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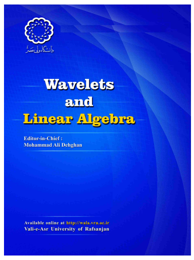

Minimizing increasing convex along rays functions based on abstract convexity

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Abstract

In this paper, a problem whose cost function and constraints are increasing convex along rays is considered. For solving such problems, an algorithm is presented that is inspired by the generalized Cutting Angle Method. A set that contains the optimal solution of the mentioned problem is defined. Some numerical examples are presented to confirm the validity and accuracy of the algorithm.

1. Introduction

The most common and well-known area of global and local optimization is convex programming. The fundamental tool in the study of convex optimization problems is subgradient, which

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plays both local and global roles. First, a subgradient of a convex function f at a point x carries out a local approximation of f in a neighborhood of x [1]. Second, the subgradient allows us to construct an affine function h, which does not exceed f over the entire space and coincides with f at x. This affine function is called a support function [18].

We cannot extend many results in convex programming to nonconvex programming, therefore convex programming alone cannot help to solve many problems of optimization in real world, so there is a clear need to develop special tools for solving nonconvex problems.

A function f is said to be abstract convex if it can be represented as the upper envelope of a class of functions, which is sufficiently simple and called elementary functions which are studied in [18, 19, 24].

S. S. Kutateladze and A.M. Rubinov published the first book on abstract convexity and some of its applications to functional analysis in ordered spaces in 1976. The first characterizations of these functions have been presented in [19].

Abstract convexity has recently emerged as an area of research with potentially intense application to nonconvex optimization (see [21, 16, 24]).

In this paper, we will focus on one particular highly structured but nevertheless widespread class of abstract convex function so-called increasing convex-along-rays function defined on nonnegative orthant that the restriction of this function f to each ray starting at the origin is a convex function of one variable [19].

Abstract convexity has found many applications in the study of mathematical analysis and optimization problems and also it has practical and theoretical applications in economics and mechanics and various engineering problems, Also it found an interesting application to the theory of inequalities. However, the development of abstract convex analysis, like the development of its forerunner, convex analysis, was mainly driven by applications to optimization. For applications of abstract convexity, we need to describe the main objects of this theory in some concrete situations. Just as the set of linear function leads to a theory of convex functions and convex sets. [9, 19].

It is known that some classes of increasing functions are abstract convex. For example, the class of increasing and positively homogeneous (IPH) functions [6, 12] and the class of increasing and convex-along-rays functions are abstract convex [12, 20]. The class of increasing and co-radiant (ICR) functions is another class of increasing functions which are abstract convex. Abstract convexity of nonnegative ICR functions defined on cones in an ordered topological vector space, has been investigated in [6, 25].

Monotonicity plays an important role in various areas of mathematics and its applications. Most results on monotonicity have been obtained in [5, 6] for increasing and co-radiant (ICR) functions and increasing and convex-along-rays functions (see [14]). Also, some of these results have been obtained in [14] for increasing and positively homogeneous (IPH) functions. The first studies of these functions were carried out over the cones in topological vector spaces [11], some suitable extensions of these functions defined over topological vector spaces, were obtained in [11, 12]. Moreover, the first characterizations of these functions have been presented in [19].

We propose an algorithm to solve a problem with such ICAR objective function whose constraints are also ICAR functions, we have approximated such a problem from below using affine functions inspired Cutting Angle method, then converted it into a new unconstrained problem by Penalty

problem method. Since the recent problem is locally Lipschitz,the Subgradient method is used for finding descent directions and approximated stationary points, then some numerical examples are presented.
Our initial experiments have been restricted to small dimensional problems, however, the results can be generalized to the global character of the algorithm in higher dimensions.
The structure of the paper is as follows: In Section 2 we provide some definitions, notations and preliminary results related to abstract convexity . In Section 3 a set that contains the optimal solution of our problem is defined. A new algorithm to solve the mentioned problem is given in Section 4 and finally in Section 5 we approximate the solution of some numerical examples with this novel algorithm.

2. Preliminaries

In this section, first we give some definitions and results related to the abstract convexity. Throughout this paper, we assume that X is a compact set in $\mathbb{R}^n$ and the functions f and g are increasing convex along rays functions where $g : \mathbb{R}^n \to \mathbb{R}$, also $[l, x]$ stands for the inner product of vectors l and x in $\mathbb{R}^n$.

Definition 2.1. [19] Let $V \subseteq \bar{\mathbb{R}} = [-\infty, \infty]$ and X be a set, suppose that H is a nonempty set of functions $h : X \to V$. A function $f : X \to \bar{V}$ is called abstract convex with respect to H (or $H-$convex) if there exists a set $U \subset H$ such that f is the upper envelope of this set:

$$f(x) = sup\{h(x) : h \in U\}, \quad (\forall x \in X).$$

The set H will be referred to as a set of elementary functions.

Definition 2.2. [19] Let $f : X \to \bar{V}$ be a function, the set $supp(f, H) = \{h \in H : h \leq f\}$ of all $H-$minorants of f is called the support set of the function f with respect to the set of elementary functions H. Occasionally, we shall use the term lower support set instead of support set.

