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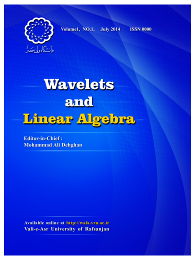

Two-wavelet constants for square integrable representations of G/H

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Abstract

In this paper we introduce two-wavelet constants for square integrable representations of homogeneous spaces. We establish the orthogonality relations for square integrable representations of homogeneous spaces which give rise to the existence of a unique self adjoint positive operator on the set of admissible wavelets. Finally, we show that this operator is a constant multiple of identity operator when G is a semidirect product group of a unimodular subgroup K and a closed subgroup H.

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1. Introduction

The purpose of this paper is to explore and develop certain results concerning two-wavelet constants in the setting of homogeneous spaces. We emphasize that in [8] the orthogonality relations for square integrable representations of locally compact groups has been proved (see also [9, 1, 15]). Our approach leads to extend orthogonality relations for square integrable representations of homogeneous spaces. In several cases the group representations are not square integrable because the group is too large and it is necessary to make the group smaller which can be performed by factoring out a suitable subgroup, that is one has to work with homogenous spaces. In [2] continuous wavelet transform and square integrable representations have been studied in the setting of homogeneous spaces. In this note we define two-wavelet constant for a square integrable representation of a homogeneous space. Also we obtain the orthogonality relations for square integrable representations of homogeneous spaces. First we need to introduce the basic definitions and fix some notations.

Let G be a locally compact group and H be a closed subgroup of G. Consider G/H as a homogeneous space on which G acts from the left and μ as a Radon measure on it. For $g \in G$ and Borel subset E of G/H, the translation μ_g of μ is defined by $\mu_g(E) = \mu(gE)$. A measure μ is said to be G-invariant if $\mu_g = \mu$, for all $g \in G$. A measure μ is called strongly quasi invariant provided that a continuous function $\lambda : G \times G/H \to (0, \infty)$ exists which satisfies

$$d\mu_g(kH) = \lambda(g, kH) d\mu(kH),$$

for all $g, k \in G$. If the functions $\lambda(g, .)$ reduce to constants, then μ is called relatively invariant under G (for a detailed account of homogeneous spaces, the reader is referred to [5, 6, 16, 3]). A rho-function for the pair (G, H) is defined to be a continuous function $\rho : G \to (0, \infty)$ which satisfies

$$\rho(gh) = \frac{\Delta_H(h)}{\Delta_G(h)} \rho(g) \quad (g \in G, h \in H),$$

where Δ_G, Δ_H are the modular functions on G and H, respectively. It is well known that (see [5]), any pair (G, H) admits a rho-function and for each rho-function ρ there is a strongly quasi invariant measure μ on G/H such that

$$\frac{d\mu_g}{d\mu}(kH) = \frac{\rho(gk)}{\rho(k)} \quad (g, k \in G).$$

We denote $L^1(G/H, \mu)$ simply by $L^1(G/H)$, where μ is a strongly quasi invariant measure on G/H. The mapping $T : L^1(G) \to L^1(G/H)$ defined by

$$Tf(gH) = \int_H \frac{f(gh)}{\rho(gh)} dh, \tag{1.1}$$

is a surjective bounded linear operator with $\|T\| \leq 1$. More precisely, for all $\phi \in L^1(G/H)$ there exists some $f \in L^1(G)$ such that $\phi = Tf$.

Let K and H be two locally compact groups with identity elements e_K and e_H, respectively and $h \mapsto \tau_h$ be a homomorphism of H into the group of automorphisms of K. Then the set $K \times H$ endowed with the operation

$$(k_1, h_1)(k_2, h_2) = (k_1 \tau_{h_1}(k_2), h_1 h_2)$$

is a group in which the identity element is (e_K, e_H) and the inverse of an element (k, h) is

