, since G is a finite set, we can choose $n > N$ such that $i_n \in I \setminus G$, then according to inequalities (2.2) and (2.3), for each $i \in I \setminus G$, we will have

$$|f(i) - l| = |f(i) - f(i_n) + f(i_n) - l| \leq |f(i) - f(i_n)| + |f(i_n) - l| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Then $\lim f = l$. □

Theorem 2.2. *Suppose $f : I \longrightarrow \mathbb{R}$, then $\lim f = l$ on I if and only if for any infinite subset D of I, $f|_D$ has a limit on D and*

$$\lim_{i\in D} f = l.$$

Proof. Suppose $\lim f = l$, then for each $\epsilon > 0$, there exists a finite subset $F \subset I$ such that for each $i \in I \setminus F$,

$$|f(i) - l| < \epsilon. \tag{2.4}$$

For each $\epsilon > 0$, put $G = F \cap D$, then $D \setminus G \subseteq I \setminus F$ and inequality (2.4) for all $i \in D \setminus G$. Therefore

$$\lim_{i\in D} f = l.$$

Conversely, since for each infinite subset of I there exists a limit and I is its subset, so it is true for I. □

Example 2.3. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and

$$f(i) = \begin{cases} 1 & i \in \mathbb{Q} \\ 0 & i \notin \mathbb{Q} \end{cases},$$

we have $f|_{\mathbb{Q}} = 1$ and $f|_{\mathbb{Q}^c} = 0$, so

$$\lim_{i\in\mathbb{Q}} f = 1 \quad , \quad \lim_{i\in\mathbb{Q}^c} f = 0.$$

According to the preceding theorem, this function has no limit on $\mathbb{R}$.

Theorem 2.4. $\lim f = l$ *if and only if there exists a sequence* $(i_n)_{n\in\mathbb{N}} \subset I$*, such that* $\lim_{n\to\infty} f(i_n) = l$ *and* $f(i) = l$ *for all* $i \in I \setminus \{i_1, i_2, \ldots\}$.

Proof. Suppose $\lim f = l$. For each $n \in \mathbb{N}$, let $\epsilon_n = \frac{1}{n}$, then there exists a finite subset $F_n \subset I$ such that $|f(i) - l| < \frac{1}{n}$ for all $i \in I \setminus F_n$. We put $F = \bigcup_{n\in\mathbb{N}} F_n$, then

$$|f(i) - l| < \frac{1}{n}, \forall i \in I \setminus F, \forall n \in \mathbb{N},$$

therefore

$$f(i) \in \bigcap_{n\in\mathbb{N}} (l - \frac{1}{n}, l + \frac{1}{n}) = \{l\}.$$

That is, $f(i) = l$ for all $i \in I \setminus F$. Since F is a countable set, it can be assumed that $F = \{i_1, i_2, \ldots\} = (i_n)_{n\in\mathbb{N}}$. For each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Also, there exists a finite subset $F_N \subset I$ such that for all $i \in I \setminus F_N$, we have

$$|f(i) - l| < \frac{1}{N} < \epsilon, \tag{2.5}$$

without loss of generality, it can be assumed that $F_N = \{i_1, i_2, \ldots, i_N\}$, thus, $F \setminus F_N \subset I \setminus F_N$. Inequality (2.5) is established for all $i_n \in F \setminus F_N$, so for each $n > N$, we have

$$|f(i_n) - l| < \epsilon.$$

Therefore, $\lim_{n\to\infty} f(i_n) = l$.

Conversely, suppose that there exists a sequence $(i_n)_{n\in\mathbb{N}} \subset I$ such that

$$\lim_{n\to\infty} f(i_n) = l,$$

and $f(i) = l$ for all $i \in I \setminus \{i_1, i_2, \ldots\}$. Then for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$|f(i_n) - l| < \epsilon.$$

Consider $F = \{i_1, i_2, \ldots\}$ and $H = \{i_1, i_2, \ldots, i_N\}$, then for all $i \in I \setminus H$, if $i \in I \setminus F$ we will have $f(i) = l$, so $|f(i) - l| = 0 < \epsilon$, and if $i \in F \setminus H$ we have $n > N$ then $|f(i_n) - l| < \epsilon$. Consequently, for each $i \in I \setminus H$, we have

$$|f(i) - l| < \epsilon.$$

So it can be conclude that $\lim f = l$. □

Lemma 2.5. *If* $\lim f = l$ *and* $\varphi : I \longrightarrow I$ *is an injective function. Then,* $\lim f \circ \varphi = l$.

