Wavelets and Linear Algebra 9(1) (2022) 1- 14

Vali-e-Asr University of Rafsanjan

Wavelets and Linear Algebra

http://wala.vru.ac.ir

Connes Amenability for Certain Product of Banach Algebras

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Article Info

Article history:
Received 5 September 2020
Accepted 15 October 2021
Available online 16 November 2022
Communicated by Ali Akbar Arefijamaal

Keywords:
Connes amenability, Banach algebras, derivation, group algebra, invariant mean.

2010 MSC:
65F05, 46L05
11Y50.

Abstract

In this paper we develop the notions of Connes amenability for certain product of Banach algebras. We give necessary and sufficient conditions for the existence of an invariant mean on the predual of Θ-Lau product $\mathcal{A} \times_\Theta \mathcal{B}$, module extension Banach algebra $\mathcal{A} \oplus \mathcal{X}$ and projective tensor product $\mathcal{A}\widehat{\otimes}\mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are dual Banach algebras with preduals $\mathcal{A}_*$ and $\mathcal{B}_*$ respectively and $\mathcal{X}$ is a normal Banach $\mathcal{A}$-bimodule with predual $\mathcal{X}_*$.

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http://doi.org/10.22072/wala.2021.135909.1301

1. Introduction

Connes amenable dual Banach algebras were introduced by Runde in [16] and turned out to be extremely interesting objects of researches. A dual Banach algebra $\mathcal{A}$ is called Connes amenable if every *weak**-continuous derivation from $\mathcal{A}$ into a normal, dual Banach $\mathcal{A}$-bimodule is inner. In [19], Runde showed that if the predual $\mathcal{A}_*$ is injective, then $\mathcal{A}$ is Connes amenable. Let E be a Banach $\mathcal{A}$-bimodule. The collection of all elements of E that module maps from $\mathcal{A}$ to E are *weak**-weakly continuous is denoted by $\sigma wc(E)$. A σwc -virtual diagonal for $\mathcal{A}$ is an element $M \in \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ such that $a.M = M.a$ and $a.\Delta_{\sigma wc} M = a$ for each $a \in \mathcal{A}$ that $\Delta : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$ is the multiplication operator. Also in [19], it is shown that a Banach algebra $\mathcal{A}$ is Connes amenable if and only if it has a so-called σwc -virtual diagonal. In [3], Daws provided some simplifications to the notion of injectivity and σwc-virtual diagonals.

Recently the authors have introduced the ϕ-version of Connes amenability of dual Banach algebra $\mathcal{A}$ that ϕ is a homomorphism from $\mathcal{A}$ onto $\mathbb{C}$ and $\phi \in \mathcal{A}_*$[6]. A dual Banach algebra $\mathcal{A}$ is ϕ-Connes amenable if, for every normal $\mathcal{A}$ -bimodule E where the left action is of the form $a.x = \phi(a)x$ $(a \in \mathcal{A}, x \in E)$, every bounded *weak**-continuous derivation $D : \mathcal{A} \to E$ is inner. The authors proved that a dual Banach algebra $\mathcal{A}$ is ϕ-Connes amenable if and only if $\mathcal{A}^{**}$ has a ϕ-invariant mean on $\mathcal{A}_*$, that is a functional $m \in \mathcal{A}^{**}$ such that $\langle m, \phi\rangle = 1$ and $\langle m, a.f\rangle = \phi(a)\langle m, f\rangle$ for all $a \in \mathcal{A}$ and $f \in \mathcal{A}_*$. All of these concepts generalized the earlier concept of amenability for Banach algebras introduced in [9], [10] and [11].

In this paper, we are going to investigate the Connes amenability and character Connes amenability for Lau product $\mathcal{A} \times_\Theta \mathcal{B}$, the module extension $\mathcal{A} \oplus \mathcal{X}$ and projective tensor product $\mathcal{A}\widehat{\otimes}\mathcal{B}$.

2. $\varphi \otimes \psi$-Connes Amenability and Some Basic Properties

Let $\mathcal{A}$ be a Banach algebra. Let $\mathcal{K}$ be a subspace of $\mathcal{A}^*$ and $\varphi \in \triangle(\mathcal{A}) \bigcap \mathcal{K}$. A linear functional m on $\mathcal{K}$ is called a mean if $\langle m, \varphi\rangle = 1$. A mean m is φ-invariant mean if $\langle m, a.f\rangle = \varphi(a)\langle m, f\rangle$ for all $a \in \mathcal{A}$ and $f \in \mathcal{K}$. Note that if $\mathcal{K} = \mathcal{A}^*$, then $\mathcal{A}$ is φ-amenable and if $\mathcal{K} = \mathcal{A}_*$, then $\mathcal{A}$ is φ-Connes amenable. See [10] and [6]. In this section we suppose that projective tensor product of two dual Banach spaces is a dual Banach space. In Theorem 2.6, we give some conditions that it takes the dual space structure.

Theorem 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. Let $\mathcal{K}$ be a subspace of $\mathcal{A}^*$ and let $\mathcal{Y}$ be a subspace of $\mathcal{B}^*$. Let $\varphi \in \triangle(\mathcal{A}) \bigcap \mathcal{K}$ and $\psi \in \triangle(\mathcal{B}) \bigcap \mathcal{Y}$. Suppose that there is a φ-invariant mean on $\mathcal{K}$ and a ψ-invariant mean on $\mathcal{Y}$. Then there is a $\varphi \otimes \psi$-invariant mean on $\mathcal{K}\widehat{\otimes}\mathcal{Y}$.*

Proof. let m_1 be a φ-invariant mean on $\mathcal{K}$ and m_2 be a ψ-invariant mean on $\mathcal{Y}$. For each $f \in \mathcal{K}$ and $g \in \mathcal{Y}$, the mapping $T(a, b) = \langle a, f\rangle\langle b, g\rangle$ is a bilinear map of $\mathcal{A} \times \mathcal{B}$ into $\mathbb{C}$. By Theorem 6 in [2], there exists a unique linear mapping $\Lambda_T : \mathcal{A}\hat{\otimes}\mathcal{B} \to \mathbb{C}$ such that $\Lambda_T(a \otimes b) = T(a, b)$. It is known that T has a continuous extension $\overline{T} : \mathcal{A}^{**} \times \mathcal{B}^{**} \to \mathbb{C}$ such that for $F \in \mathcal{A}^{**}$, $G \in \mathcal{B}^{**}$ and nets $\{a_\alpha\} \subseteq \mathcal{A}$ and $\{b_\beta\} \subseteq \mathcal{B}$ with $\lim_\alpha a_\alpha = F$ and $\lim_\beta b_\beta = G$ in the *weak**-topology,

$$\overline{T}(F, G) = \lim_\alpha \lim_\beta T(a_\alpha, b_\beta).$$

By Lemma 1.7 in [7], there exists a continuous linear mapping $\Psi : \mathcal{A}^{**}\hat{\otimes}\mathcal{B}^{**} \to (\mathcal{A}\hat{\otimes}\mathcal{B})^{**}$ such that $\langle \Psi(F \otimes G), \Lambda_T\rangle = \overline{T}(F, G)$. Choose nets $\{a_\alpha\}$ in $\mathcal{A}$ and $\{b_\beta\}$ in $\mathcal{B}$ such that $a_\alpha \to m_1$ in the *weak**-topology on $\mathcal{A}^{**}$ and $b_\beta \to m_2$ in the *weak**-topology on $\mathcal{B}^{**}$. We have

$$\begin{aligned}
\langle \Psi(m_1 \otimes m_2), (a \otimes b).(f \otimes g)\rangle &= \langle \Psi(m_1 \otimes m_2), a.f \otimes b.g\rangle \\
&= \lim_\alpha \lim_\beta a.f \otimes b.g\langle a_\alpha, b_\beta\rangle \\
&= \lim_\alpha \lim_\beta \langle a.f, a_\alpha\rangle\langle b.g, b_\beta\rangle \\
&= \langle a.f, m_1\rangle\langle b.g, m_2\rangle \\
&= \varphi(a)\psi(b)\langle f, m_1\rangle\langle g, m_2\rangle \\
&= (\varphi \otimes \psi)(a \otimes b)\langle f, m_1\rangle\langle g, m_2\rangle \\
&= (\varphi \otimes \psi)(a \otimes b) f \otimes g\langle m_1, m_2\rangle \\
&= (\varphi \otimes \psi)(a \otimes b)\langle \Psi(m_1 \otimes m_2), f \otimes g\rangle
\end{aligned}$$

for all $a \otimes b \in \mathcal{A}\hat{\otimes}\mathcal{B}$. Therefore $\Psi(m_1 \otimes m_2)$ is a $\varphi \otimes \psi$-invariant mean on $\mathcal{K}\hat{\otimes}\mathcal{Y}$. □