$$(k, h)^{-1} = (\tau_{h^{-1}}(k^{-1}), h^{-1}).$$

This group, which is denoted by $K \times_\tau H$, is called the semidirect product of K and H with respect to τ. Equip $K \times H$ with the product topology. If the mapping $(k, h) \mapsto \tau_h(k)$ from $K \times H$ onto K is continuous, then the semidirect product $K \times_\tau H$ of K and H with respect to τ is a locally compact group [10]. It has been shown in [13] that if G is the semidirect product of two locally compact groups K, H respectively, then any $x \in G$ has a unique decomposition $x = kh$, for $k \in K, h \in H$ and G/H has a relatively invariant Radon measure, which arises from the rho function $\rho : G \to (0, \infty)$ defined by $\rho(x) = \frac{\Delta_H(h)}{\Delta_G(h)}$, where $x \in G, h \in H$, and $x = kh$ for some $k \in K$. Moreover, there exists a left Haar measure dk on K for which

$$\int_G f(x)dx = \int_K \int_H f(kh)\delta(h)dhdk, \quad f \in L^1(G), \tag{1.2}$$

where $\delta : H \to (0, \infty)$ is a homomorphism such that $\delta(h) = \frac{\Delta_G(h)}{\Delta_H(h)}$ [4, 13, 11, 16].

In this paper we study two-wavelet constants and the orthogonality relations for a square integrable representation ϖ of a homogeneous space G/H which results to the existence of a unique self adjoint positive operator on the set of admissible wavelets for ϖ. This paper is organized as follows.

In Section 2, two-wavelet constant is introduced and it is shown that there exists a unique self adjoint positive operator on the set of admissible wavelets which it grantees orthogonality relations. In Section 3, we show that if G is a semidirect product of two locally compact groups H, K and K is unimodular, then the unique self adjoint positive operator on the set of admissible wavelets is a constant multiple of identity operator. Finally we give some examples which support our discussion.

Throughout this paper we assume that G is a locally compact group and H is a compact subgroup of G.

2. Two-Wavelet Constant

An admissible wavelet in the setting of homogeneous space G/H is studied by the authors in [2]. For the readers' convenience we need to recall some basic concepts from [2] concerning the theory of unitary representations and continuous wavelet transforms on homogeneous spaces.

A continuous unitary representation of a homogeneous space G/H is a map ϖ from G/H into the group $U(\mathcal{H})$, all unitary operators on some nonzero Hilbert space $\mathcal{H}$, for which the function $gH \mapsto < \varpi(gH)x, y >$ is continuous , for each $x, y \in \mathcal{H}$ and

$$\varpi(gkH) = \varpi(gH)\varpi(kH), \quad \varpi(g^{-1}H) = \varpi(gH)^*,$$

for each $g, k \in G$. Moreover, a closed subspace M of $\mathcal{H}$ is said to be invariant with respect to ϖ if $\varpi(gH)M \subseteq M$, for all $g \in G$. A continuous unitary representation is called irreducible if the only invariant subspaces of $\mathcal{H}$ are $\{0\}$ and $\mathcal{H}$ (in the sequel we always mean by a representation, a continuous unitary representation). An irreducible representation ϖ of G/H on $\mathcal{H}$ is said to be *square integrable* if there exists a nonzero element $\zeta \in \mathcal{H}$ such that

$$\int_{G/H} \frac{\rho(e)}{\rho(g)} \ |<\zeta, \varpi(gH)\zeta>|^2 d\mu(gH) < \infty, \tag{2.1}$$

where μ is a relatively invariant measure on G/H which arises from a rho function $\rho : G \to (0, \infty)$. If ζ satisfies (2.1), it is called an *admissible vector* for ϖ in Hilbert space $\mathcal{H}$. An admissible vector $\zeta \in \mathcal{H}$ is called *admissible wavelet* if $\|\zeta\| = 1$. In this case, we define the wavelet constant c_ζ as

$$c_\zeta := \int_{G/H} \frac{\rho(e)}{\rho(g)} \ |<\zeta, \varpi(gH)\zeta>|^2 d\mu(gH). \tag{2.2}$$