Proof. Since $\lim f = l$, for each $\epsilon > 0$, there exists a finite subset $F \subset I$ such that for all $i \in I \setminus F$,

$$|f(i) - l| < \epsilon. \tag{2.6}$$

Pick $H := \varphi^{-1}(F)$ then H is finite. For all $i \in I \setminus H$, we have $\varphi(i) \notin F$, therefore by (2.6),

$$|f \circ \varphi(i) - l| = |f(\varphi(i)) - l| < \epsilon,$$

which means that $\lim f \circ \varphi = l$. □

In the previous lemma, the injective condition is necessary. Because for example if we consider $\varphi : [0,1] \to [0,1]$ and define

$$\varphi(i) = \begin{cases} 1 & \text{if } i \in \{1, \frac{1}{2}, \ldots\} \\ i & \text{otherwise} \end{cases},$$

and define $f : [0,1] \to \mathbb{R}$ by

$$f(i) = \begin{cases} i & \text{if } i \in \{1, \frac{1}{2}, \ldots\} \\ 0 & \text{otherwise} \end{cases},$$

then

$$f \circ \varphi(i) = \begin{cases} 1 & \text{if } i \in \{1, \frac{1}{2}, \ldots\} \\ 0 & \text{otherwise} \end{cases},$$

so, $\lim f = 0$, but $\lim f \circ \varphi$ does not exist.

Theorem 2.6. $(c(I), \|\cdot\|_\infty)$ *is a Banach space.*

Proof. Since $c(I) \subseteq l^\infty(I)$ it is sufficient to show that $c(I)$ is closed in $l^\infty(I)$. Let $(f_n)_{n\in\mathbb{N}} \subset c(I)$ be a sequence converging to $f \in l^\infty(I)$. Then for each $n \in \mathbb{N}$ there exists $l_n \in \mathbb{R}$ such that $\lim f_n = l_n$. First, it is shown that $(l_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Clearly for $n, m \in \mathbb{N}$

$$|l_n - l_m| = |\lim f_n - \lim f_m| = |\lim(f_n - f_m)| \leqslant \|f_n - f_m\|_\infty.$$

Thus, $(l_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, so there exists $l \in \mathbb{R}$ such that $\lim l_n = l$. Then, for each $\epsilon > 0$ there is $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$

$$|l_n - l| < \frac{\epsilon}{3}. \tag{2.7}$$

We show that that $\lim f = l$. For each $\epsilon > 0$ there is $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$ we have

$$\|f_n - f\|_\infty < \frac{\epsilon}{3}. \tag{2.8}$$

On the other hand $\lim f_n = l_n$, so for each $\epsilon > 0$ there exists a finite subset $F_n \subset I$ such that for all $i \in I \setminus F_n$ we have

$$|f_n(i) - l_n| < \frac{\epsilon}{3}, \tag{2.9}$$

By (2.7), (2.8), and (2.9) it can be shown that for sufficiently large $n \in \mathbb{N}$ and $i \in I \setminus F_n$

$$|f(i) - l| \leqslant |f(i) - f_n| + |f_n(i) - l_n| + |l_n - l| \\ \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$

then, $\lim f = l$. So, $c(I)$ is complete with the norm $\|\cdot\|_\infty$ and it is a Banach space. □

Definition 2.7. For $f \in c(I)$ define $\|f\|_D = \|f - (\lim f)\mathbf{e}\|_d + |\lim f|$ which is called the D-norm on $c(I)$, such that, $\mathbf{e}(i) = 1$ for all $i \in I$. The D-norm on $c(I)$ is an extension of the $d-$norm on $c_0(I)$.