Definition 2.3. [19] Let $Q \subseteq \mathbb{R}^n$ be a conic set. A function $f : Q \to (-\infty, +\infty]$ is called convex-along-rays (briefly, CAR) if, for each $x \in Q$, the function $f_x(t) := f(tx), t \in [0, +\infty)$ is convex.
Denote by R_x the closed ray $\{tx : t \geq 0\}$ starting at the origin and passing through a point x. We can rephrase the definition of convexity-along-rays in the following form. A function $f : Q \to (-\infty, +\infty]$ is convex-along-rays if its restriction to each ray R_x with $x \in Q$ is a convex function.

The class of ICAR functions is very broad. For example an increasing positively homogeneous function of degree $m \geq 1$ and an increasing convex function defined on $\mathbb{R}^n_+$ and a family of functions $f(x) = x_1^{\alpha_1}.x_2^{\alpha_2}... x_n^{\alpha_n}$ for $x \in \mathbb{R}^n_+$ ($\sum \alpha_i \geq 1$ and $\alpha_i \geq 0$) are ICAR.
abstract convexity is studied based on classes of elementary functions, consisting of the so-called min-type functions. Instead of linear functions $x \to \sum l_i x_i$ defined on $\mathbb{R}^n$, we consider functions of the form $x \to min_{i \in I_+(l)} l_i x_i$ with $I_+(l) = \{i : l_i > 0\}$, defined either on the cone $\mathbb{R}^n_+$ of vectors with positive coordinates or on the cone $\mathbb{R}^n_+$ of vectors with nonnegative coordinates. Abstract convexity with respect to this class of elementary functions leads to the theory of monotone functions

defined on cones $\mathbb{R}^n_+$ and $\mathbb{R}^n_+$.
Min-type functions $l(x)$ with $l = (l_1, l_2, ..., l_n)$ and $x = (x_1, x_2, ..., x_n)$ have a following general form:

$$(x) =< l, x >:= min_{i\in I_+(l)} l_i x_i \;\; with \;\; I_+(l) = \{i : l_i > 0\}.$$

Definition 2.4. [19] Let L be a set of all min-type functions defined on $\mathbb{R}^n_+$ where $< l, x >:= min_{i\in I_+(l)} l_i x_i$ *with* $I_+(l) = \{i : l_i > 0\}$, the L-subdifferential of an ICAR functions f at point $x \in dom f$ is defined by:

$$\partial_L f(x) = \{\ell \in L :< \ell, y > - < \ell, x >\leq (f(y) - f(x))\}, \quad \forall y \in \mathbb{R}^n_+.$$

Proposition 2.5. *[19] Let f be an ICAR function and $y \in \mathbb{R}^n_+ \setminus \{0\}$ be a vector, such that $(1 + \varepsilon)y \in dom f$ for some $\varepsilon > 0$. Then there exists $\ell \in \mathbb{R}^n_+$ with the property :*

$$< \ell, x > - < \ell, y >\leq (f(x) - f(y)), \quad \forall x \in \mathbb{R}^n_+.$$

In particular, the vector $l = u/y$ with $u \in \partial f_y(1)$ enjoys this property, where:

$$(u/y)_i = \begin{cases} 0 & \text{if } y_i = 0, \\ u_i/y_i & \text{if } y_i \neq 0. \end{cases} \tag{2.1}$$

Let $f : \mathbb{R}^n_+ \to \mathbb{R}$ be a convex-along-rays function, It is well known that $\partial f_y(t)$ coincides with the segment $[(f'_y)_-(t), (f'_y)_+(t)]$, where $(f'_y)_-(t)$ and $(f'_y)_+(t)$ are the left and right derivatives of the function f_y at point t, respectively. An easy calculation ([19]) shows that $(f'_y)_-(1) = -f'(y, -y)$ and $(f'_y)_+(1) = f'(y, y)$, where

$$f'(y, u) := \lim_{\alpha\to 0^+} (\frac{f(y + \alpha u) - f(y)}{\alpha})$$

is the directional derivative of function f at point y in the direction u. Thus $\partial f_y(1) = [-f'(y, -y), f'(y, y)]$.

Proposition 2.6. *[19] Let f be an ICAR function and $x \in \mathbb{R}^n_+ \setminus \{0\}$ a point such that $(1+\varepsilon)x \in dom f$ for some $\varepsilon > 0$, and also let L be a set of all min-type functions defined on $\mathbb{R}^n_+$ where $< l, x >:= min_{i\in I_+(l)} l_i x_i$ with $I_+(l) = \{i : l_i > 0\}$. Then the subdifferential $\partial_L f(x)$ is nonempty and:*

$$\partial_L f(x) \supset \{u/x : u \in \partial f_x(1)\}$$

where $f_x(t) = f(tx)$.