We call c_ζ the *wavelet constant* associated to the admissible wavelet ζ. It is worthwhile to note that there is a close relation between the representations of homogeneous spaces G/H, where H is a compact subgroup of G, and the representations of G. More precisely if ϖ is a representation of G/H, then it defines a representation π of G in which the subgroup H is considered to be contained in kernel of π. Conversely, any representation π of G which is trivial on H induces a representation ϖ of G/H, by letting $\varpi(gH) = \pi(g)$.
Let ϖ be a representation of G/H on a Hilbert space $\mathcal{H}$ and ζ be an admissible wavelet for ϖ. We define the *continuous wavelet transform associated to the admissible wavelet* ζ as the linear operator $W_\zeta : \mathcal{H} \to C(G/H)$ defined by

$$(W_\zeta x)(gH) = \frac{1}{\sqrt{c_\zeta}} (\frac{\rho(e)}{\rho(g)})^{1/2} < x, \varpi(gH)\zeta >,$$

for all $x \in \mathcal{H}, g \in G$ where c_ζ is the wavelet constant associated to ζ as in (2.2). Note that if ϖ is a square integrable representation of G/H on $\mathcal{H}$ and ζ is an admissible wavelet for ϖ, then W_ζ is a bounded linear operator from $\mathcal{H}$ into $L^2(G/H)$.

Our aim in this section is to obtain the orthogonality relations for ϖ and to show the existence of a self adjoint positive operator on the set of the admissible wavelets which satisfies the orthogonality relations.

Theorem 2.1. *(orthogonality relations) Let ϖ be a square integrable representation of G/H on Hilbert space $\mathcal{H}$ and ζ, ξ be two admissible wavelets for ϖ. Then the following hold.*

(i) For all x ,y in $\mathcal{H}$,

$$\int_{G/H} \frac{\rho(e)}{\rho(g)} < x, \varpi(gH)\zeta >< \varpi(gH)\xi, y > d\mu(gH) = c_{\zeta,\xi} < x, y > \tag{2.3}$$

where

$$c_{\zeta,\xi} = \int_{G/H} \frac{\rho(e)}{\rho(g)} < \zeta, \varpi(gH)\zeta >< \varpi(gH)\xi, \zeta > d\mu(gH). \tag{2.4}$$

We call $c_{\zeta,\xi}$ two-wavelet constant.
(ii) There exists a unique self adjoint positive operator C on $\mathcal{H}$ such that:

(a) The set of all admissible wavelets coincides with the domain of C.
(b) For all x ,y in $\mathcal{H}$

$$\int_{G/H} \frac{\rho(e)}{\rho(g)} < x, \varpi(gH)\zeta >< \varpi(gH)\xi, y > d\mu(gH) =< C\xi, C\zeta >< x, y > .$$

Proof. To prove (i), let ζ be an admissible wavelet. Define the linear operator

$$A_\zeta : M \to L^2(G/H), \;\; A_\zeta(x)(gH) = (\frac{\rho(e)}{\rho(g)})^{1/2} < x, \varpi(gH)\zeta >, \tag{2.5}$$

where

$$M = \{x \in \mathcal{H}, \;\; \int_{G/H} \frac{\rho(e)}{\rho(g)} \;| < x, \varpi(gH)\zeta > |^2 d\mu(gH) < \infty\}.$$

As has been shown in [2, Lemma 2.5], M is a closed invariant subspace of $\mathcal{H}$ under ϖ and by the irreducibility of ϖ we have $M = \mathcal{H}$. Since A_ζ is an intertwining operator for ϖ and ℓ, where ℓ is the left regular representation of G on $L^2(G/H)$ such that $\ell(g)\varphi = (\frac{\rho(e)}{\rho(g)})^{1/2} l_g\varphi, \; l_g\varphi(\acute{g}H) = \varphi(g^{-1}\acute{g}H)$ (see more details in [2]), we get

$$\begin{aligned} < A_\xi^* A_\zeta \varpi(gH)x, y > &= < A_\xi^* \ell(g) A_\zeta x, y > \\ &= < \ell(g) A_\zeta x, A_\xi y > \\ &= < A_\zeta x, \ell(g^{-1}) A_\xi y > \\ &= < A_\zeta x, A_\xi \varpi(g^{-1}H) y > \\ &= < \varpi(gH) A_\xi^* A_\zeta x, y > \end{aligned}$$