Example 2.2. Let $luc(S)$ be the bounded left uniformly continuous functions on discrete semigroup S. Consider $\mathbb{N}$ with trivial character 1. There is a 1-mean on $luc(\mathbb{N})$. But $l^1(\mathbb{N})$ is not amenable. This is equivalent to there is not a 1-mean on $l^\infty(\mathbb{N})$. Therefore, for character $\varphi = 1$, there is a $\varphi \otimes \varphi$-invariant mean on subspace $luc(S)\hat{\otimes}luc(S)$ of $l^\infty(S)\hat{\otimes}l^\infty(S)$, but there is not a $\varphi \otimes \varphi$-mean on $l^\infty(S)\hat{\otimes}l^\infty(S) = (l^1(S)\hat{\otimes}l^1(S))^*$.

In the following, we establish several criteria for $(\mathcal{A}\hat{\otimes}\mathcal{B})^*$ to possess a $\varphi \otimes \psi$-invariant mean that $\mathcal{A}$ and $\mathcal{B}$ are dual Banach algebras with preduals $\mathcal{A}_*$ and $\mathcal{B}_*$ respectively, $\varphi \in \triangle(\mathcal{A}) \bigcap \mathcal{A}_*$ and $\psi \in \triangle(\mathcal{B}) \bigcap \mathcal{B}_*$. Recall that $\mathcal{A}_* \subset \sigma wc(\mathcal{A}^*)$, $\mathcal{B}_* \subset \sigma wc(\mathcal{B}^*)$ [19, Corollary 4.6] and $\mathcal{A}\hat{\otimes}\mathcal{B}$ can be embedded in $\sigma wc((\mathcal{A}\hat{\otimes}\mathcal{B})^*)^*$.

Theorem 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be dual Banach algebras with preduals $\mathcal{A}_*$ and $\mathcal{B}_*$ respectively. Let $\mathcal{A}\widehat{\otimes}\mathcal{B}$ be dual Banach algebra. Let $\varphi \in \triangle(\mathcal{A}) \bigcap \mathcal{A}_*$ and $\psi \in \triangle(\mathcal{B}) \bigcap \mathcal{B}_*$. Then there is a φ-invariant mean on $\mathcal{A}_*$ and a ψ-invariant mean on $\mathcal{B}_*$ if and only if there is a $\varphi \otimes \psi$-invariant mean on $\sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{B})^*)$.*

Proof. By [6, Theorem 2.3] it is sufficient to $\mathcal{A}\widehat{\otimes}\mathcal{B}$ be $\varphi \otimes \psi$-Connes amenable. Let $m \in \mathcal{A}$ be a φ-Connes mean on $\mathcal{A}_*$ and let $n \in \mathcal{B}$ be ψ-Connes mean on $\mathcal{B}_*$. Take E as a normal $\mathcal{A}\widehat{\otimes}\mathcal{B}$-bimodule such that $(a \otimes b).x = \varphi \otimes \psi(a \otimes b)x$ for all $a \in \mathcal{A}, b \in \mathcal{B}, x \in E$ and let $D : \mathcal{A}\widehat{\otimes}\mathcal{B} \to E$ be a bounded *weak**-continuous derivation defined by

$$D((a_1 \otimes b_1)(a_2 \otimes b_2)) = D(a_1 \otimes b_1).(a_2 \otimes b_2) + (a_1 \otimes b_1).D(a_2 \otimes b_2),$$

for all $a_1, a_2 \in \mathcal{A}, b_1, b_2 \in \mathcal{B}$. From [19, Corollary 4.6], D^* maps E_* into $\sigma wc((\mathcal{A}\hat{\otimes}\mathcal{B})^*)$. Assume that d be the restriction of D^* to E_*. For all $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $f \in E_*$, we get for the left action of $(\mathcal{A}\widehat{\otimes}\mathcal{B})$ on E_*

$$\begin{aligned}
\langle a_2 \otimes b_2, d((a_1 \otimes b_1).f)\rangle &= \langle a_2 \otimes b_2, D^*|_{E_*}((a_1 \otimes b_1).f)\rangle = \langle D(a_2 \otimes b_2), (a_1 \otimes b_1).f\rangle \\
&= \langle D(a_2 \otimes b_2).(a_1 \otimes b_1), f\rangle \\
&= \langle D((a_2 \otimes b_2)(a_1 \otimes b_1)), f\rangle - \varphi \otimes \psi(a_2 \otimes b_2)\langle D(a_1 \otimes b_1), f\rangle \\
&= \langle a_2 \otimes b_2, (a_1 \otimes b_1).d(f)\rangle - \varphi \otimes \psi(a_2 \otimes b_2)\langle f, D(a_1 \otimes b_1)\rangle.
\end{aligned}$$

Therefore $d((a \otimes b).f) = (a \otimes b).d(f) - \langle f, D(a \otimes b)\rangle \varphi \otimes \psi$ for all $a \in \mathcal{A}$,$b \in \mathcal{B}$ and $f \in E_*$. Now take $D' = d^* : \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{B})^*)^* \longrightarrow E$ and consider those elements of $\sigma wc((\mathcal{A}\hat{\otimes}\mathcal{B})^*)^*$ that lie in $\mathcal{A}\hat{\otimes}\mathcal{B}$ and set $x' = D'(m \otimes n) \in E$. It follows that

$$\begin{aligned}\langle x'.(a \otimes b), f\rangle &= \langle (a \otimes b).f, D'(m \otimes n)\rangle = \langle d((a \otimes b).f), m \otimes n\rangle \\ &= \langle (a \otimes b).d(f), m \otimes n\rangle - \langle f, D(a \otimes b)\rangle \langle \varphi \otimes \psi, m \otimes n\rangle \\ &= \langle (a \otimes b).d(f), m \otimes n\rangle - \langle f, D(a \otimes b)\rangle \langle m, \varphi\rangle \langle n, \psi\rangle \\ &= \varphi \otimes \psi(a \otimes b)\langle d(f), m \otimes n\rangle - \langle f, D(a \otimes b)\rangle \\ &= \varphi \otimes \psi(a \otimes b)\langle f, x'\rangle - \langle f, D(a \otimes b)\rangle.\end{aligned}$$

Hence,

$$\begin{aligned}D(a \otimes b) &= \varphi \otimes \psi(a \otimes b).x' - x'.(a \otimes b) \\ &= (a \otimes b).x' - x'.(a \otimes b) = ad_{x'}(a \otimes b),\end{aligned}$$

for all $a \in \mathcal{A}$, $b \in \mathcal{B}$, as required.

