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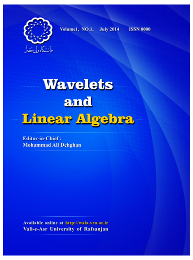

Property (T) for C*-dynamical systems

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Abstract

In this paper, we introduce a notion of property (T) for a C^*-dynamical system $(\mathcal{A}, \mathcal{G}, \alpha)$ consisting of a unital C^*-algebra $\mathcal{A}$, a locally compact group $\mathcal{G}$, and an action α on $\mathcal{A}$. As a result, we show that if $\mathcal{A}$ has strong property (T) and $\mathcal{G}$ has Kazhdan's property (T), then the triple $(\mathcal{A}, \mathcal{G}, \alpha)$ has property (T).

1. Introduction

A unital C^*-algebra $\mathcal{A}$ has property (T) if there exist a finite subset $\mathcal{F}$ of $\mathcal{A}$ and $\varepsilon > 0$, such that for every Hilbert bimodule on $\mathcal{A}$ with a unit $(\mathcal{F}, \varepsilon)$-central vector, there is a non-zero central

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vector (see [1]). This property is similar to the property (T) for locally compact groups, which is defined by D. Kazhdan in [7]. A locally compact group $\mathcal{G}$ has property (T) if, whenever a unitary representation $(\pi, \mathcal{H})$ of $\mathcal{G}$ almost has invariant vectors, $\mathcal{H}$ has a non-zero invariant vector. It is proved in [1] that a countable discrete group $\mathcal{G}$ has property (T) if and only if its full (or equivalently reduced) group C^*-algebra has property (T). In [3], property (T) for a von Neumann algebra was introduced, it is shown that a discrete ICC-group $\mathcal{G}$ has property (T) if and only if the von Neumann algebra generated by the left regular representation of $\mathcal{G}$ has property (T).

In this paper, if V and W are Hilbert spaces, $V \otimes W$ denotes their Hilbert space tensor product. If V and W are algebras, $V \odot W$ denotes their algebraic tensor product. If V and W are C^*-algebras, then $V \otimes_{\min} W$ will denote their C^*-tensor product with respect to the minimal (spatial) C^*-norm and $V \otimes_{\max} W$ will denote their C^*-tensor product with respect to the maximal C^*-norm. Also, if V is a Hilbert space we denote by $\mathcal{L}(V)$ the unital C^*-algebra of bounded linear operators on V.

The paper is organised as follows. In Section 2, we recall some definitions and results in the framework of C^*-dynamical systems which are used in this paper.

In Section 3, we define a notion of property (T) for an arbitrary C^*-dynamical system $(\mathcal{A}, \mathcal{G}, \alpha)$. We show that if $\mathcal{A}$ has strong property (T) and $\mathcal{G}$ has property (T), then $(\mathcal{A}, \mathcal{G}, \alpha)$ has property (T). We will also show that if $\mathcal{G}$ is a discrete group and $(\mathcal{A}, \mathcal{G}, \alpha)$ has property (T), then its C^*-crossed product has property (T) as a unital C^*-algebra. Furthermore, we show that if $\mathcal{A}$ is a commutative unital C^*-algebra, $\mathcal{G}$ is a countable discrete group such that there exists a faithful representation of $\mathcal{A}$ to the Hilbert space $\ell^2(\mathcal{G})$, then property (T) of $C^*_r(\mathcal{G}) \otimes_{\min} \mathcal{A}$ implies property (T) of $\mathcal{G}$, where $C^*_r(\mathcal{G})$ is the reduced group C^*-algebra of $\mathcal{G}$.

Our basic references for C^*-algebras are [5, 8, 9]. A good reference for C^*-dynamical systems is [10]. For a survey on Kazhdan's property (T) one can refer to [2].

2. Preliminaries and Basic Concepts

A *C^*-dynamical system* (or a *dynamical system*) is a triple $(\mathcal{A}, \mathcal{G}, \alpha)$, where $\mathcal{A}$ is a unital C^*-algebra, $\mathcal{G}$ is a locally compact group, and α is a continuous homomorphism from $\mathcal{G}$ into the group of all $*$-automorphisms of $\mathcal{A}$. Note that the continuity condition on α amounts to the statement that $\gamma \mapsto \alpha_\gamma(a)$ is continuous for all $a \in \mathcal{A}$.