So, $A_\xi^* A_\zeta \varpi(gH) = \varpi(gH) A_\xi^* A_\zeta$ for every $g \in G$. By Schur's Lemma for homogeneous spaces [2] there exists $c_{\zeta,\xi}$ such that $A_\xi^* A_\zeta = c_{\zeta,\xi} I$. This means that for every x, y in $\mathcal{H}$,

$$\int_{G/H} \frac{\rho(e)}{\rho(g)} < x, \varpi(gH)\zeta >< \varpi(gH)\xi, y > d\mu(gH) = c_{\zeta,\xi} < x, y >,$$

where

$$c_{\zeta,\xi} = \frac{1}{\|x\|^2} \int_{G/H} \frac{\rho(e)}{\rho(g)} < x, \varpi(gH)\zeta >< \varpi(gH)\xi, x > d\mu(gH). \tag{2.6}$$

For (ii), we denote the set of admissible wavelets by $\mathfrak{A}$. Define

$$q : \mathfrak{A} \times \mathfrak{A} \to \mathbb{C}, \;\; q(\zeta, \xi) = c_{\zeta,\xi}, \tag{2.7}$$

for $\zeta, \xi \in \mathfrak{A}$. It is clear that $q(\zeta, \xi)$ is a positive, symmetric quadratic form with domain $\mathfrak{A}$. To show that q is closed, consider the norm $\|.\|_q$ on $\mathfrak{A}$ defined by,

$$\|\zeta\|_q^2 = \|\zeta\|^2 + q(\zeta, \zeta). \tag{2.8}$$

Let $\{\zeta_k\}$ be a Cauchy sequence in $\mathfrak{A}$ with respect to the norm $\|.\|_q$. Then $\{\zeta_k\}$ converges to ζ in $\mathcal{H}$, as $k \to \infty$. For any $x \in \mathcal{H}$ and $k \in \mathbb{N}$, we define the function ψ_k^x on G/H by

$$\psi_k^x(gH) = \frac{1}{\|x\|}.(\frac{\rho(e)}{\rho(g)})^{1/2} < x, \varpi(gH)\zeta_k > . \tag{2.9}$$

Then by (2.8), (2.9) we get

$$\begin{aligned} \|\psi_j^x - \psi_k^x\|^2_{L^2(G/H)} &= \tfrac{1}{\|x\|^2} \int_{G/H} \tfrac{\rho(e)}{\rho(g)} \ | < x, \varpi(gH)(\zeta_j - \zeta_k) > |^2 d\mu(gH) \\ &= q(\zeta_j - \zeta_k, \zeta_j - \zeta_k) \\ &\leq \|\zeta_j - \zeta_k\|_q^2 \to 0, \end{aligned}$$

as $j, k \to \infty$. Thus $\{\psi_k^x\}_{k=1}^{\infty}$ is a Cauchy sequence in $L^2(G/H)$ for all $x \in \mathcal{H}$ which implies that $\psi_k^x \to \psi$ for some ψ in $L^2(G/H)$ as $k \to \infty$. Also by the Schwartz inequality

$$\begin{aligned} \psi_k^x(gH) &= \tfrac{1}{\|x\|}.(\tfrac{\rho(e)}{\rho(g)})^{1/2} < x, \varpi(gH)\zeta_k > \\ &\to \tfrac{1}{\|x\|}.(\tfrac{\rho(e)}{\rho(g)})^{1/2} < x, \varpi(gH)\zeta >, \end{aligned}$$

as $k \to \infty$. Then

$$\frac{1}{\|x\|}(\frac{\rho(e)}{\rho(g)})^{1/2} < x, \varpi(gH)\zeta >= \psi(gH),$$

and hence

$$\int_{G/H} \frac{\rho(e)}{\rho(g)} \ | < x, \varpi(gH)\zeta > |^2 d\mu(gH) < \infty,$$

for all $x \in \mathcal{H}$. Therefore

$$\int_{G/H} \frac{\rho(e)}{\rho(g)} \ | < \zeta, \varpi(gH)\zeta > |^2 d\mu(gH) < \infty,$$

i.e $\zeta \in \mathfrak{A}$. Furthermore,

$$\begin{aligned} \|\zeta_k - \zeta\|_q^2 &= \|\zeta_k - \zeta\|^2 + q(\zeta_k - \zeta, \zeta_k - \zeta) \\ &= \|\zeta_k - \zeta\|^2 + \tfrac{1}{\|x\|^2} \int_{G/H} \tfrac{\rho(e)}{\rho(g)} \ | < x, \varpi(gH)(\zeta_k - \zeta) > |^2 d\mu(gH) \to 0 \end{aligned}$$

as $k \to \infty$. It implies that $\mathfrak{A}$ is complete and then q is closed. By the second representation theorem [14], there exists a unique self adjoint positive operator C with domain $\mathfrak{A}$ such that $q(\zeta, \xi) = c_{\zeta,\xi} =< C\zeta, C\xi >$ and the proof is complete. □

In the following proposition, we introduce orthogonal subspaces of $L^2(G/H)$.