Conversely, as [10, Theorem 3.3] define $m_\varphi(f) = m(\varphi \otimes f)$ for $f \in \mathcal{A}_*$ and $n_\psi(g) = n(g \otimes \psi)$ for $g \in \mathcal{B}_*$. It is easy to see that m_φ is a φ-invariant mean on $\mathcal{A}_*$ and n_ψ is a ψ-invariant mean on $\mathcal{B}_*$. □

Example 2.4. It is known that $M(G)\hat{\otimes}M(G) = M(G \times G) = (C_0(G \times G))^* = (C_0(G)\hat{\otimes}C_0(G))^*$. In [18] and [17], it is shown that G is amenable if and only if $M(G)$ is Connes amenable. By [6, Theorem 2.3] and Theorem 2.1, for $\varphi \in C_0(G)$ there is a $\varphi \otimes \varphi$-invariant mean on $C_0(G)\hat{\otimes}C_0(G)$. From [9], G is amenable if and only if $L^1(G)$ is amenable. Also from [2, Proposition 5.4], for any $\varphi \in L^\infty(G)$ there is a φ-invariant mean on $L^\infty(G)$. All in all, for each amenable group G and $\varphi \in C_0(G)$ there is a $\varphi \otimes \varphi$-invariant mean on $C_0(G)\hat{\otimes}C_0(G)$ if and only if there is a $\varphi \otimes \varphi$-invariant mean on $L^\infty(G)\hat{\otimes}L^\infty(G)$.

Example 2.5. Set $\mathcal{A}=\begin{pmatrix} 0 & 0 \\ \mathbb{C} & \mathbb{C} \end{pmatrix}$. With the usual matrix multiplication and l^1-norm, $\mathcal{A}$ is a Banach algebra. Since the complex numbers set is a dual Banach algebra, $\mathcal{A}$ is a dual Banach algebra. We define the following map

$$\varphi : \mathcal{A} \to \mathbb{C}; \qquad \varphi\begin{pmatrix} 0 & 0 \\ z & t \end{pmatrix} = t,$$

for all $z, t \in \mathbb{C}$. Note that φ is norm continuous and $weak^*$-continuous. Let $u = \begin{pmatrix} 0 & 0 \\ 1 & \frac{1}{i} \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ i & i \end{pmatrix} \in \mathcal{A}\widehat{\otimes}\mathcal{A}$. Note that $u \in (\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}$ is a φ- diagonal for $\mathcal{A}$. It is clear that $u \in \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)^*$ and then u is a $\varphi-\sigma wc$-virtual diagonal for $\mathcal{A}$. All in all there is a $\varphi\otimes\varphi$-invariant mean on $(\mathcal{A}\widehat{\otimes}\mathcal{A})^*$ and on predual $(\mathcal{A}\widehat{\otimes}\mathcal{A})_*$. For more details see [13] and [6].

Let E be a Banach space. Then E has the approximation property if there is a net $(S_\alpha)_\alpha$ in $F(E)$; bounded finite rank operators on E, such that $S_\alpha \to E$ uniformly on compact subsets of E.

E has the Radon-Nikodym property if for each finite measure space (Ω, S, μ) and each bounded linear operator $T : L^1(\Omega, S, \mu) \to E$ there is a bounded μ-measurable function $\phi : \Omega \to E$ such that $Tf = \int f\phi d\mu$. The following is [4, Theorem 1.6.16].

Theorem 2.6. *Let E and F be Banach spaces such that*

(i) *E^* or F^* has the approximation property, and*

(ii) *E^* or F^* has the Radon-Nikodym property.*

Then $\kappa_{E,F} : E^*\hat{\otimes}F^* \to (E \otimes_w F)^*,\ \kappa_{E,F}(g, h)(x, y) = \langle x, g\rangle\langle y, h\rangle$ is an isometric isomorphism.

Let S be a weakly cancellative semigroup, then $l^1(S)$ is a dual Banach algebra with predual $c_0(S)$. It is known that $l^1(S)$ and $c_0(S)$ have the approximation property, $l^1(S)$ has the Radon-Nikodym property and $c_0(S)$ lacks the Radon-Nikodym property. By Theorem 5.14 in [3], for a cancellative semigroup S and character $\varphi \in c_0(S)$, there is a $\varphi\otimes\varphi$-invariant mean on $c_0(S)\otimes_w c_0(S)$ if and only if there is a $\varphi \otimes \varphi$-invariant mean on $l^\infty(S)\hat{\otimes}l^\infty(S)$. On the other hand, if S is a weakly cancellative semigroup S that $l^1(S)$ is not amenable and $l^1(S)$ is Connes amenable, then there is a $\varphi \otimes \varphi$-invariant mean on $c_0(S) \otimes_w c_0(S)$ but it is not a $\varphi \otimes \varphi$-invariant mean on $l^\infty(S)\hat{\otimes}l^\infty(S)$.

Theorem 2.7. *Let $\mathcal{A} = (\mathcal{A}_*)^*$, $\mathcal{B} = (\mathcal{B}_*)^*$ and $\mathcal{A}\widehat{\otimes}\mathcal{B}$ be dual Banach algebras, and let $\varphi \in \Delta_{\omega^*}(\mathcal{A}) \bigcap \mathcal{A}_*$, $\psi \in \Delta_{\omega^*}(\mathcal{B}) \bigcap \mathcal{B}_*$. Let I and J be closed two-sided ideals of $\mathcal{A}$ and $\mathcal{B}$ respectively and $\varphi|_I \neq 0$, $\psi|_J \neq 0$. Let I and J be dual Banach algebras. Let $I\widehat{\otimes}J$ be a dual Banach algebra that is $\varphi \otimes \psi|_{I\widehat{\otimes}J}$-Connes amenable, then $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is $\varphi \otimes \psi$-Connes amenable.*

Proof. Let E be a normal $\mathcal{A}\widehat{\otimes}\mathcal{B}$-bimodule such that $(a \otimes b).x = \varphi \otimes \psi(a \otimes b)x$ for all $a \in \mathcal{A}, b \in \mathcal{B}, x \in E$ and suppose that $D : \mathcal{A}\widehat{\otimes}\mathcal{B} \to E$ is a bounded *weak**-continuous derivation. Clearly, $D|_{I\widehat{\otimes}J}$ is a *weak**-continuous derivation. Since $I\widehat{\otimes}J$ is $\varphi \otimes \psi|_{I\widehat{\otimes}J}$-Connes amenable, there exists $x_0 \in E$ such that

$$D(i \otimes j) = (i \otimes j).x_0 - x_0.(i \otimes j) = \varphi \otimes \psi(i \otimes j)x_0 - x_0.(i \otimes j).$$

for all $i \in I, j \in J$. Choose $i_0 \otimes j_0 \in I\widehat{\otimes}J$ with $\varphi(i_0) = 1 = \psi(j_0)$, and put $x = x_0.(i_0 \otimes j_0)$. Now for $(a \otimes b) \in \mathcal{A}\widehat{\otimes}\mathcal{B}$, we obtain