Definition 2.7. A function $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous at a point $x \in \mathbb{R}^n$ if there exist scalars $K > 0$ and $\varepsilon > 0$ such that:

$$|f(y) - f(z)| \leq K\|y - z\|, \;\; \forall \, y, z \in B(x; \varepsilon).$$

Function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be locally Lipschitz continuous on a set $U \subseteq \mathbb{R}^n$ if it is locally Lipschitz continuous at every point belonging to the set U.

Definition 2.8. A point $x \in \mathbb{R}^n$ satisfying $0 \in \partial f(x)$ is called a stationary point of f.

3. Characterizing a set which contains the optimal solution

In this section, we want to find a set contains the optimal solution of our problem which has the following form:

$$\begin{array}{c} \min f(x) \\ s.t. \\ g(x) \le 0, \\ x \in X, \end{array} \tag{P}$$

where X is a compact set in $\mathbb{R}^n_+$ and functions $f(x)$ and $g(x)$ are increasing convex along rays. For every point $x \in X$, consider:

$$\begin{array}{c} \min f_x(t) \\ s.t. \\ g_x(t) \le 0, \\ t \ge 0, \end{array} \tag{P_x}$$

where the functions $f_x(t)$ and $g_x(t)$ are convex for all $t \ge 0,\ x \in X$ and and the feasible region of problem P_x for $x \in X$ is:

$$U_x := \{t \ge 0 : \quad g_x(t) \le 0\ \}.$$

We define the set S as follow:

$$S := \{x \in X : \min_{t \in U_x}(f_x(t)) = f_x(1)\}. \tag{3.1}$$

Lemma 3.1. *The set S in (3.2) is nonempty.*

Proof. Consider an arbitrary and fixed point $\hat{x} \in X$, suppose that the optimal solution of $P_{\hat{x}}(t)$ is $\hat{t}$, .i.e:

$$min_{t \in U_{\hat{x}}}(f_{\hat{x}}(t)) = f_{\hat{x}}(\hat{t}).$$

We assume $\hat{t}\hat{x} := \bar{x}$, since the rays $\{t\hat{x} : \quad t \ge 0\}$ and $\{t\bar{x} : \quad t \ge 0\}$ are equal, the optimal value of two following problems:

$$\begin{array}{c} min f_{\hat{x}}(t) \\ s.t \quad g_{\hat{x}}(t) \le 0, \\ t \ge 0, \end{array}$$

and:

$$\begin{array}{c} min f_{\bar{x}}(t) \\ s.t \quad g_{\bar{x}}(t) \le 0, \\ t \ge 0, \end{array}$$

are equal. so we have:

$$min_{t \in U_{\bar{x}}}(f_{\bar{x}}(t)) = min_{t \in U_{\hat{x}}}(f_{\hat{x}}(t)) = f_{\hat{x}}(\hat{t}) = f(\hat{t}\hat{x}) = f(\bar{x}) = f_{\bar{x}}(1),$$

consequently, $\bar{x} \in S$ and S is not empty.

□

Lemma 3.2. *Suppose that $\tilde{x}$ is an optimal solution for problem (P) and consider the set:*

$$S := \{x \in X : \min_{t \in U_x}(f_x(t)) = f_x(1)\},$$

then $\tilde{x} \in S$.

Proof. We must show that $t = 1$ is the optimal solution for the problem $(P_{\tilde{x}})$. let $t_0 \geq 0$ is the optimal solution for the problem $(P_{\tilde{x}})$, Since $\tilde{x}$ is an optimal solution of problem (P), on ray $\{\lambda\tilde{x} : \lambda \geq 0\}$ we have:

$$f(\tilde{x}) \leq f_{\tilde{x}}(t_0),$$

if $f(\tilde{x}) < f_{\tilde{x}}(t_0)$ then it is a contradiction, because it means that t_0 is not optimal for problem $(P_{\tilde{x}})$. Therefore $f_{\tilde{x}}(1) = f(\tilde{x}) = f_{\tilde{x}}(t_0)$ and it can be concluded that $t = 1$ is an optimal solution for $(P_{\tilde{x}})$, so $\tilde{x} \in S$.

□

Let L be the set of all min-type functions $l(x) =< l, x >$ with $l \in \mathbb{R}^n_+$, where:

$$< l, x > \ = \min_{i \in I_+(l)} l_i x_i \ \ with \ \ I_+(l) = \{i : l_i > 0\}.$$

Vertical shifts of functions $\ell \in L$, which are in the form $h(x) = \ell(x) - c$, for all $x \in X$ with $\ell \in L$ and $c \in \mathbb{R}$, are called $L-$affine functions or abstract affine functions with respect to L.

Definition 3.3. Consider a point $x_0 \in f := \{x \in X : f(x) < \infty\}$.The set $\partial^*_{H_L} f(x)$ is defined as follows:

$$\partial^*_{H_L} f(x_0) := \{h \in supp(f, H) : h(x_0) = f(x_0)\},$$

where H_L is the set of all L-affine functions.