Proposition 2.2. *Let ϖ_1 and ϖ_2 be two square integrable representations of G/H on Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ with admissible wavelets ζ and ξ, respectively. If ϖ_1 and ϖ_2 are not equivalent, then the images of W_ζ and W_ξ are orthogonal subspaces of $L^2(G/H)$. That is*

$$< W_\zeta x_1, W_\xi x_2 >= 0,$$

for all $x_1 \in \mathcal{H}_1, \ \ x_2 \in \mathcal{H}_2$.

Proof. Define the linear mapping $\beta : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathbb{C}$ as follows,

$$\beta(x_1, x_2) :=< W_\zeta x_1, W_\xi x_2 >_{L^2(G/H)}= \int_{G/H} (W_\zeta x_1)(gH)\overline{(W_\xi x_2)(gH)}d\mu(gH).$$

Since the wavelet transform is continuous, it follows that

$$\begin{aligned} |\beta(x_1, x_2)| &= | < W_\zeta x_1, W_\xi x_2 > | \\ &\leq ||W_\zeta x_1||_{L^2(G/H)}||W_\xi x_2||_{L^2(G/H)} \\ &\leq ||W_\zeta||||W_\xi||||x_1||||x_2||, \end{aligned}$$

By the Little Riesz Representation Theorem (see 12.5, in [17]), we can conclude that, there is a continuous linear map $S : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\beta(x_1, x_2) =< S x_1, x_2 > .$$

Using the fact that continuous wavelet transform is intertwining, we get

$$\begin{aligned} < S\varpi_1(gH)x_1, x_2 > &= \beta(\varpi_1(gH)x_1, x_2) \\ &= \int_{G/H}(W_\zeta\varpi_1(gH)x_1)(kH)\overline{(W_\xi x_2)(kH)}d\mu(kH) \\ &= \int_{G/H}(\ell(g)W_\zeta x_1)(kH)\overline{(W_\xi x_2)(kH)}d\mu(kH) \\ &= \int_{G/H}(\tfrac{\rho(e)}{\rho(g)})^{1/2}(W_\zeta x_1)(g^{-1}kH)\overline{(W_\xi x_2)(kH)}d\mu(kH) \\ &= \int_{G/H}(\tfrac{\rho(e)}{\rho(g)})^{1/2}(W_\zeta x_1)(kH)\overline{(W_\xi x_2)(gkH)}(\tfrac{\rho(g)}{\rho(e)})d\mu(kH) \\ &= \int_{G/H}(\tfrac{\rho(g)}{\rho(e)})^{1/2}(W_\zeta x_1)(kH)\overline{(W_\xi x_2)(gkH)}d\mu(kH) \\ &= \int_{G/H}(W_\zeta x_1)(kH)\overline{(\ell(g^{-1})W_\xi x_2)(kH)}d\mu(kH) \\ &= \int_{G/H}(W_\zeta x_1)(kH)\overline{(W_\xi\varpi_2(g^{-1}H)x_2)(kH)}d\mu(kH) \\ &= \beta(x_1, \varpi_2(g^{-1}H)x_2) \\ &= < S x_1, \varpi_2(gH)^* x_2 > \\ &= < \varpi_2(gH)S x_1, x_2 > \end{aligned}$$

for all $x_1 \in \mathcal{H}_1, x_2 \in \mathcal{H}_2$. It follows that $S\varpi_1(gH) = \varpi_2(gH)S$. So by Schur's Lemma for homogeneous spaces [2], we conclude that S is zero when ϖ_1 is not equivalent to ϖ_2. Thus $< W_\zeta x_1, W_\xi x_2 >= 0$, for all $x_1 \in \mathcal{H}_1, x_2 \in \mathcal{H}_2$. □