$$
\begin{aligned}
(a\otimes b).x - x.(a\otimes b) &= \varphi\otimes\psi(a\otimes b)x - x.(a\otimes b)\\
&= \varphi\otimes\psi(a\otimes b)x_0.(i_0\otimes j_0) - x_0.(i_0\otimes j_0).(a\otimes b)\\
&- \varphi\otimes\psi((a\otimes b)(i_0\otimes j_0))x_0 + \varphi\otimes\psi((a\otimes b)(i_0\otimes j_0))x_0\\
&= -\varphi\otimes\psi(a\otimes b)(\varphi\otimes\psi(i_0\otimes j_0)x_0 - x_0.(i_0\otimes j_0))\\
&+ \varphi\otimes\psi((a\otimes b)(i_0\otimes j_0))x_0 - x_0.(i_0\otimes j_0).(a\otimes b)\\
&= -\varphi\otimes\psi(a\otimes b)D\mid_{I\widehat{\otimes}J}(i_0\otimes j_0) + D\mid_{I\widehat{\otimes}J}((i_0\otimes j_0)(a\otimes b))\\
&= -\varphi\otimes\psi(a\otimes b)D\mid_{I\widehat{\otimes}J}(i_0\otimes j_0) + (i_0\otimes j_0).D(a\otimes b)\\
&+ D\mid_{I\widehat{\otimes}J}(i_0\otimes j_0).(a\otimes b)\\
&= -\varphi\otimes\psi(a\otimes b)D\mid_{I\widehat{\otimes}J}(i_0\otimes j_0) + D(a\otimes b)\\
&+ D\mid_{I\widehat{\otimes}J}(i_0\otimes j_0).(a\otimes b)\\
&= D(a\otimes b) + (\varphi\otimes\psi(i_0\otimes j_0)x_0 - x_0.(i_0\otimes j_0)).(a\otimes b)\\
&- \varphi\otimes\psi(a\otimes b)(\varphi\otimes\psi(i_0\otimes j_0).x_0 - x_0.(i_0\otimes j_0))\\
&= D(a\otimes b) + (x_0 - x_0.(i_0\otimes j_0)).(a\otimes b)\\
&- \varphi\otimes\psi(a\otimes b)(x_0 - x_0.(i_0\otimes j_0)).
\end{aligned}
$$

Therefore $D(a\otimes b) = \varphi\otimes\psi(a\otimes b)x_0 - x_0.(a\otimes b) = (a\otimes b).x_0 - x_0.(a\otimes b)$ is inner. □

Theorem 2.8. *Let $\mathcal{A} = (\mathcal{A}_*)^*$, $\mathcal{B} = (\mathcal{B}_*)^*$ and $\mathcal{A}\widehat{\otimes}\mathcal{B}$ be dual Banach algebras, and let $\varphi \in \Delta_{\omega^*}(\mathcal{A})\bigcap\mathcal{A}_*$, $\psi \in \Delta_{\omega^*}(\mathcal{B})\bigcap\mathcal{B}_*$. Let I and J be closed two-sided ideals of $\mathcal{A}$ and $\mathcal{B}$ with bounded approximate identities such that $\varphi\mid_I$ and $\psi\mid_J$ are non-zero and $I\widehat{\otimes}J$ is dual Banach algebra. If $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is $\varphi\otimes\psi$-Connes amenable, then $I\widehat{\otimes}J$ is $\varphi\otimes\psi\mid_{I\widehat{\otimes}J}$-Connes amenable.*

Proof. Let $\mathcal{A}\widehat{\otimes}\mathcal{B}$ be $\varphi\otimes\psi$-Connes amenable. By hypothesis and [6, Lemma 3.3] we show that for each normal $I\widehat{\otimes}J$-bimodule E such that predual E_* is pseudo-unital and the left action of $I\widehat{\otimes}J$ on E is $(a\otimes b).x = \varphi\otimes\psi(a\otimes b)x$ $(a\in I, b\in J, x\in E)$, every bounded *weak**-continuous derivation $D : I\widehat{\otimes}J \to E$ is inner. It is easy to see that E is normal $\mathcal{A}\widehat{\otimes}\mathcal{B}$-bimodule (see [20, Proposition 2.1.6]). Let $\{a_\alpha\}\subseteq\mathcal{A}$ and $\{b_\beta\}\subseteq\mathcal{B}$ be two nets that $\{a_\alpha\}\to a$ and $\{b_\beta\}\to b$ in the *weak**-topology. We consider $\{a_\alpha\otimes b_\beta\}_{\alpha,\beta}$ as a net in $\mathcal{A}\widehat{\otimes}\mathcal{B}$ such that $\{a_\alpha\otimes b_\beta\}\to a\otimes b$ in the *weak**-topology. Let $x\in E$ and pick $x_0\in E_*$. There are $i\otimes j\in I\widehat{\otimes}J$ and $y_0\in E_*$ such that $x_0 = (i\otimes j).y_0$. We obtain

$$
\begin{aligned}
\lim_\alpha\lim_\beta\langle x.(a_\alpha\otimes b_\beta), x_0\rangle &= \lim_\alpha\lim_\beta\langle x.(a_\alpha\otimes b_\beta), (i\otimes j).y_0\rangle\\
&= \lim_\alpha\lim_\beta\langle x.(a_\alpha\otimes b_\beta).(i\otimes j), y_0\rangle = \langle x.(a\otimes b), x_0\rangle.
\end{aligned}
$$

Now let (e_α) and (e_β) be bounded approximate identities of $\mathcal{A}$ and $\mathcal{B}$ respectively. Define

$$
\widetilde{D} : \mathcal{A}\widehat{\otimes}\mathcal{B} \to E, \qquad a\otimes b \mapsto \omega^* - \lim_\alpha(D((a\otimes b)(e_\alpha\otimes e_\beta)) - (a\otimes b).D(e_\alpha\otimes e_\beta)).
$$

It is clear that $\widetilde{D}$ is a continuous derivation. To see that $\widetilde{D}$ is continuous with respect to *weak**-

topology, we have

$$\begin{aligned}\lim_\alpha \lim_\beta \langle \widetilde{D}(a_\alpha \otimes b_\beta), x_0\rangle &= \lim_\alpha \lim_\beta \langle \widetilde{D}(a_\alpha \otimes b_\beta), (i\otimes j).y_0\rangle \\ &= \lim_\alpha \lim_\beta \langle \widetilde{D}((a_\alpha \otimes b_\beta).(i\otimes j)) - (a_\alpha \otimes b_\beta).\widetilde{D}(i\otimes j), y_0\rangle \\ &= \langle \widetilde{D}((a\otimes b).(i\otimes j)) - (a\otimes b).\widetilde{D}(i\otimes j), y_0\rangle \\ &= \langle \widetilde{D}(a\otimes b).(i\otimes j), y_0\rangle = \langle \widetilde{D}(a\otimes b), x_0\rangle,\end{aligned}$$

because D is *weak**-continuous and E is a normal $\mathcal{A}\widehat{\otimes}\mathcal{B}$-bimodule by definition. Therefore $\widetilde{D}$ is *weak**-continuous. From $\varphi\otimes\psi$-Connes amenability of $\mathcal{A}\widehat{\otimes}\mathcal{B}$ we conclude that $\widetilde{D}$ is inner, and so D is inner. □

3. Connes amenability of Lau product $\mathcal{A}\times_\Theta \mathcal{B}$

Let $\mathcal{A}$ be a unital dual Banach algebra with predual $\mathcal{A}_*$ and let $\mathcal{B}$ be a dual Banach algebra with predual $\mathcal{B}_*$. Let $\theta \in \mathcal{B}_* \cap \triangle(\mathcal{B})$ and consider algebra homomorphism $\Theta : \mathcal{B} \to \mathcal{A}$ with $\Theta(b) = \theta(b)e_{\mathcal{A}}$. The Θ-Lau product $\mathcal{A}\times_\Theta \mathcal{B}$ is defined with

$$(a_1, b_1).(a_2, b_2) = (a_1.a_2 + a_1.\Theta(b_2) + \Theta(b_1).a_2, b_1 b_2)$$

and the norm $\|(a_1, b_1)\| = \|a_1\| + \|b_1\|$ for all $a_1, a_2 \in \mathcal{A}$ and $b_1, b_2 \in \mathcal{B}$. This definition is a certain case of product presented in [12], [14], [13] and [1]. Since $\theta \in \mathcal{B}_* \cap \triangle(\mathcal{B})$, then $\mathcal{A}\times_\Theta \mathcal{B}$ is a dual Banach algebra with predual $\mathcal{A}_* \times \mathcal{B}_*$. It is known that $(\mathcal{A}\times_\Theta \mathcal{B})^*$ can be identified with $\mathcal{A}^* \times \mathcal{B}^*$ that $\langle (f, g), (a, b)\rangle = f(a) + g(b)$ for all $a \in \mathcal{A}, b \in \mathcal{B}$ and $f \in \mathcal{A}^*, g \in \mathcal{B}^*$ and the second dual of $(\mathcal{A}\times_\Theta \mathcal{B})^{**}$ can be identified with $\mathcal{A}^{**} \times_{\Theta^{**}} \mathcal{B}^{**}$. Consider $\mathcal{A}$ with $\mathcal{A}\times\{0\}$, then $\mathcal{A}$ is a closed ideal in $\mathcal{A}\times_\Theta \mathcal{B}$ and $(\mathcal{A}\times_\Theta \mathcal{B})/\mathcal{A}$ is isometric isomorphism with $\mathcal{B}$ [14]. In the following we study Connes amenability of $\mathcal{A}\times_\Theta \mathcal{B}$.