Proposition 3.4. *[19] Let $f : \mathbb{R}^n_+ \to \mathbb{R}$ be an ICAR function such that for each $x \in \mathbb{R}^n_+$, the function f_x defined by $f_x(t) = f(tx)$ is lower semicontinuous on $\mathbb{R}^n_+$. Then a function f is abstract convex with respect to the set H_L of all functions h defined by $h(x) =< l, x > -c$.*

Let f be an ICAR function. Proposition 2.6 shows that the $L-$subdifferential $\partial_L f(x)$ is nonempty and contains the function $l^x := \dfrac{f'(x,x)}{x}$, where:

$$l^x(y) = \min_{i \in I_+(x)} \frac{f'(x,x)}{x_i} y_i = f'(x,x) \min_{i \in I_+(x)} \frac{y_i}{x_i}.$$

According to Proposition 7.1 in ([19]) the set $\partial^*_{H_L} f(x)$ is nonempty for all $x \in \mathbb{R}^n_+$ and this set contains function h_x, where:

$$h_x(y) = f'(x,x) \min_{i \in I_+(x)} \frac{y_i}{x_i} - (f'(x,x) - f(x)).$$

It is clear that $h_x(x) = f(x)$ and also $h_x(y) \leq f(y)$ for all $y \in X$. Indeed, let L be the set of all min-type functions $l(x) = \ < l, x >$ with $l \in \mathbb{R}^n_+$, where $< l, x >= \min_{i \in I_+(l)} l_i x_i$ with $I_+(l) = \{i : l_i > 0\}$.

since $\partial f_y(1) = [-f'(y,-y), f'(y,y)]$ and $\partial_L f(x) \supset \{u/x : u \in \partial f_x(1)\}$ we have $\partial_L f(x) \supset \{f'(x,x)/x : x \in X\}$. Therefore:

$$< f'(x,x)/x, y > - < f'(x,x)/x, x > \leq (f(y) - f(x)), \quad \forall y \in \mathbb{R}^n_+,$$

thus:

$$f'(x,x) \min_{i \in I_+(x)} \frac{y_i}{x_i} - f'(x,x) + f(x) \leq f(y), \forall y \in \mathbb{R}^n_+$$
$$\to h_x(y) \leq f(y). \tag{3.2}$$

Proposition 3.5. *Let f be an ICAR function defined on $\mathbb{R}^n_+$ and $\bar{x} \neq 0$ a point such that $\lambda\bar{x} \in dom$ f for some $\lambda > 1$, then $f(\bar{x}) = liminf_{k\to\infty} f(x_k)$ whenever $x_k \to \bar{x}$.*

By using h as an element of the set $\partial^*_{H_L} f(x)$, we can represent the cutting angle method in the following form.

***Algorithm*1 :**
***Step*0.** Set $k := 0$ and Choose an arbitrary point $x^0 \in X$.
***Step*1.** Calculate a vector l^k and $\bar{l}^k$ with coordinates l^k_i and $\bar{l}^k_i$:

$$l^k_i := \begin{cases} \dfrac{f'(x^k, x^k)}{x^k_i}, & , x^k_i \neq 0, \\ 0 & , x^k_i = 0. \end{cases} \tag{3.3}$$

$$\bar{l}^k_i := \begin{cases} \dfrac{g'(x^k, x^k)}{x^k_i} & , x^k_i \neq 0, \\ 0 & , x^k_i = 0. \end{cases} \tag{3.4}$$

where x^k_i is the $i-$th coordinate of vector x^k .
***Step*2.** Define concave functions h_k and $\bar{h}_k$ by:

$$h_k(x) := \min_{i \in I_+(x^k)} l^k_i x_i - (f'(x^k, x^k) - f(x^k)),$$

$$\bar{h}_k(x) := \min_{i \in I_+(x^k)} \bar{l}^k_i x_i - (g'(x^k, x^k) - g(x^k)).$$

***Step*3.** Set:

$$f_k(x) := max_{0\leq j\leq k} h_j(x) \ \ and \ \ g_k(x) := max_{0\leq j\leq k} \bar{h}_j(x),$$

and solve the following problem by algorithm 3:

$$\begin{array}{c} \min f_k(x) \\ s.t. \\ g_k(x) \leq 0, \\ x \in X. \end{array} \tag{P_k}$$

***Step*4.** Let x^{k+1} be a solution of problem (P_k), using the existing software packages or the algorithm for solving subproblem in [22] (section 5) find $x^{k+1}_s \in S$ by solving the convex subproblem

$(P_{x^{k+1}}(t))$ and set $x^{k+1} := x_s^{k+1}$ and go to Step 1.