3. orthogonality relations for some special homogeneous spaces

In this section G is considered as the semidirect product of a locally compact group K and a compact group H. Let ϖ be a representation of G/H in which $G = K \times_\tau H$. Then it defines a representation of K by letting $\pi(k) = \varpi(gH)$, where $g = kh$. In [13] it is shown that $\rho(g) = \frac{\Delta_H(h)}{\Delta_G(h)} = 1$, where $g \in G, h \in H$ and $g = kh$ for some $k \in K$. Note that in this case the condition for admissibility is reduced to $\int_{G/H} |<\zeta, \varpi(gH)\zeta>|^2 d\mu(gH) < \infty$. Now, we show that the unique self adjoint operator C in Theorem 2.1 is a constant multiple of the identity operator if and only if K is unimodular. To this end we establish the following lemmas. The next lemma shows that the set of all admissible wavelets for ϖ contains in the set of all admissible wavelets for π and the second lemma states that the set of all admissible wavelets is invariant under ϖ.

Lemma 3.1. *Let K, H be two locally compact groups and $G = K \times_\tau H$ and ϖ, π be the representations of G/H and K, respectively on a Hilbert space $\mathcal{H}$. If ζ is an admissible wavelet for ϖ then ζ is an admissible wavelet for π.*

Proof. Let ϖ be an irreducible representation of G/H on $\mathcal{H}$. This representation defines an irreducible representation π of K on $\mathcal{H}$ by $\pi(k) = \varpi(gH)$, where $g = kh$. Using Weil's formula, (1.2), and the fact that T defined as in (1.1) is onto, we have

$$\begin{aligned}
\int_{G/H} |<\zeta, \varpi(gH)\zeta>|^2 d\mu(gH) &= \int_{G/H} T(\varphi)(gH) d\mu(gH) \\
&= \int_G \varphi(g) dg \\
&= \int_K \int_H \varphi(kh) dh dk \\
&= \int_K T(\varphi)(kH) dk \\
&= \int_K |<\zeta, \varpi(kH)\zeta>|^2 dk \\
&= \int_K |<\zeta, \pi(k)\zeta>|^2 dk,
\end{aligned}$$

for some $\varphi \in L^1(G)$. Therefore, if ζ is an admissible wavelet for ϖ then ζ is an admissible wavelet for π. □

Lemma 3.2. *Let K, H be two locally compact groups and $G = K \times_\tau H$ and ϖ, $\mathcal{H}$ be as in Lemma3.1. Then the set of all admissible wavelets is invariant under ϖ.*

Proof. Denote the set of all admissible wavelets for ϖ by $\mathfrak{A}$. That is

$$\mathfrak{A} = \{\zeta \in \mathcal{H}, \int_{G/H} |<\zeta, \varpi(gH)\zeta>|^2 d\mu(gH) < \infty\}.$$

Consider the representation π of K as $\pi(k) = \varpi(gH)$, in which $g = kh$. Let the Haar measure on H be normalized. Since π is trivial on H, we have

$$\begin{aligned}
\frac{1}{\Delta(k_1)} \int_K |<\zeta, \omega(k)\zeta>|^2 dk[1ex] &= \int_K |<\zeta, \omega(k_1^{-1}kk_1)\zeta>|^2 dk \\
&= \int_K |<\zeta, \omega(k_1^{-1})\omega(k)\omega(k_1)\zeta>|^2 dk \\
&= \int_K |<\omega(k_1)\zeta, \omega(k)\omega(k_1)\zeta>|^2 dk \\
&= \int_K \int_H |<\omega(k_1h_1)\zeta, \omega(kh)\omega(k_1h_1)\zeta>|^2 dh dk \\
&= \int_K \int_H |<\omega(g_1)\zeta, \omega(g)\omega(g_1)\zeta>|^2 dh dk \\
&= \int_G \varphi(g) dg \\
&= \int_{G/H} \int_H \varphi(g\eta) d\eta d\mu(gH) \\
&= \int_{G/H} \int_H |<\omega(k_1h_1)\zeta, \omega(kh\eta)\omega(k_1h_1)\zeta>|^2 d\eta d\mu(gH) \\
&= \int_{G/H} |<\varpi(g_1H)\zeta, \varpi(gH)\varpi(g_1H)\zeta>|^2 d\mu(gH),
\end{aligned}$$

for some $\varphi \in L^1(G)$, where $g_1 = k_1 h_1$. Thus the set $\mathfrak{A}$ is invariant under ϖ. □

Now we are ready to prove that the unique self adjoint operator C in Theorem 2.1 is a constant multiple of the identity operator if and only if K is unimodular.