Theorem 3.1. *Let $\mathcal{A}$ be a unital dual Banach algebra with predual $\mathcal{A}_*$ and let $\mathcal{B}$ be a dual Banach algebra with predual $\mathcal{B}_*$. Then $\mathcal{A}\times_\Theta \mathcal{B}$ is Connes amenable if and only if $\mathcal{A}$ and $\mathcal{B}$ are Connes amenable.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be Connes amenable dual Banach algebras, let E be a normal $\mathcal{A}\times_\Theta \mathcal{B}$-bimodule and let $D : \mathcal{A}\times_\Theta \mathcal{B} \to E$ be a bounded *weak**-continuous derivation. Consider $D' = D : \mathcal{A}\times_\Theta \{0\} \to E$. Since $\mathcal{A}$ is Connes amenable, then there exists $\phi \in E$ such that $D' = ad_\phi$ and $\tilde{D} = D' - ad_\phi$ vanishes on $\mathcal{A}$. So $\tilde{D}$ has a continuous extension $D_1 : \frac{\mathcal{A}\times_\Theta\mathcal{B}}{\mathcal{A}} \to E$. Recall that $\frac{\mathcal{A}\times_\Theta\mathcal{B}}{\mathcal{A}} \simeq \mathcal{B}$. Define

$$F = \{(a, 0).\ x - x.\ (a, 0) : a \in \mathcal{A},\ x \in E\}.$$

Note that F is a normal $\{0\}\times_\Theta \mathcal{B}$-bimodule and $D_1(\mathcal{B}) \perp F$. In fact,

$$\begin{aligned}D_1((a, 0)\times_\Theta (0, b)) &= D_1(a, 0).(0, b) + (a, 0).D_1(0, b) = (a, 0).D_1(0, b)\\ D_1((0, b)\times_\Theta (a, 0)) &= D_1(0, b).(a, 0) + (0, b).D_1(a, 0) = D_1(0, b).(a, 0).\end{aligned}$$

Then

$$\langle D_1(0,b),(a,0).\ x - x.\ (a,0)\rangle = \langle D_1(0,b).(a,0) - (a,0).D_1(0,b), x\rangle.$$

It is easy to see that $F^{\perp}$ is normal dual Banach $\mathcal{B}$-bimodule with predual $\frac{E_*}{F}$. Since $\mathcal{B}$ is Connes amenable, there exists $\mu \in F^{\perp}$ such that $D_1(0,b) = ad_{\mu}$. We conclude that $\tilde{D}$ is inner and so D is inner.

Let $\mathcal{A} \times_{\Theta} \mathcal{B}$ be Connes amenable. We show that $\mathcal{A}$ is Connes amenable. Let E be a normal dual Banach $\mathcal{A}$-bimodule and let $D : \mathcal{A} \to E$ be a bounded *weak**-continuous derivation. By [6, Lemma 3.3], there is no loss of generality if we suppose that the predual of E is pseudo-unital. Clearly, E_* is a $\mathcal{A} \times_{\Theta} \mathcal{B}$-bimodule. Pick $e_* \in E_*$. Since E_* is pseudo-unital, there exists $(c,0) \in \mathcal{A} \times_{\Theta} \{0\}$ and $g_* \in E_*$ such that $e_* = (c,0).g_*$. Define $(a,b).e_* = (a,b).(c,0).g_*$ for all $(a,b) \in \mathcal{A} \times_{\Theta} \mathcal{B}$. We claim that E is a normal $\mathcal{A} \times_{\Theta} \mathcal{B}$-bimodule. Indeed, let $\{(a_\alpha, b_\alpha)\}$ be a net in $\mathcal{A} \times_{\Theta} \mathcal{B}$ such that $(a_\alpha, b_\alpha) \to (a,b)$ in the *weak**-topology of $\mathcal{A} \times_{\Theta} \mathcal{B}$ and let $e \in E$. It follows that

$$\begin{aligned}
\lim_{\alpha}\langle e.(a_\alpha, b_\alpha), e_*\rangle &= \lim_{\alpha}\langle e.(a_\alpha, b_\alpha), (c,0).g_*\rangle = \lim_{\alpha}\langle e.(a_\alpha, b_\alpha)(c,0), g_*\rangle \\
&= \lim_{\alpha}\langle e.(a_\alpha c + a_\alpha.\Theta(0) + \Theta(b_\alpha).c, b_\alpha.0), g_*\rangle \\
&= \lim_{\alpha}\langle e.(a_\alpha c + \theta(b_\alpha)e_{\mathcal{A}}.c, 0), g_*\rangle \\
&= \langle e.(ac + \theta(b)e_{\mathcal{A}}.c, 0), g_*\rangle \\
&= \langle e.(a,b)(c,0), g_*\rangle \\
&= \langle e.(a,b), (c,0).g_*\rangle \\
&= \langle e.(a,b), e_*\rangle.
\end{aligned}$$

To extend D, let

$$\tilde{D} : \mathcal{A} \times_{\Theta} \mathcal{B} \to E, \quad (a,b) \mapsto D((a,b)(e_{\mathcal{A}}, 0)) - (a,b).\ D(e_{\mathcal{A}}, 0).$$

It is clear that $\tilde{D}$ is a continuous derivation. To see that $\tilde{D}$ is *weak**-continuous, again let $\{(a_\alpha, b_\alpha)\}$ be a net in $\mathcal{A} \times_{\Theta} \mathcal{B}$ such that $(a_\alpha, b_\alpha) \to (a,b)$ in the *weak**-topology of $\mathcal{A} \times_{\Theta} \mathcal{B}$. For $e_* \in E_*$, let $(c,0) \in \mathcal{A} \times_{\Theta} \{0\}$ and $g_* \in E_*$ be such that $e_* = (c,0).g_*$. Then

$$\begin{aligned}
\lim_{\alpha}\langle \tilde{D}(a_\alpha, b_\alpha), e_*\rangle &= \lim_{\alpha}\langle \tilde{D}(a_\alpha, b_\alpha), (c,0).g_*\rangle \\
&= \lim_{\alpha}\langle \tilde{D}(a_\alpha, b_\alpha).(c,0), g_*\rangle \\
&= \lim_{\alpha}\langle \tilde{D}((a_\alpha, b_\alpha).(c,0)) - (a_\alpha, b_\alpha).\tilde{D}(c,0), g_*\rangle \\
&= \lim_{\alpha}\langle D(a_\alpha c + a_\alpha.\Theta(0) + \Theta(b_\alpha).c, 0) - (a_\alpha, b_\alpha).D(c,0), g_*\rangle \\
&= \langle D(ac + a.\Theta(0) + \Theta(b).c, 0) - (a,b).D(c,0), g_*\rangle \\
&= \langle D((a,b).(c,0)) - (a,b).D(c,0), g_*\rangle \\
&= \langle \tilde{D}((a,b).(c,0)) - (a,b).\tilde{D}(c,0), g_*\rangle \\
&= \langle \tilde{D}(a,b).(c,0), g_*\rangle \\
&= \langle \tilde{D}(a,b), (c,0).g_*\rangle \\
&= \langle \tilde{D}(a,b), e_*\rangle,
\end{aligned}$$