Let $(\mathcal{A}, \mathcal{G}, \alpha)$ be a dynamical system such that $\mathcal{G}$ is a discrete group. Let $\mathcal{K}(\mathcal{G}, \mathcal{A})$ be the algebra of all $\mathcal{A}$-valued functions with finite support endowed with the following twisted convolution as product, involution and norm:

$$xy(t) = \sum_{\gamma} x(\gamma)\alpha_\gamma(y(\gamma^{-1}t)), \qquad x^*(t) = \alpha_t(x(t^{-1})^*), \qquad \|x\|_1 = \sum_{\gamma} \|x(\gamma)\|,$$

where $x, y \in \mathcal{K}(\mathcal{G}, \mathcal{A})$ and $t \in \mathcal{G}$. The algebra $\mathcal{K}(\mathcal{G}, \mathcal{A})$ becomes a normed $*$-algebra and we denote its completion by $\ell^1(\mathcal{G}, \mathcal{A})$. The algebra $\mathcal{A}$ is regarded as a subalgebra of $\mathcal{K}(\mathcal{G}, \mathcal{A})$ with the same unit element in which each arbitrary element $a \in \mathcal{A}$ can be thought as a function on $\mathcal{G}$ subject to the conditions $a(e) = a$ and $a(\gamma) = 0$ for $\gamma \neq e$, where e is the unit of $\mathcal{G}$.

The unital Banach $*$-algebra $\ell^1(\mathcal{G}, \mathcal{A})$ has a faithful representation and we call the C^*-envelope of $\ell^1(\mathcal{G}, \mathcal{A})$ the *C^*-crossed product* of $\mathcal{A}$ by $\mathcal{G}$ with respect to the action α and write as $\mathcal{A} \times_\alpha \mathcal{G}$. Let δ_γ be the unitary element of $\ell^1(\mathcal{G}, \mathcal{A})$ such that $\delta_\gamma(\gamma) = 1$ and $\delta_\gamma(t) = 0$ if $t \neq \gamma$. The element

δ_γ belongs to $\mathcal{A} \times_\alpha \mathcal{G}$ and satisfies $\delta_\gamma a \delta_\gamma^* = \alpha_\gamma(a)$. An element x in $\mathcal{K}(\mathcal{G}, \mathcal{A})$ can be written as $x = \sum_\gamma x(\gamma)\delta_\gamma$.

A pair (μ, π) consisting of a representation μ of $\mathcal{A}$ and a unitary representation π of $\mathcal{G}$ on the same Hilbert space $\mathcal{H}$ is called a *covariant representation* of $(\mathcal{A}, \mathcal{G}, \alpha)$ if for all $a \in \mathcal{A}$ and $\gamma \in \mathcal{G}$ we have

$$\pi(\gamma)\mu(a) = \mu(\alpha_\gamma(a))\pi(\gamma).$$

Consider two covariant representations (μ_1, π_1) and (μ_2, π_2) on the Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, respectively. We say that (μ_1, π_1) and (μ_2, π_2) are *equivalent* if there exists a unitary operator $W : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$W\mu_1(a) = \mu_2(a)W, \qquad W\pi_1(\gamma) = \pi_2(\gamma)W,$$

for all $a \in \mathcal{A}$ and $\gamma \in \mathcal{G}$.

Consider a faithful representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$. Define a representation of $\mathcal{A}$ as well as a unitary representation of $\mathcal{G}$ on the Hilbert space $\ell^2(\mathcal{G}, \mathcal{H})$ by

$$\pi_\alpha(a)\widetilde{\xi}(\gamma) = \alpha_{\gamma^{-1}}(a) \cdot \widetilde{\xi}(\gamma), \qquad \lambda_\alpha(\gamma)\widetilde{\xi}(t) = \widetilde{\xi}(\gamma^{-1}t),$$

where $a \in \mathcal{A}, \widetilde{\xi} \in \ell^2(\mathcal{G}, \mathcal{H})$ and $\gamma, t \in \mathcal{G}$. We say that $(\pi_\alpha, \lambda_\alpha)$ is a *regular representation* of $(\mathcal{A}, \mathcal{G}, \alpha)$.

The *reduced C^*-crossed product* $\mathcal{A} \times_{\alpha r} \mathcal{G}$ is the C^*-algebra on $\ell^2(\mathcal{G}, \mathcal{H})$ generated by the family of $\{\pi_\alpha(a), \lambda_\alpha(\gamma) \mid a \in \mathcal{A}, \gamma \in \mathcal{G}\}$. Note that this definition is independent of the choice of the space $\mathcal{H}$.

If $\mathcal{A} = \mathbb{C}$ and α is trivial, then $\ell^1(\mathcal{G}, \mathcal{A})$ coincide with $\ell^1(\mathcal{G})$ and λ_α is the regular representation on the Hilbert space $\ell^2(\mathcal{G})$. In this case, $\mathcal{A} \times_\alpha \mathcal{G}$ is the group C^*-algebra $C^*(\mathcal{G})$ and $\mathcal{A} \times_{\alpha r} \mathcal{G}$ is the reduced group C^*-algebra $C_r^*(\mathcal{G})$.