Theorem 3.3. *Let K, H be two locally compact groups and $G = K \times_\tau H$. Then the unique self adjoint positive operator C in Theorem 2.1 is a constant multiple of the identity operator if and only if K is unimodular.*

Proof. In Theorem 2.1, we have shown that there exists a unique self adjoint positive operator C such that $q(\zeta, \xi) = c_{\zeta,\xi} =< C\zeta, C\xi >$, where $c_{\zeta,\xi}$ is as in (2.4), for admissible wavelets ζ, ξ . So we have

$$\begin{aligned} q(\varpi(gH)\zeta, \varpi(gH)\xi) &= q(\pi(k)\zeta, \pi(k)\xi) \\ &= \tfrac{1}{\|\upsilon\|^2} \int_K \overline{< \pi(k_1 k)\xi, \upsilon >} < \pi(k_1 k)\zeta, \upsilon > dk_1 \\ &= \tfrac{\Delta(k^{-1})}{\|\upsilon\|^2} \int_K \overline{< \pi(k_1)\xi, \upsilon >} < \pi(k_1)\zeta, \upsilon > dk_1. \end{aligned}$$

Thus

$$< C\varpi(gH)\zeta, C\varpi(gH)\xi >= \frac{1}{\Delta(k)} q(\zeta, \xi) = \frac{1}{\Delta(k)} < C\zeta, C\xi > . \tag{3.1}$$

Also, domain C^2 is invariant. Indeed, for $\zeta \in D(C^2)$, we have $\zeta \in D(C)$ such that $C\zeta \in D(C)$ and by invariance of $D(C)$ we get $\varpi(gH)\zeta \in D(C)$. Then by (3.1)

$$< C\varpi(gH)\zeta, C\varpi(gH)\xi >= \frac{1}{\Delta(k)} q(\zeta, \xi) = \frac{1}{\Delta(k)} < \zeta, C^2\xi >, \tag{3.2}$$

which implies that $C\varpi(gH)\xi \in D(C)$. That is $\varpi(gH)\xi \in D(C^2)$ and C^2 is densely defined. Also, (3.2) implies that

$$C^2\varpi(gH) = \frac{1}{\Delta(k)} \varpi(gH)C^2.$$

By using Shur's Lemma [2], K is unimodular, i.e. $\Delta(k) = 1$ if and only if $C = \lambda I$. □

Now, we conclude that if ζ is an admissible wavelet for ϖ, then every vector in $\mathcal{H}$ is an admissible wavelet provided that K is unimodular.

Proposition 3.4. *Let $G = K \times_\tau H$ such that K is a unimodular, H is a compact subgroup of G and ϖ is a representation of G/H. If there exists an admissible vector in $\mathcal{H}$ for ϖ, then all vectors of $\mathcal{H}$ are admissible.*

Proof. By Lemma 3.2 and irreducibility ϖ, it is enough to show that $\mathfrak{A}$ is closed. By unimodularity of K we have

$$< C\varpi(gH)\zeta, C\varpi(gH)\xi >=< C\zeta, C\xi >,$$

for $\zeta, \xi \in \mathfrak{A}$. Then $C\varpi(gH) = \varpi(gH)C$. Let $\mathfrak{A}$ be the Hilbert space equipped with the inner product of which the induced norm $\|.\|_q$ defined as in (2.8). By the extended Schur's Lemma [2] there exists a positive number α such that

$$\|C\zeta\|^2 = \alpha\|\zeta\|_q^2 = \alpha\|\zeta\|^2 + \alpha\|C\zeta\|^2, \ \ \zeta \in \mathfrak{A}.$$

It implies that $\alpha < 1$ and

$$\|C\zeta\|^2 = \frac{\alpha}{1-\alpha}\|\zeta\|^2, \quad \zeta \in \mathfrak{A}. \tag{3.3}$$