because D is *weak**-continuous, E is a $\mathcal{A} \times_\Theta \{0\}$-bimodule and $\mathcal{A}$ is normal dual Banach $\mathcal{A}$-bimodule. From the Connes amenability of $\mathcal{A} \times_\Theta \mathcal{B}$, we conclude that $\tilde{D}$ and hence D is inner.
Let $\mathcal{A} \times_\Theta \mathcal{B}$ be Connes amenable. We show $\mathcal{B}$ is Connes amenable. By [19], there exists $M = (\alpha,\beta) \otimes (\alpha,\beta) \in \sigma wc(((\mathcal{A} \times_\Theta \mathcal{B})\hat{\otimes}(\mathcal{A} \times_\Theta \mathcal{B}))^*)^*$ such that $M.(a,b) = (a,b).M$ and $\Delta_{\sigma wc}(M).(a,b) = (a,b)$ for all $(a,b) \in \mathcal{A} \times_\Theta \mathcal{B}$. Put $M' = (0,\beta) \otimes (0,\beta)$. It is easy to see that $M' \in \sigma wc((\mathcal{B}\hat{\otimes}\mathcal{B})^*)^*$ is a σwc- virtual diagonal for $\mathcal{B}$. □

In [14], it is shown that

$$\Delta(\mathcal{A} \times_\Theta \mathcal{B}) = \{(\psi,\theta) : \psi \in \Delta(\mathcal{A})\} \cup \{(0,\varphi) : \varphi \in \Delta(\mathcal{B})\}.$$

With the help of it we can prove the next result.

Theorem 3.2. *Let $\mathcal{A}$ be a unital dual Banach algebra with predual $\mathcal{A}_*$ and let $\mathcal{B}$ be a dual Banach algebra with predual $\mathcal{B}_*$. Let $\psi \in \Delta(\mathcal{A}) \cap \mathcal{A}_*$ and $\theta \in \Delta(\mathcal{B}) \cap \mathcal{B}_*$. Then*

(i) *$\mathcal{A} \times_\Theta \mathcal{B}$ is (ψ,θ)-Connes amenable if and only if $\mathcal{A}$ is ψ-Connes amenable.*

(ii) *$\mathcal{A} \times_\Theta \mathcal{B}$ is $(0,\theta)$-Connes amenable if and only if $\mathcal{B}$ is θ-Connes amenable.*

Proof. (i) Suppose that m is a ψ-invariant mean on $\mathcal{A}_*$ and define $n \in \mathcal{A}^{**} \times_{\Theta^{**}} \mathcal{B}^{**}$ by $n = (m,0)$. Then $\langle (m,0),(\psi,\theta)\rangle = \langle m,\psi\rangle + \langle 0,\theta\rangle = 1$. On the other hand, choose a net $\{m_\alpha\}$ in $\mathcal{A}$ with the property that $m_\alpha \to m$ in the *weak**-topology. We have

$$\begin{aligned}
\langle (m,0).(a,b),(f,g)\rangle &= \lim_\alpha \langle (m_\alpha,0).(a,b),(f,g)\rangle \\
&= \lim_\alpha \langle (m_\alpha.\, a + m_\alpha.\, \Theta(b) + \Theta(0).\, a, 0.\, b),(f,g)\rangle \\
&= \lim_\alpha \langle (\psi(a)\, m_\alpha + \theta(b)\, m_\alpha.\, e_{\mathcal{A}}, 0),(f,g)\rangle \\
&= \lim_\alpha (\psi(a) + \theta(b))\langle (m_\alpha,0),(f,g)\rangle \\
&= (\psi,\theta)(a,b)\langle (m,0),(f,g)\rangle
\end{aligned}$$

for all $(f,g) \in \mathcal{A}_* \times \mathcal{B}_*$. So $(m,0)$ is a (ψ,θ)-invariant mean.

Conversely, let E be a normal $\mathcal{A}$-bimodule such that $a.e = \psi(a)e (a \in \mathcal{A},\ e \in E)$ and let $D : \mathcal{A} \to E$ be a bounded *weak**-continuous derivation. Suppose that the predual of E is pseudo-unital. As in the proof of Theorem 3.1, it is easy to see that E is a normal $\mathcal{A} \times_\Theta \mathcal{B}$-bimodule. It is known that there exist $g \in E$ and $c \in \mathcal{A}$ such that $e = (c,0).g$. Then

$$\begin{aligned}
(a,b).e &= (a,b).(c,0).g = (ac + \theta(b).c.e_{\mathcal{A}}, 0).g \\
&= (\psi(ac) + \theta(b)\psi(c))g = ((\psi(a) + \theta(b))\psi(c)g \\
&= (\psi(a) + \theta(b))(c,0).g = (\psi,\theta)(a,b).e.
\end{aligned}$$

Now set

$$\tilde{D} : \mathcal{A} \times_\Theta \mathcal{B} \to E,\ (a,b) \mapsto D((a,b)(e_{\mathcal{A}},0)) - (a,b).D(e_{\mathcal{A}},0).$$

As in the proof of Theorem 3.1, we obtain $\tilde{D}$ is inner.

(ii) Suppose that (m_1, m_2) is a $(0,\theta)$-invariant mean on $\mathcal{A}_* \times_\Theta \mathcal{B}_*$. Then

$$\langle (m_1, m_2).(a,b), (f,g)\rangle = \theta(b)\langle (m_1, m_2), (f,g)\rangle, \quad \langle (m_1, m_2), (0,\theta)\rangle = 1$$

for all $f \in \mathcal{A}_*, g \in \mathcal{B}_*$. It is easy to see that

$$\langle (m_1.a + m_1.\theta(b) + \theta(m_1).a, m_2.b), (f,g)\rangle = \langle (\theta(b)m_1, \theta(b)m_2), (f,g)\rangle.$$

So $\langle m_2.b, g\rangle = \theta(b)\langle m_2, g\rangle$ and $\langle m_2, \theta\rangle = 1$. Conversely, let $\mathcal{B}$ be θ-Connes amenable and m be θ-invariant mean on $\mathcal{B}_*$. We claim that $(-e_{\mathcal{A}}, m)$ is a $(0,\theta)$-invariant mean on $\mathcal{A}_* \times \mathcal{B}_*$. Let $\{m_\alpha\}$ be a net in $\mathcal{A}$ that $m_\alpha \to m$ in the *weak**-topology. So

$$\begin{aligned}
\langle (-e_{\mathcal{A}}, m).(a,b), (f,g)\rangle &= \lim_\alpha \langle (-e_{\mathcal{A}}, m_\alpha).(a,b), (f,g)\rangle \\
&= \lim_\alpha \langle (-e_{\mathcal{A}}.a - e_{\mathcal{A}}.\Theta(b) + \Theta(m_\alpha).a, m_\alpha.b), (f,g)\rangle \\
&= \lim_\alpha \langle (-e_{\mathcal{A}}.a - e_{\mathcal{A}}.\theta(b).e_{\mathcal{A}} + \theta(m_\alpha).e_{\mathcal{A}}.a, m_\alpha.b), (f,g)\rangle \\
&= \langle (-e_{\mathcal{A}}.a - e_{\mathcal{A}}.\theta(b).e_{\mathcal{A}} + \theta(m).e_{\mathcal{A}}.a, m.b), (f,g)\rangle \\
&= \langle (-e_{\mathcal{A}}\theta(b), \theta(b)m), (f,g)\rangle \\
&= (0,\theta)(a,b)\langle (-e_{\mathcal{A}}, m), (f,g)\rangle
\end{aligned}$$

for all $(f,g) \in \mathcal{A}_* \times \mathcal{B}_*$ and $(a,b) \in \mathcal{A} \times_\Theta \mathcal{B}$.