Now, a proof of the convergence of the algorithm 1 is given under an additional assumption and in order to study the convergence, we introduce the following quantities (for the classical convex cases which were considered by A.M. Rubinov):

$$\mu_k = f_k(x_s^k) \ \ (k = 0, 1, 2, ..), \ \ \lambda_k = f_{k-1}(x^k) \ \ (k = 1, 2, ..).$$

Some properties of functions f_k and numbers λ_k and μ_k are as follows:
1) $f_0(x) \le f_1(x) \le f_2(x) \le ... \le f_k(x) \le ... \le f(x), \quad \forall x \in X.$
Since $f_k(x) := \max_{0\le j\le k} h_j(x)$ and by (3.2) we have $h_j(x) \le f(x)$ for all $j = 0, 1, ..., k$ and $x \in X$, we conclude that $\max_{0\le j\le k} h_j(x) \le f(x)$.
2) $\mu_k = f(x_s^k) = h_k(x_s^k)$. Indeed:

$$f_k(x_s^k) \le f(x_s^k) = h_k(x_s^k) \le \max_{0\le i\le k} h_i(x_s^k) = f_k(x_s^k).$$

3) The inequality $\lambda_k \le \lambda_{k+1}$ holds for all k. In fact:

$$\begin{aligned}\lambda_k = f_{k-1}(x^k) &= \min_{g_{k-1}(x)\le 0 \ \ x\in X} f_{k-1}(x) = \min_{g_{k-1}(x)\le 0 \ \ x\in X} \max_{0\le i\le k-1} h_i(x)\\ &\le \min_{g_{k-1}(x)\le 0 \ \ x\in X} \max_{0\le i\le k} h_i(x) \le \min_{g_k(x)\le 0 \ \ x\in X} \max_{0\le i\le k} h_i(x)\\ &= \min_{g_k(x)\le 0 \ \ x\in X} f_k(x) = f_k(x^{k+1}) = \lambda_{k+1}.\end{aligned}$$

4) Sequence $\{\lambda_k\}$ is increasing and bounded from above. Therefore, $\lim_{k\to\infty} \lambda_k$ exists. Indeed:

$$\lambda_k = f_{k-1}(x^k) = \min_{g_{k-1}(x)\le 0 \ \ x\in X} f_{k-1}(x)$$

$$\le \min_{g_{k-1}(x)\le 0 \ \ x\in X} f(x) \le \min_{g(x)\le 0 \ \ x\in X} f(x) \le f(x^0), \forall k = 1, 2, ...$$

5) $\lambda_k \le \min_{g(x)\le 0 \ \ x\in X} f(x) \le \mu_k$ for all k=1,2, ... Indeed for a solution u of the problem (P), we have:

$$\min_{g(x)\le 0 \ \ x\in X} f(x) = f(u) \ge f_{k-1}(u) \ge \min_{g_{k-1}(x)\le 0 \ \ x\in X} f_{k-1}(x) = \lambda_k.$$

On the other hand, $\min_{g(x)\le 0 \ \ x\in X} f(x) \le f(x_s^k) = \mu_k$.

Lemma 3.6. *If we have $x^k = x^{k+1} \in S$ for some k, then x^k is a solution of problem P.*

Proof. In this case, we can take $h_k = h_{k+1}$, therefore:

$$\lambda_{k+1} = f_k(x^{k+1}) = f_k(x_s^{k+1}) = \max_{0\le i\le k} h_i(x_s^{k+1}) = \max_{0\le i\le k+1} h_i(x_s^{k+1}) = f_{k+1}(x_s^{k+1}) = \mu_{k+1}.$$

Property (5) shows that $f(x^k) = \min_{g(x)\le 0 \ \ x\in X} f(x)$. □

In the following, we give a condition which provides the convergence of algorithm 1 whenever the produced sequence is infnite.

Remark 3.7. In step 4 of algorithm 1, we take the feasible point x_s^{k+1} such that tx_s^{k+1} is not feasible for all scaler $t < 1$ and therefore, we have $x_s^{k+1} \leq x^{k+1}$ for all feasible solutions x^{k+1}.
Since $f_k(.)$ is an increasing function, we may write $f_k(x_s^{k+1}) \leq f_k(x^{k+1})$, also in problem $min_{g_k(x)\leq 0 \quad x\in X} f_k(x) = f_k(x^{k+1})$ we have $g_k(x_s^{k+1}) \leq g(x_s^{k+1}) \leq 0$, thus x_s^{k+1} is feasible for this problem and $f_k(x^{k+1}) \leq f_k(x_s^{k+1})$, so we conclude $f_k(x_s^{k+1}) = f_k(x^{k+1})$.
We also suppose that $X \subset (dom\ f \cap dom\ g)$ and for each $x \in X$ there exists $x_s \in S$.

Proposition 3.8. *Assume that the set L consists of concave functions defined on an open convex set, which contains the compact set X. Let the sequence $\{x_s^k\}$, produced by algorithm 1, is infinite. Suppose that the directional derivatives $u \to h'_k(x, u)$ of the concave functions h_k at points $x \in X$ are uniformly bounded on set X:*

$$\|h'_k(x, .)\| = \max_{\|u\|\leq 1} |h'_k(x, u)| \leq r < +\infty, x \in X, k = 0, 1, \dots .$$

Then each limit point $\bar{x}_s$ of the sequence $\{x_s^k\}$ is a solution of problem (P).