Lemma 2.8. *The D-norm is a norm on $c(I)$.*

Proof. For each $f \in c(I)$, if $\|f\|_D = 0$, then $\|f - (\lim f)\mathbf{e}\|_d + |\lim f| = 0$, so $\|f - (\lim f)\mathbf{e}\|_d = 0$ and $|\lim f| = 0$. Since $f - (\lim f)\mathbf{e} \in c_0(I)$ and $\|\cdot\|_d$ is a norm on $c_0(I)$, then $f = 0$. Also, for every $f, g \in c(I)$, we have

$$\begin{aligned} \|f + g\|_D &= \|(f + g) - (\lim(f + g))\mathbf{e}\|_d + |\lim(f + g)| \\ &= \|(f - (\lim f)\mathbf{e}) + (g - (\lim g)\mathbf{e})\|_d + |\lim f + \lim g| \\ &\leq \|f - (\lim f)\mathbf{e}\|_d + \|g - (\lim g)\mathbf{e}\|_d + |\lim f| + |\lim g| \\ &\leq \|f\|_D + \|g\|_D. \end{aligned}$$

Finally, for each $r \in \mathbb{R}$ and $f \in c(I)$, we have $\|rf\|_D = |r|\|f\|_D$. Therefore $\|\cdot\|_D$ is a norm on $c(I)$. □

Theorem 2.9. *$(c(I), \|\cdot\|_D)$ is a Banach space.*

Proof. For each $f \in c(I)$, we have $f - (\lim f)\mathbf{e} \in c_0(I)$ and

$$\|\cdot\|_\infty \leq \|\cdot\|_d \leq 2\|\cdot\|_\infty,$$

then

$$\|f - (\lim f)\mathbf{e}\|_\infty \leq \|f - (\lim f)\mathbf{e}\|_d \leq 2\|f - (\lim f)\mathbf{e}\|_\infty.$$

On the other hand, we have $|\lim f| \leq \|f\|_\infty$, therefore,

$$\begin{aligned} \|f\|_D = \|f - (\lim f)\mathbf{e}\|_d + |\lim f| &\leq 2\|f - (\lim f)\mathbf{e}\|_\infty + \|f\|_\infty \\ &\leq 2\|f\|_\infty + 2\|(\lim f)\mathbf{e}\|_\infty + \|f\|_\infty \\ &\leq 2\|f\|_\infty + 2\|f\|_\infty + \|f\|_\infty = 5\|f\|_\infty. \end{aligned}$$

Also

$$\begin{aligned} \|f\|_\infty = \|f - (\lim f)\mathbf{e} + (\lim f)\mathbf{e}\|_\infty &\leq \|f - (\lim f)\mathbf{e}\|_\infty + \|(\lim f)\mathbf{e}\|_\infty \\ &\leq \|f - (\lim f)\mathbf{e}\|_d + |\lim f| \\ &= \|f\|_D, \end{aligned}$$

thus

$$\|f\|_\infty \leq \|f\|_D \leq 5\|f\|_\infty.$$

Consequently, two norms are equivalent, so $c(I)$ is a Banach space. □

Lemma 2.10. *The space $c_1(I)$ is dense in $c(I)$ with the D-norm.*

Proof. It is sufficient to show that $c_1(I)$ is dense in $c(I)$ with respect to the norm $\|\cdot\|_\infty$. Let $f \in c(I)$, then there exists $l \in \mathbb{R}$ such that $\lim f = l$. Therefore, for all $n \in \mathbb{N}$ there exists a finite subsets F_n such that for all $i \in I \setminus F_n$,

$$|f(i) - l| < \frac{1}{n}, \tag{2.10}$$

Define

$$f_n(i) = \begin{cases} f(i) & i \in F_n \\ l & i \notin F_n \end{cases},$$

then $f_n \in c_1(I)$. We show that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. If $i \in I$, then, $i \in F_n$ or $i \in I \setminus F_n$. In the first case, we have $f_n(i) = f(i)$; then, $|f_n(i) - f(i)| = 0 < \epsilon$. Secondly, according to (2.10),

$$|f(i) - f_n(i)| = |f(i) - l| < \frac{1}{n}, \quad \forall i \in I \setminus F_n.$$