□

4. Connes amenability of module extension Banach algebras

In this section we study Connes amenability and character Connes amenability of module extension of a dual Banach algebra $\mathcal{A}$. Let $\mathcal{A}$ be a dual Banach algebra with predual $\mathcal{A}_*$ and let $\mathcal{X}$ be a normal Banach $\mathcal{A}$-bimodule with predual $\mathcal{X}_*$. Module extension Banach algebra $\mathcal{A} \oplus \mathcal{X}$ with the algebra product is defined as follows:

$$(a,x)(a',x') = (aa', ax' + xa') \quad (a,a' \in \mathcal{A}, x,x' \in \mathcal{X})$$

and the norm $\|(a,x)\| = \|a\| + \|x\|$. It is known that $\mathcal{A} \oplus \mathcal{X}$ is a dual Banach algebra with predual $\mathcal{A}_* \oplus_\infty \mathcal{X}_*$, where $\oplus_\infty$ denotes l_∞-direct sum of Banach $\mathcal{A}$-modules. The dual and the second dual of $\mathcal{A} \oplus \mathcal{X}$ is identified with $(\mathcal{A} \oplus \mathcal{X})^* = \mathcal{A}^* \oplus_\infty \mathcal{X}^*$ and $(\mathcal{A} \oplus \mathcal{X})^{**} = \mathcal{A}^{**} \oplus \mathcal{X}^{**}$ respectively. For more details see [21].

We write $B(\mathcal{A}, \mathcal{A}^*)$ for Banach space of bounded linear maps from $\mathcal{A}$ to $\mathcal{A}^*$. It is known that $B(\mathcal{A}, \mathcal{A}^*)$ is a Banach $\mathcal{A}$-bimodule with module multiplications determined by

$$a.T(b) = T(ba), \;\; T.a(b) = T(ab) \quad (a,b \in \mathcal{A}, T \in B(\mathcal{A}, \mathcal{A}^*))$$

It is standard that $B(\mathcal{A}, \mathcal{A}^*) = (\mathcal{A}\hat{\otimes}\mathcal{A})^*$ where $\langle T, x \otimes y\rangle = \langle T(y), x\rangle$. It is known that a Banach algebra $\mathcal{A}$ with bounded approximate identity is amenable if and only if the Banach $\mathcal{A}$-bimodule $\mathcal{A}^*$ is injective [8]. This is equivalent to the following short exact sequence splits:

$$0 \longrightarrow \mathcal{A}^* \overset{\longrightarrow}{\dashleftarrow} B(\mathcal{A}, \mathcal{A}^*) \longrightarrow B(\mathcal{A}, \mathcal{A}^*)/\Delta^*_{\mathcal{A}}(\mathcal{A}^*) \longrightarrow 0$$

In other words there exists a bounded bimodule homomorphism P from $B(\mathcal{A}, \mathcal{A}^*)$ to $\mathcal{A}_*$ such that $Po\Delta^*_{\mathcal{A}} = I_{\mathcal{A}^*}$. In [3], it is proved that unital dual Banach algebra $\mathcal{A}$ with predual $\mathcal{A}_*$ is Connes amenable if and only if the following short exact sequence splits:

$$0 \longrightarrow \mathcal{A}_* \overset{\longrightarrow}{\leftarrow\!-\!-} \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*) \longrightarrow \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)/\Delta^*_{\mathcal{A}}(\mathcal{A}_*) \longrightarrow 0$$

Theorem 4.1. *Let $\mathcal{A}$ be a dual Banach algebra with predual $\mathcal{A}_*$ and let $\mathcal{X}$ be a normal Banach $\mathcal{A}$-bimodule with predual $\mathcal{X}_*$ that its predual is pseudo-unital. Then $\mathcal{A} \oplus \mathcal{X}$ is Connes amenable if and only if $\mathcal{A}$ is Connes amenable.*

Proof. It is easy to see that if $\mathcal{A} \oplus \mathcal{X}$ is Connes amenable, then $\mathcal{A}$ is Connes amenable.

Conversely, let $\mathcal{A}$ be Connes amenable. By Theorem 4.8 in [19], there exist $M \in \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ such that $a.M = M.a$ and $a.\Delta_{\sigma wc}M = a$ for all $a \in \mathcal{A}$. Let $\{m_\alpha \otimes n_\alpha\}$ be a bounded net in $\mathcal{A} \otimes \mathcal{A}$ such that $(m_\alpha \otimes n_\alpha) \to M$ in the *weak**-topology. Write $(m_\alpha \otimes n_\alpha) = (m_\alpha, 0) \otimes (n_\alpha, 0)$ and consider $(m_\alpha \otimes n_\alpha)$ as an element of $(\mathcal{A} \oplus \mathcal{X}) \otimes (\mathcal{A} \oplus \mathcal{X})$. Then we have $(a, 0).\Delta_{\sigma wc}((m_\alpha, 0) \otimes (n_\alpha, 0)) = (a, 0)$. Consider the following admissible short exact sequence of $(\mathcal{A} \oplus \mathcal{X})$-bimodules:

$$\begin{gathered} 0 \longrightarrow \mathcal{A}_* \oplus \mathcal{X}_* \overset{\longrightarrow}{\leftarrow\!-\!-} \sigma wc(((\mathcal{A} \oplus \mathcal{X})\hat{\otimes}(\mathcal{A} \oplus \mathcal{X}))^*) \to \\ \sigma wc(((\mathcal{A} \oplus \mathcal{X})\hat{\otimes}(\mathcal{A} \oplus \mathcal{X}))^*)/\Delta^*_{(\mathcal{A}\oplus\mathcal{X})}(\mathcal{A}_* \oplus \mathcal{X}_*) \to 0 \end{gathered}$$

Define $P : \sigma wc(B(\mathcal{A} \oplus \mathcal{X}, (\mathcal{A} \oplus \mathcal{X})^*)) \to \mathcal{A}_* \oplus \mathcal{X}_*$, $\langle P(T), (a, x)\rangle = \lim_\alpha \langle (m_\alpha, 0) \otimes (n_\alpha, 0), (a, x).T\rangle$. Let (a_α, x_α) be a bounded net in $\mathcal{A} \oplus \mathcal{X}$ which tends to (a, x) in the *weak**- topology of $\mathcal{A} \oplus \mathcal{X}$. Since $T \in \sigma wc(B(\mathcal{A} \oplus \mathcal{X}, (\mathcal{A} \oplus \mathcal{X})^*))$, then $(a_\alpha, x_\alpha).T \to (a, x).T$ in the weak topology. So $(a_\alpha, x_\alpha).T \to (a, x).T$ in the *weak**-topology. We conclude that P maps $\sigma wc(B(\mathcal{A} \oplus \mathcal{X}, (\mathcal{A} \oplus \mathcal{X})^*))$ into $\mathcal{A}_* \oplus \mathcal{X}_*$. Since $\mathcal{X}_*$ is pseudo-unital, then

$$\begin{aligned} \langle Po\Delta^*_{(\mathcal{A}\oplus\mathcal{X})}(a_*, x_*), (a, x)\rangle &= \lim_\alpha \langle (m_\alpha, 0) \otimes (n_\alpha, 0), (a, x).\Delta^*_{(\mathcal{A}\oplus\mathcal{X})}(a_*, x_*)\rangle \\ &= \lim_\alpha \langle (m_\alpha, 0) \otimes (n_\alpha, 0), \Delta^*_{(\mathcal{A}\oplus\mathcal{X})}((a, x).(a_*, x_*))\rangle \\ &= \lim_\alpha \langle \Delta^{**}_{(\mathcal{A}\oplus\mathcal{X})}((m_\alpha, 0) \otimes (n_\alpha, 0)), (a, x)(a_*, x_*)\rangle \\ &= \lim_\alpha \langle \Delta^{**}_{(\mathcal{A}\oplus\mathcal{X})}((m_\alpha, 0) \otimes (n_\alpha, 0)).(a, x), (a_*, x_*)\rangle \\ &= \langle (a, x), (a_*, x_*)\rangle. \end{aligned}$$