Using (3.3) and density of $\mathfrak{A}$ in $\mathcal{H}$, we can extend $C : \mathfrak{A} \to \mathcal{H}$ to a bounded linear operator $\tilde{C} : \mathcal{H} \to \mathcal{H}$. Now let $\{\zeta_k\}_{k\in\mathbb{N}}$ is a sequence of elements in $\mathfrak{A}$ such that $\zeta_k \to \zeta$ in $\mathcal{H}$ as $k \to \infty$. Then $C\zeta_k \to \tilde{C}\zeta$ in $L^2(G/H)$ as $k \to \infty$. Since C is a closed linear operator with domain $\mathfrak{A}$, it follows that $\zeta \in \mathfrak{A}$. Therefore $\mathfrak{A}$ is a closed subspace. □

Here we intend to support our technical considerations developed in this note by giving some examples.

Example 3.5. Let N be an even number and G be the group with two generators a and b satisfying $|a| = N, |b| = 3$ and $ba = ab^2$ i.e.

$$G =< a, b; \ \ a^N = b^3 = e, \ \ ba = ab^2 > .$$

Also, let $H =< b >, \ \ K =< a >$ and

$$\varpi : G/H \to U(L^2(G/H)), \ \ \varpi(a^i b^j H)\psi(a^n H) = \psi(a^{n-i}H)$$

be a representation for all $0 \leq i, n \leq N-1$ and $0 \leq \ j \leq 2$. It is easy to check that ϖ admits an admissible vector. Therefore we can define a continuous wavelet transform

$$(W_\psi\varphi)(a^i b^j H) = \frac{1}{\sqrt{c_\psi}}\sum_{n=0}^{N-1}\varphi(a^n H)\overline{\psi(a^{n-i}H)}.$$

for all $0 \leq n \leq N-1$ and admissible vector ψ in $L^2(G/H)$. Since $L^2(G/H)$ is isometry isomorphism with $L^2(K)$, the representation $\pi : K \to L^2(K), \ \ \pi(a^j)f(a^n) = f(a^{n-j})$ is a square integrable representation. For two admissible wavelets $f, g \in L^2(K)$, the two-wavelet constant is as follows:

$$c_{f,g} = \sum_{i=0}^{N-1}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}(\overline{f(a^n))}f(a^{n-i})(f(a_m)\overline{g(a^{m-i}})).$$

Since K is unimodular, by Theorem 3.3 the self adjoint positive operator C on the set of admissible wavelets is a constant multiple of identity operator. Note that, in this example $G \neq H \times_\tau K$ since K is not a normal subgroup of G. But H is a closed normal subgroup of G and this causes $W_\psi\varphi$ to be constant on left cosets of H [12].

Example 3.6. Let G be the Weyl-Heisenberg group $(WH)^n$ and $H = \{(0, 0, t), \ \ t \in \mathbb{R}/2\pi\mathbb{Z}\}$. The Euclidean space $\mathbb{R}^n \times \mathbb{R}^n$ can be considered as homogenous space of $(WH)^n$ and $\frac{(WH)^n}{H} = \mathbb{R}^n \times \mathbb{R}^n$ which admits the Lebesgue measure. The representation

$$\varpi : \mathbb{R}^n \times \mathbb{R}^n \to U(L^2(\mathbb{R}^n)), \ \ (\varpi(q, p)\varphi)(x) = e^{i(px-qp)}\varphi(x-q),$$

where $x \in \mathbb{R}^n, \varphi \in L^2(\mathbb{R}^n)$ is square integrable (see [18]) and for two admissible wavelets $\zeta, \xi \in L^2(\mathbb{R}^n)$, two-wavelet constant is as follows

$$\begin{aligned} c_{\zeta,\xi} &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} < \zeta, \varpi(q,p)\zeta >< \varpi(q,p)\xi, \zeta > dqdp \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} < \zeta, \zeta_{q,p} >< \xi_{q,p}, \zeta > dqdp, \end{aligned}$$

where $\zeta_{q,p}(x) = e^{ipx}\zeta(x-q), \quad x \in \mathbb{R}^n$. Since $\mathbb{R}^n \times \mathbb{R}^n$ is unimodular, by Theorem 3.3 the self adjoint positive operator C on the set of admissible wavelets is a constant multiple of identity operator.

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