Therefore,

$$\|f_n - f\|_\infty = \sup_{i \in I} |f_n(i) - f(i)| < \frac{1}{n}.$$

That is $\|f_n - f\|_\infty \to 0$ as $n \to \infty$, which implies that $c_1(I)$ is dense in $c(I)$. □

It can be easily indicated that $\|f\|_D = \mathrm{diam}(f) + |\lim f|$, then

$$\|f\|_\infty \leq \|f\|_D \leq 3\|f\|_\infty.$$

If $f \in c(I)$ define $\|f\| = \mathrm{diam}(f) + \|f\|_\infty$, it is easily indicated that $\|\cdot\|$ is a norm on $c(I)$ and it is equivalent with the $D-$ norm.

In the following, the extreme points of the unit balls of $c_0(I)$ and $c(I)$ are determined. Also some orthogonal vectors are obtained in $c(I)$.

Lemma 2.11. *For $f \in c_0(I)$ we have $f \in extU(c_0(I))$ if and only if there exists a non-empty finite subset $F \subset I$ such that*

$$f = \pm \sum_{i \in F} e_i.$$

Proof. Let $f \in \mathrm{ext}\, U(c_0(I))$. We claim that there is no $i \in I$ such that,

$$\inf(f) < f(i) < \sup(f). \tag{2.11}$$

Suppose there exists $i_0 \in I$ such that $\inf(f) < f(i_0) < \sup(f)$. We put $0 < \alpha < \min\{\sup(f) - f(i_0), f(i_0) - \inf(f)\}$ and

$$g(i) = \begin{cases} f(i_0) + \alpha & i = i_0 \\ f(i) & i \neq i_0 \end{cases}, \quad h(i) = \begin{cases} f(i_0) - \alpha & i = i_0 \\ f(i) & i \neq i_0 \end{cases}. \tag{2.12}$$

Because $g(i_0) \neq h(i_0)$, we have $g \neq h$. On the other hand,

$$\inf(f) \leq f(i_0) \leq f(i_0) + \alpha \leq f(i_0) + \sup(f) - f(i_0) = \sup(f),$$

and

$$\sup(f) \geq f(i_0) \geq f(i_0) - \alpha \geq f(i_0) + \inf(f) - f(i_0) = \inf(f).$$

Then, $\inf(f) \leq g(i_0) \leq \sup(f)$ and $\inf(f) \leq h(i_0) \leq \sup(f)$. Therefore

$$\|f\|_d = \|g\|_d = \|h\|_d = 1,$$

and $f = \frac{1}{2}(g + h)$, that $g, h \in U(c_0(I))$. This is a contradiction, thus the function f takes at most two values. This shows that there exists a set $F \subset I$ such that

$$f(i) = \begin{cases} \alpha & i \in F \\ \beta & i \in I \setminus F \end{cases}.$$

We claim that only one of the two sets F or $I \setminus F$ can be infinite. Suppose F and $I \setminus F$ are infinite. Since $\lim f = 0$ then by Theorem (2.2), we have

$$\lim_{i \in F} f(i) = \alpha = 0, \lim_{i \in I \setminus F} f(i) = \beta = 0,$$

that is $f = 0$, which is impossible. Without loss of generality, we assume that F is finite, thus $I \setminus F$ is infinite. Then based on Theorem (2.2) we will have $\beta = 0$. On the other hand $f \neq 0$, then the set F is non-empty. So the function f will be as follows:

$$f(i) = \begin{cases} \alpha & i \in F \\ 0 & i \in I \setminus F \end{cases}.$$

Since $f \in \text{ext}\, U(c_0(I))$, we have $\|f\|_d = 1$. Consequently $|\alpha| = 1$, hence

$$f = \pm \sum_{i \in F} e_i.$$

Conversely, assume that $f = \sum_{i \in F} e_i$ for some finite set $F \subset I$. Then clearly $\|f\|_d = 1$. Suppose $f = \frac{1}{2}(g + h)$, and