Proof. Let $x_s^{k_j} \to \bar{x}_s$ for each j, consider the function h_i with $i \leq k_{j-1}$ and find a supergradient a_i of the concave function h_i at the point x^{k_j} . Let $b_i(x) = h_i(x_s^{k_j}) + [a_i, x - x_s^{k_j}]$, then $b_i(x) \geq h_i(x)$ for all $x \in X$ and $h_i(x_s^{k_j}) = b_i(x_s^{k_j})$. We have:

$$\mu_{k_{j-1}} = f_{k_{j-1}}(x_s^{k_{j-1}}) = \max_{0\leq i\leq k_{j-1}} h_i(x_s^{k_{j-1}}) \leq \max_{0\leq i\leq k_{j-1}} b_i(x_s^{k_{j-1}})$$

$$= \max_{0\leq i\leq k_{j-1}} (h_i(x_s^{k_j}) + [a_i, x_s^{k_{j-1}} - x_s^{k_j}]) \leq \max_{0\leq i\leq k_{j-1}} (h_i(x_s^{k_j})) + \max_{0\leq i\leq k_{j-1}} \|a_i\|.\|x_s^{k_{j-1}} - x_s^{k_j}\|$$

$$= f_{k_{j-1}}(x_s^{k_j}) + \max_{0\leq i\leq k_{j-1}} \|a_i\|.\|x_s^{k_{j-1}} - x_s^{k_j}\|.$$

Let $\bar{\partial} h_i(x_s^{k_j})$ be the superdifferential of concave function h_i at point $x_s^{k_j}$. Since $a_i \in \bar{\partial} h_i(x_s^{k_j})$, we have:

$$h'_i(x_s^{k_j}, u) = \min\{[l, u], l \in \bar{\partial} h_i(x_s^{k_j})\} \leq [a_i, u], \quad \forall u \in \mathbb{R}^n.$$

Hence:

$$-h'_i(x_s^{k_j}, -u) \geq -[a_i, -u] = [a_i, u].$$

According to assumption we have:

$$\|a_i\| = \max_{\|u\|\leq 1}[a_i, u] \leq \max_{\|u\|\leq 1} -h'_i(x_s^{k_j}, -u) \leq \max_{\|u\|\leq 1} |h'_i(x_s^{k_j}, u)| \leq r.$$

Let $\lim_{k\to\infty} \lambda_k = \lambda$ Since $\|x_s^{k_{j-1}} - x_s^{k_j}\| \to 0$, we have (by Remark 3.7) *limsup* $\mu_{k_j} \leq \lambda$. On the other hand, $\mu_{k_j} \geq \lambda_{k_{j-1}}$ implies *liminf* $\mu_{k_j} \geq \lambda$.Thus $\mu_{k_j} = f(x_s^{k_j}) \to \lambda$.
Since $x_s^{k_j} \to \bar{x}_s$ and functions f and g are *ICAR*. by proposition 3.1, we have: $f(\bar{x}_s) = liminf_{k\to\infty} f(x_s^{k_j})$, whenever $x_s^{k_j} \to \bar{x}_s$.
Furthermore, $f(x_s^{k_j}) \to \lambda$, thus $\lambda = f(\bar{x}_s) \leq min_{g(x)\leq 0 \quad x\in X} f(x) \leq f(\bar{x}_s)$ therefore:

$$\min_{g(x)\leq 0 \quad x\in X} f(x) = f(\bar{x}_s).$$

For function $g(x)$ we have $liminf_{x_s^{k_j} \to \bar{x}_s} g(x_s^{k_j}) = g(\bar{x}_s)$ and since $g(x_s^{k_j}) \leq 0$, $limsup_{x_s^{k_j} \to \bar{x}_s} g(x_s^{k_j}) \leq 0$, and we have:

$$g(\bar{x}_s) = liminf_{x_s^{k_j} \to \bar{x}_s} g(x_s^{k_j}) \leq limsup_{x_s^{k_j} \to \bar{x}_s} g(x_s^{k_j}) \leq 0$$

therefore point $\bar{x}_s$ is feasible and the optimal solution for problem (P). □

4. Solving Subproblem (P_k)

The subproblem obtained in algorithm 1 is going to be solved, which is in the following form:

$$\begin{array}{c} \min f_k(x) \\ s.t. \\ g_k(x) \leq 0, \\ x \in X. \end{array} \qquad (P_k)$$

where $f_k(.)$, $g_k(.)$ are noconvex locally Lipschitz functions and the set X is compact.
In the following, we define exact penalty function for (P_k) and also the Subgradient method (see [3]) is used for solving the subproblem (P_k).
For simplicity of presentation, we have used a single scalar penalty parameter r for the constraints [4]. The exact penalty function formulation is as follows:

$$F_r(x) = f_k(x) + r(max\{0, g_k(x)\}), \qquad (F_r)$$

where $r \geq 0$.
Approximate minimizer x_r of function F_r can be computed using the subgradient method for nonconvex nonsmooth optimization, [3]. The main difficulty of penalty function methods lies in choosing the initial value and the updating strategy for the penalty parameter r. If the value remains too small, the unconstrained problem F_r may produce a solution which is not feasible for the original problem. On the other hand, if the value becomes too large, the unconstrained problem F_r will be ill-conditioned. Moreover, both too large and too small penalty parameter could present numerical difficulties.
The pseudo-code for exact penalty method is given in the following. [4].
Exact Penalty Algorithm
Initialize $x_0 \in \mathbb{R}^n_+$, $r_1 \geq 0, \varepsilon \geq 0$ and $k = 1$;
While the termination condition is not met, find an approximate minimizer x_k of problem:

$$F_{r_k}(x) = f_k(x) + r_k(max\{0, g_k(x)\}),$$

starting at x_k using Algorithm 3.
If $F_{r_k}(x_k) - f_k(x_k) \leq \varepsilon$, then STOP with the approximate solution x_k;
Else Choose new penalty parameter $r_{k+1} > r_k$;
Choose new starting point x_{k+1};
End if,
End While,
Return final solution x_k;

End Exact Penalty algorithm.
Now we want to find descent directions for problem F_r by the following algorithm [3]. Let $x \in \mathbb{R}^n_+$ be a given point,$c_1 \in (0, 1)$ and $\delta > 0$ be given numbers.
Algorithm2 : Computation of the descent direction:
Step1. Select any $d_1 \in S_1 := \{x \in \mathbb{R}^n : \|x\| = 1\}$ and compute a subgradient $v_1 \in \partial F_{r_k}(x)$ such that $F'_{r_k}(x, d_1) = [v_1, d_1]$. Set $\tilde{v_1} := v_1$ and $k := 1$.
Step2. Solve the following problem:

$$minimize \;\; Q_k(\lambda) := \|\lambda v_k + (1 - \lambda)\tilde{v_k}\|^2,$$

such that $\lambda \in [0, 1]$. Let $\bar{\lambda}_k$ be a solution of this problem. Set $\bar{v_k} := \bar{\lambda}_k v_k + (1 - \bar{\lambda}_k)\tilde{v_k}$.
Step3. (Stopping criterion) If $\|\bar{v_k}\| < \delta_1$ then stop, otherwise go to Step 4.
Step4. Compute the search direction by $\bar{d_k} = -\frac{\bar{v_k}}{\|\bar{v_k}\|}$.
Step5. If $F'_{r_k}(x, \bar{d_k}) \leq -c_1\|\bar{v_k}\|$, then stop. Otherwise go to Step 6.
Step6. Compute a subgradient $u \in \partial F_{r_k}(x)$ such that $F'_{r_k}(x, \bar{d_k}) =< u, \bar{d_k} >$.
Set $v_{k+1} := u, \tilde{v}_{k+1} := \bar{v_k}, k := k + 1$ and go to Step 2.
we will apply algorithm 2 to find descent directions because it allows us to design a simple line search procedure.
Let $c_1 \in (0, 1), c_2 \in (0, c_1]$ be given numbers. An algorithm for finding approximate stationary points satisfying $0 \in \partial F_{r_k}(x) + B_{\varepsilon+\delta}$ proceeds as follows, [3].
Algorithm3 : Computation of the approximate stationary points of Problem F_r.
Step1. Select any starting point $x_0 \in \mathbb{R}^n_+$ and set $k := 0$.
Step2. Apply algorithm 2 to comput the descent direction at $x = x_k$ for given $\delta > 0$ and $c_1 \in (0, 1)$ for Problem F_r.
Step3. If $\|\bar{v_k}\| < \delta_2$, then stop, otherwise go to Step 4.
Step4. Compute $x_{k+1} = xk + \sigma_k d_k$, where σ_k is defined as follows:

$$\sigma_k = Argmax\,\{\sigma \geq 0 : F_{r_k}(x_k + \sigma\bar{d_k}) - F_{r_k}(x_k) \leq -c_2\sigma\|\bar{v_k}\|\}.$$

Set $k := k + 1$ and go to Step 2.
Algorithm 3 terminates after finite number of iterations [3].

5. Numerical Examples

In this section, we will consider some numerical examples using the mentioned algorithm and draw their diagram to see how the algorithm works. These examples are presented to demonstrate the accuracy and efficiency of the the algorithms.

Parameters were chosen as follows:

$$\varepsilon = 10^{-7}, c_1 = 10^{-3}, c_2 = 10^{-4}, \delta_1 = 10^{-3}, \delta_2 = 10^{-5}, r_1 = 10.$$

By using matlab software, solutions in high accuracy were obtained and the proposed algorithm is evaluated by five examples, and the numerical results are summarized in Table 1.

Example 5.1. Consider the problem:

$$min\ \ 50\sqrt{xy}$$

$$s.t\quad 2/3\sqrt{(4xy-1)^3}\leq 10\quad 0.5\leq x\ \leq\ 5,\ 0.5\leq y\leq\ 5.$$

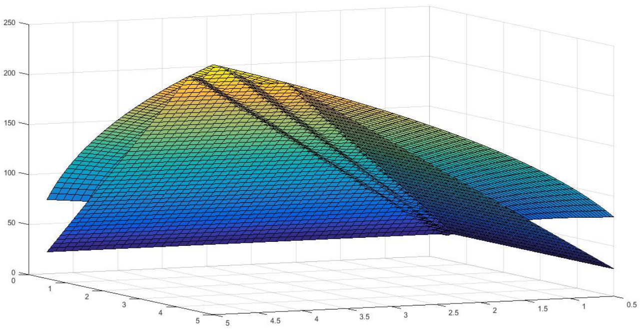

Figure 1: Diagram of example 5.1.