Also a copy of the argument given in the proof of Proposition 4.4 in [3] shows that P is a $\mathcal{A} \oplus \mathcal{X}$-module homomorphism. This implies that the short exact sequence splits. □

In [5], the authors showed that $\Delta(\mathcal{A} \oplus \mathcal{X}) = \Delta(\mathcal{A}) \times \{0\}$. They proved that if $\mathcal{A} \oplus \mathcal{X}$ is $(\psi, 0)$-amenable then $\mathcal{A}$ is ψ-amenable and the converse also holds in the case where $\mathcal{X}A = 0$. It would be interesting to know whether the result extends to $(\psi, 0)$-Connes amenability. The following theorem is an analog of Theorem 1.1 in [10].

Theorem 4.2. *Let $\mathcal{A}$ be a dual Banach algebra with predual $\mathcal{A}_*$ and $\psi \in \Delta(\mathcal{A}) \cap \mathcal{A}_*$. Let $\mathcal{X}$ be a normal Banach $\mathcal{A}$-bimodule with predual $\mathcal{X}_*$. Then:*

(i) *If $\mathcal{A} \oplus \mathcal{X}$ is $(\psi, 0)$-Connes amenable, then $\mathcal{A}$ is ψ-Connes amenable.*

(ii) *If* $\mathcal{X}\mathcal{A} = 0$ *and* $\mathcal{A}$ *is* ψ*-Connes amenable, then* $\mathcal{A} \oplus \mathcal{X}$ *is* $(\psi, 0)$*-Connes amenable.*

Proof. (i) Let $m = (P, Q) \in (\mathcal{A} \oplus \mathcal{X})^{**}$ be a $(\psi, 0)$-invariant mean on $(\mathcal{A}_* \oplus \mathcal{X}_*)$. Let $P : \mathcal{A} \oplus \mathcal{X} \to \mathcal{A}$ be the projection map. It is known that $P^*(a.f) = a.P^*(f)$ for all $f \in \mathcal{A}_*, a \in \mathcal{A}$. Then

$$\begin{aligned} \langle P^{**}(P, Q), a.\, f\rangle &= \langle (P, Q), P^*(a.\, f)\rangle = \langle (P, Q), (a, 0).\, P^*(f)\rangle \\ &= (\psi, 0)(a, 0)\langle (P, Q), P^*(f)\rangle = (\psi, 0)(a, 0)\langle P^{**}(P, Q), f\rangle \end{aligned}$$

for all $f \in \mathcal{A}_*, a \in \mathcal{A}$. On the other hand

$$\langle P^{**}(P, Q), \psi\rangle = \langle (P, Q), P^*(\psi)\rangle = \langle (P, Q), (\psi, 0)\rangle = 1$$

Consequently $P^{**}(P, Q)$ is a $(\psi, 0)$-invariant mean on $\mathcal{A}_*$.

(ii) Suppose there exists $m \in \mathcal{A}^{**}$ such that $\langle m, \psi\rangle = 1$ and $\langle m, a.f\rangle = \psi(a)\langle m, f\rangle$ for all $f \in \mathcal{A}_*$ and $a \in \mathcal{A}$. Let E_* be a $\mathcal{A}$-bimodule with $e.(a, x) = (\psi, 0)(a, x)e$ for all $a \in \mathcal{A}, x \in \mathcal{X}, e \in E_*$ and let $D : \mathcal{A} \oplus \mathcal{X} \to E$ be a bounded *weak**-continuous derivation. Let $D^* : E^* \to (\mathcal{A} \oplus \mathcal{X})^*$ denotes the adjoint of D. Then D^* maps the predual E_* of E^* into $\mathcal{A}_* \oplus \mathcal{X}_*$ (see section 3.14 in [15]). Let $\pi : (\mathcal{A} \oplus \mathcal{X})^{**} \to (\mathcal{A} \oplus \mathcal{X})$ be the Dixmier projection. For all $(a, x), (b, x') \in \mathcal{A} \oplus \mathcal{X}$ and $e \in E_*$,

$$\begin{aligned} \langle D^*(e.(a, x)), (b, x')\rangle &= \langle e.(a, x), D(b, x')\rangle \\ &= (\psi, 0)(a, x)\langle e, D(b, x')\rangle \\ &= (\psi, 0)(a, x)\langle D^*(e), (b, x')\rangle. \end{aligned}$$

Then $D^*(e.(a, x)) = (\psi, 0)(a, x)D^*(e)$. Now put $h = Do\pi(m, 0)$. Then for all $(a, x) \in \mathcal{A} \oplus \mathcal{X}$ and $e \in E_*$,

$$\begin{aligned} \langle e, (a, x).h\rangle &= \langle e, (a, x).Do\pi(m, 0)\rangle \\ &= \langle D^*(e.(a, x)), \pi(m, 0)\rangle = (\psi, 0)(a, x)\langle D^*(e), \pi(m, 0)\rangle \\ &= (\psi, 0)(a, x)\langle e, Do\pi(m, 0)\rangle = (\psi, 0)(a, x)\langle e, h\rangle \end{aligned}$$

and hence $(a, x).h = (\psi, 0)(a, x)h$. Since D is a derivation, for the left action of $\mathcal{A} \oplus \mathcal{X}$ on E we get

$$\begin{aligned} \langle D^*((a, x).e), (b, x')\rangle &= \langle (a, x).e, D(b, x')\rangle \\ &= \langle e, D(b, x').(a, x)\rangle \\ &= \langle e, D((b, x').(a, x))\rangle - \langle e, (b, x').D(a, x)\rangle \\ &= \langle D^*(e), (b, x')(a, x)\rangle - \langle D(a, x), e.(b, x')\rangle \\ &= \langle (a, x)D^*(e), (b, x')\rangle - (\psi, 0)(b, x')\langle e, D(a, x)\rangle \end{aligned}$$

for all $(b, x') \in \mathcal{A} \oplus \mathcal{X}$. This implies that

$$D^*((a, x).e) = (a, x)D^*(e) - (\psi, 0)\langle e, D(a, x)\rangle.$$

It follows that

$$\begin{aligned}\langle e, h.(a.x)\rangle &= \langle e, Do\pi(m,0).(a,x)\rangle\\ &= \langle (a,x).e, Do\pi(m,0)\rangle\\ &= \langle D^*((a,x).e), \pi(m,0)\rangle\\ &= \langle (a,x)D^*(e) - (\psi,0)\langle e, D(a,x)\rangle, \pi(m,0)\rangle\\ &= \langle \pi(m,0).(a,x), D^*(e)\rangle - \langle \pi(m,0), (\psi,0)\rangle\langle e, D(a,x)\rangle\\ &= (\psi,0)(a,x)\langle \pi(m,0), D^*(e)\rangle - \langle e, D(a,x)\rangle\\ &= (\psi,0)(a,x)\langle Do\pi(m), e\rangle - \langle e, D(a,x)\rangle,\end{aligned}$$

because $\mathcal{X}A = 0$. This shows that $D(a,x) = (\psi,0)(a,x).h - h.(a,x) = (a,x).h - h.(a,x)$. Thus D is inner.

□

Acknowledgments

The authors would like to thank the referee for his/her careful reading of the paper and for many valuable suggestions.

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