$$\|g\|_d, \|h\|_d \leq 1.$$

In this case, for each $i \in F$, $g(i) + h(i) = 2$ and for each $i \in I \setminus F$, $f(i) + g(i) = 0$. We claim that for each $i \in F$, $g(i) = h(i) = 1$ and for each $i \in I \setminus F$, $f(i) = g(i) = 0$. Suppose there exists $i_0 \in F$ such that $g(i_0) > 1$ or $h(i_0) > 1$. Considering $\lim g = 0$, we have $\inf(g) \leq 0 \leq \sup(g)$, then

$$\|g\|_d = \sup(g) - \inf(g) > g(i_0) - 0 > 1.$$

That is a contradiction. Similarly, for the case $h(i_0) > 1$, a contradiction is obtained. So for each $i \in F$,

$$g(i) = h(i) = 1.$$

If $j_0 \in I \setminus F$, exists such that $g(j_0) = -h(j_0) \neq 0$, now if $g(j_0) < 0$, then

$$\|g\|_d = \sup(g) - \inf(g) \geq 1 - g(j_0) > 1,$$

which is a contradiction. In the case of $g(j_0) > 0$, we have $h(j_0) < 0$. Similar to the previous case it can be shown $\|h\|_d > 1$, so $g = h$. Consequently $f = \sum_{i\in F} e_i \in \text{ext}\, U(c_0(I))$. Similarly, we deduce that, $f = -\sum_{i\in F} e_i \in \text{ext}\, U(c_0(I))$. □

Theorem 2.12. *$f \in$ ext $U(c(I))$ if and only if there exists a non-empty finite $F \subset I$ such that $f = \pm \sum_{i\in F} e_i$ or $f = \pm \mathbf{e}$.*

Proof. We first show that $\pm\mathbf{e} \in \text{ext}\, U(c(I))$. Suppose $g, h \in U(c(I))$ such that $\mathbf{e} = \frac{1}{2}(g + h)$ and $\|h\|_D, \|g\|_D \leq 1$. Therefore, $h + g = 2\mathbf{e}$. We claim that $g(i) = h(i) = 1$, for each $i \in I$. Assuming there exists $i_0 \in I$ such that $g(i_0) > 1$ or $h(i_0) > 1$, then we have $\|g\|_D \geq \|g\|_\infty \geq g(i_0) > 1$. That is a contradiction. Similarly the case where $h(i_0) > 1$, also leads to a contradiction. So, $g = h$ which concludes that $\mathbf{e} \in \text{ext}\, U(c(I))$. Similarly, we deduce that, $-\mathbf{e} \in \text{ext}\, U(c(I))$.

If $F \subset I$ is finite and $f = \sum_{i\in F} e_i$, we show that $f \in \text{ext}\, U(c(I))$. Suppose $f = \frac{1}{2}(g + h)$, where $g, h \in c(I)$ and $\|h\|_D, \|g\|_D \leq 1$. Then for all $i \in F$, we get

$$g(i) + h(i) = 2,$$

and for each $i \in I \setminus F$, we conclude $g(i) + h(i) = 0$. We show that $h(i) = g(i) = 1$, for all $i \in F$, and $h(i) = g(i) = 0$, for all $i \in I \setminus F$. Otherwise, there exists $i_0 \in F$ such that $g(i_0) > 1$ or $h(i_0) > 1$. Then $\|g\|_D \geq \|g\|_\infty \geq g(i_0) > 1$ or $\|h\|_D = \|h\|_\infty \geq h(i_0) > 1$. This is a contradiction. Thus for each $i \in F$, we have

$$g(i) = h(i) = 1. \tag{2.13}$$

Now it is enough to show $g(i) = h(i) = 0$, for each $i \in I \setminus F$. Suppose $i_0 \in I \setminus F$, such that $g(i_0) = -h(i_0) \neq 0$. If $g(i_0) < 0$, then

$$\|g\|_D = \text{diam}(g) + |\lim g| \geq 1 - g(i_0) > 1.$$

If $g(i_0) > 0$ then $\|h\|_D > 1$ which is a contradiction. Thus, for each $i \in I \setminus F$,

$$g(i) = h(i) = 0. \tag{2.14}$$

Based on (2.13) and (2.14), we have $g = h$. So, $f = \sum_{i\in F} e_i \in \text{ext}\, U(c(I))$, and similarly, we deduce that $-\sum_{i\in F} e_i \in \text{ext}\, U(c(I))$.