Example 5.2. Consider the following problem:

$$min\ \ max(x+y+20, 10\sqrt{xy})$$

$$s.t\quad x^2+y^2\leq 20,\quad 0.5\leq x\ \leq\ 5,\ 0.5\leq y\leq\ 5.$$

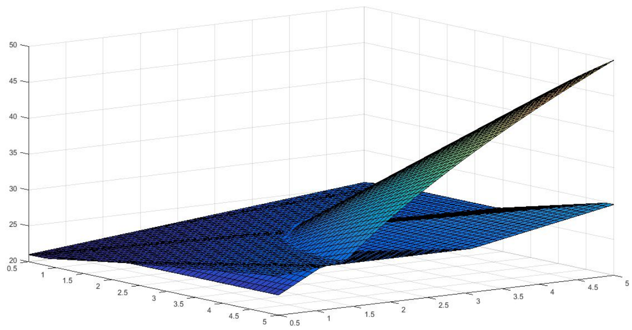

Figure 2: Diagram of example 5.2.

Example 5.3. Consider the problem:

$$min\ \ max(\sqrt{(x^3y^5)},\ \sqrt{(x^2y^6)},\ \sqrt{(x^6y^2)})$$

$$s.t\quad x^2+y^2\leq 20,\quad (x,y)\in X,$$

where X is a parallelogram with vertices $(1,2),(2,1),(3,2)$ *and* $(4,1)$.

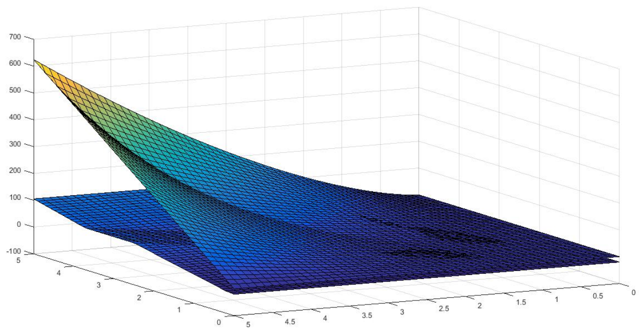

Figure 3: Diagram of example 5.3.

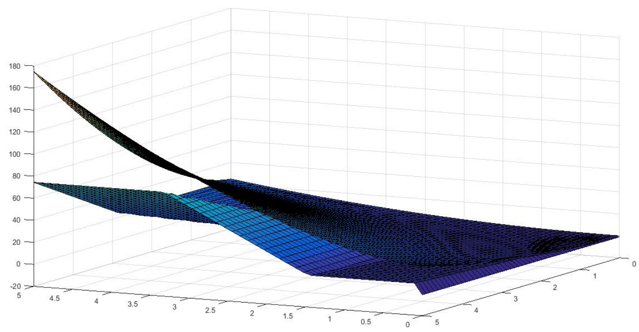

Figure 4: Diagram of example 5.4.

Example 5.4. The following problem is considered:

$$min \;\; x^2 + y^2 + (xy)^{3/2}$$

$$s.t \quad x^3 + y^3 + 5(xy)^{3/2} \leq 100, \quad (x, y) \in X$$

Where X is a trapezius with vertices $(0, 1), (0, 3), (3, 0)$ *and* $(9, 0)$.

Example 5.5. The following problem is considered:

$$min \;\; \sqrt{xy^2z^3}$$

$$s.t \quad \sqrt{xy} + \sqrt{xz} + \sqrt{zy} \leq 50, \quad 1 \leq x \leq 5, \; 1 \leq y \leq 5, \; 1 \leq z \leq 5$$

Note that in Table 1, Opt.solution and Opt.value are the optimal solution and the optimal value obtained by the proposed algorithm, respectively. Time is in second and shows the duration to reach the optimal solutions in the proposed algorithm.
From the above discussions, it can be concluded that the mentioned algorithm, which is derived from the Cutting Angle Method, can also be used to solve problems with increasing convex along rays constraints and many problems in the fields of economic, engineering and mathematical analysis can be solved with this method.

Table 1: Obtained results by the proposed algorithms

Examples	Opt. solution	Opt. value	Starting point	Iter.	Time(Sec.)
Example 5.1	(0.5000,0.5000)	25	(1,1)	5	12.12
Example 5.2	(1.0000,1.0000,1.0000)	1	(1.5,2,2.5)	3	14.27
Example 5.3	(0.5000,0.5000)	21	(2,1)	5	11.77
Example 5.4	(2.0000,1.0000)	8	(3,1.5)	3	13.51
Example 5.5	(0.0330,0.9890)	0.9851	(1,3)	5	15.69

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