Conversely, if $f \in \text{ext}\, U(c(I))$. Then as in Lemma (2.11), we see that f takes at most two values, that is there exists a finite set $F \subset I$ and real numbers of α, β such that $|\alpha| \leq 1, |\beta| \leq 1$ and

$$f(i) = \begin{cases} \alpha & i \in F \\ \beta & i \notin F \end{cases}.$$

In the following, it will be demonstrated that either $\alpha = \beta = \pm 1$ or $\alpha = \pm 1$, $\beta = 0$. If $\beta = 1$, then $\|f\|_D = 1 = \beta - \alpha + \beta$; we conclude, $\alpha = 1$. In the same way, $\beta = -1$ implies that $\alpha = -1$. In the case that $\alpha = \pm 1$, we show that either $\beta = 0$ or $\beta = \pm 1$. Assuming $\alpha = 1$ and $0 < \beta < 1$, we put $0 < \alpha_0 < \min\{1-\beta, \beta\}$ and

$$g(i) = \begin{cases} 1 & i \in F \\ \beta + \alpha_0 & i \notin F \end{cases}, \quad h(i) = \begin{cases} 1 & i \in F \\ \beta - \alpha_0 & i \notin F \end{cases}.$$

So, $\|g\|_D = 1 - \beta - \alpha_0 + \beta + \alpha_0 = 1$, $\|h\|_D = 1 - \beta + \alpha_0 + \beta - \alpha_0 = 1$, and $f = \frac{1}{2}(g+h)$, which is a contradiction. For the case that $\alpha = 1$ and $\beta < 0$, we have $\operatorname{diam}(f) = 1 - \beta > 1$ which is a contradiction. Consequently, if $\alpha = 1$ then $\beta = 0$ or $\beta = 1$. Similarly, it can be shown that if $\alpha = -1$ then $\beta = 0$ or $\beta = -1$.

Now, it suffices to show that if $0 \leq |\alpha| < 1$ and $0 < |\beta| < 1$ then $f \notin \operatorname{ext} U(c(I))$. Suppose $0 < \beta \leq \alpha < 1$, and $f \in \operatorname{ext} U(c(I))$, then we have

$$1 = \|f\|_D = \alpha - \beta + \beta.$$

So, $\alpha = 1$, which is a contradiction. let $0 \leq \alpha \leq \beta < 1$, and $f \in \operatorname{ext} U(c(I))$. Then there exists $n \in \mathbb{N}$ such that $\alpha \leq \frac{n-1}{n+1}$, and $1 = \|f\|_D = \beta - \alpha + \beta = 2\beta - \alpha$. So, $\frac{n+1}{n}\beta \geq \alpha + \frac{2}{n}\beta$, $\beta = \frac{\alpha+1}{2}$. Putting

$$g(i) = \begin{cases} \alpha + \dfrac{2}{n}\beta & i \in F \\ \dfrac{n+1}{n}\beta & i \notin F \end{cases}, \quad h(i) = \begin{cases} \alpha - \dfrac{2}{n}\beta & i \in F \\ \dfrac{n-1}{n}\beta & i \notin F \end{cases}.$$

Thus, $\|g\|_D = \dfrac{n+1}{n}\beta - \alpha - \dfrac{2}{n}\beta + \dfrac{n+1}{n}\beta = 2\beta - \alpha = 1$, $\|h\|_D = 1$, and $f = \frac{1}{2}(g+h)$, $h \neq g$, which is a contradiction. With a similar argument for state $-1 < \alpha < 0 < \beta < 1$, we can show that $f \notin \operatorname{ext} U(c(I))$. If $-1 < \alpha < \beta < 0$, $-1 < \beta < \alpha < 0$ or $-1 < \beta < 0 < \alpha < 1$, than $f \notin \operatorname{ext} U(c(I))$. Because if we have $f \in \operatorname{ext} U(c(I))$, then $-f \in \operatorname{ext} U(c(I))$, and by the previous argument, we come to a contradiction. So we have $f = \pm \mathbf{e}$ or $f = \pm \sum_{i \in F} e_i$ for non-empty finite $F \subset I$. □

Example 2.13. Let $I = \mathbb{N}$. Then, by the previous theorem, and Lemma (2.11), we have $e_i = (0, \ldots, 1, 0, \ldots) \in \operatorname{ext} U(c_0(\mathbb{N}))$, $e_i \in \operatorname{ext} U(c(\mathbb{N}))$ and $(1, 1, \ldots) \in \operatorname{ext} U(c(\mathbb{N}))$, which $c(\mathbb{N}) = c_0 = \{(x_n)_{n \in \mathbb{N}} : \lim_{n \to \infty} x_n = 0\}$.

Finally, we identify some orthogonal elements in the Banach space $c(I)$. For example if $i \neq j$, then e_i is orthogonal to e_j, because $\|e_i\|_D = 1$ and for every $\lambda \in \mathbb{R}$, we have

$$(e_i + \lambda e_j)(r) = \begin{cases} 1 & r = i \\ \lambda & r = j \\ 0 & r \neq i, j \end{cases}.$$

Assuming $\lambda \geq 1$, we have $\|e_i + \lambda e_j\|_D = \lambda$. Consequently $1 = \|e_i\|_D \leq \|e_i + \lambda e_j\|_D$. If $0 \leq \lambda < 1$,then we have $1 = \|e_i\|_D \leq \|e_i + \lambda e_j\|_D = 1$. Now if $\lambda < 0$ then $\inf(e_i + \lambda e_j) = \lambda$, which, implies that $1 = \|e_i\|_D \leq 1 - \lambda = \|e_i + \lambda e_j\|_D$.

If there exists $i \in I$ sach that $\min(f) < f(i) < \max(f)$, then f is orthogonal to e_i, because for every $r \in I$ and any $\lambda \in \mathbb{R}$.

$$(f+\lambda e_i)(r)=\begin{cases} f(i)+\lambda & r=i \\ f(r) & r\neq i \end{cases}.$$

If $f(i)+\lambda > \max(f)$, then $\max(f) < \max(f+\lambda e_i)$ and $\min(f)=\min(f+\lambda e_i)$, on the other hand,

$$\lim f = \lim(f+\lambda e_i),$$

and hence

$$\|f\|_D \leq \|(f+\lambda e_i)\|_D.$$

If $f(i)+\lambda < \min(f)$ then $\min(f) > \min(f+\lambda e_i)$ and $\max(f)=\max(f+\lambda e_i)$. Therefore

$$\|f\|_D \leq \|(f+\lambda e_i)\|_D.$$

If $\min(f) \leq f(i)+\lambda \leq \max(f)$, since $\min(f) < f(i) < \max(f)$, we have

$$\max(f)=\max(f+\lambda e_i), \min(f)=\min(f+\lambda e_i)$$

and consequently

$$\|f\|_D = \|(f+\lambda e_i)\|_D.$$

According to the above relations, the function f is orthogonal to e_i. Also the function f is orthogonal to the element u in the linear space of

$$\{e_i : \min(f) < f(i) < \max(f)\}.$$

Indeed, if $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$ and $u=\alpha_1 e_{i_1}+\alpha_2 e_{i_2}+\cdots+\alpha_k e_{i_k}$. Then for each $\lambda \in \mathbb{R}$ and $r \in I$, we have

$$(f+\lambda u)(r)=\begin{cases} f(r)+\lambda\alpha_r & r=i_r \\ f(r) & r\neq i_r \end{cases}.$$

If there exists $i_r \in I$ such that $f(i_r+\lambda\alpha_r) > \max(f)$ or $f(i_r+\lambda\alpha_r) < \min(f)$, then $\max(f+\lambda u) > \max(f)$ or $\min(f+\lambda u) < \min(f)$. In such cases, it can be similarly shown that $\min(f)=\min(f+\lambda u)$ or $\max(f)=\max(f+\lambda u)$. So in these cases $\|f\|_D \leq \|(f+\lambda u)\|_D$ and so f is orthogonal to u. In the case of $\min(f) \leq f(i_r)+\lambda\alpha_r \leq \max(f)$, it can be concluded that the function f is orthogonal to u.

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