3. Property (T) for a dynamical system

A *Hilbert bimodule* on a unital C^*-algebra $\mathcal{A}$ (or a *Hilbert $\mathcal{A}$-bimodule*) is a Hilbert space $\mathcal{H}$ carrying two commuting actions, one from $\mathcal{A}$ and one from the opposite algebra $\mathcal{A}^0$ (see [1]). In other words, there exists a representation from $\mathcal{A} \otimes_{\max} \mathcal{A}^0$ to $\mathcal{L}(\mathcal{H})$. If $\mathcal{H}$ is a Hilbert $\mathcal{A}$-bimodule, we will write $a \cdot \xi \cdot b$ for all $a, b \in \mathcal{A}$ and $\xi \in \mathcal{H}$, to denote the module actions.

A *tracial state* on a unital C^*-algebra $\mathcal{A}$ is a positive linear functional $Tr : \mathcal{A} \to \mathbb{C}$ such that $Tr(ab) = Tr(ba)$ for all $a, b \in \mathcal{A}$ and $Tr(1) = 1$.

Definition 3.1. (see [1]) Let $\mathcal{B} \subset \mathcal{A}$ be a C^*-subalgebra containing the identity of a unital C^*-algebra $\mathcal{A}$. The pair $(\mathcal{A}, \mathcal{B})$ has *property (T)* if there exist a finite subset $\mathcal{F}$ of $\mathcal{A}$ and $\varepsilon > 0$ such that the following property holds: if a Hilbert bimodule $\mathcal{H}$ on $\mathcal{A}$ contains a unit vector $\xi \in \mathcal{H}$ which is $(\mathcal{F}, \varepsilon)$-central, that is:

$$\max_{a \in \mathcal{F}} \|a \cdot \xi - \xi \cdot a\| < \varepsilon,$$

then $\mathcal{H}$ has a non-zero $\mathcal{B}$-central vector, that is, a non-zero vector $\eta \in \mathcal{H}$ such that

$$b \cdot \eta = \eta \cdot b,$$

for all $b \in \mathcal{B}$. Moreover, $\mathcal{A}$ has *property (T)* if the pair $(\mathcal{A}, \mathcal{A})$ has such property.

It is clear that if $\mathcal{A}$ has property (T), then the pair $(\mathcal{A}, \mathcal{B})$ has, too. As an example, if $\mathcal{H}$ is any Hilbert space and $\mathcal{B} \subset \mathcal{L}(\mathcal{H})$ a unital C^*-subalgebra, then $(\mathcal{L}(\mathcal{H}), \mathcal{B})$ has property (T) (see [4]).

Note that Definition 3.1 comes from the original definition of property (T) for groups. Let $\mathcal{G}$ be a locally compact group and $\mathcal{N}$ a closed subgroup. The pair $(\mathcal{G}, \mathcal{N})$ has *property (T)* if there exist a compact subset Q of $\mathcal{G}$ and $r > 0$ such that the following property holds: if a unitary representation $(\pi, \mathcal{H})$ of $\mathcal{G}$ contains a unit vector $\xi \in \mathcal{H}$ which is (Q, r)-invariant, that is:

$$\sup_{\gamma \in Q} \|\pi(\gamma)(\xi) - \xi\| < r,$$

then $\mathcal{H}$ has a non-zero $\mathcal{N}$-invariant vector, that is, there is a non-zero vector $\eta \in \mathcal{H}$ such that

$$\pi(\gamma)(\eta) = \eta,$$

for all $\gamma \in \mathcal{N}$. Moreover, $\mathcal{G}$ has *property (T)* if the pair $(\mathcal{G}, \mathcal{G})$ has property (T). An example of a pair with property (T) is the pair $(SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$, where $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ is the semi-direct product for the natural action of $SL_2(\mathbb{Z})$ on $\mathbb{Z}^2$.

In the following, we give definition of a covariant birepresentation on a dynamical system and apply it to study the property (T) on the dynamical systems.

Let $(\mathcal{A}, \mathcal{G}, \alpha)$ be a dynamical system. A triple $(\mathcal{H}, \pi_1, \pi_2)$ consisting of a Hilbert bimodule $\mathcal{H}$ on $\mathcal{A}$ and two commuting unitary representations π_1, π_2 of $\mathcal{G}$ on the same Hilbert space $\mathcal{H}$ is called a *covariant birepresentation* of $(\mathcal{A}, \mathcal{G}, \alpha)$ if we have

$$\pi_1(\gamma_1)\pi_2(\gamma_2)(a \cdot \xi \cdot b) = \alpha_{\gamma_1}(a) \cdot \pi_1(\gamma_1)\pi_2(\gamma_2)(\xi) \cdot \alpha_{\gamma_2}(b),$$

for all $a, b \in \mathcal{A}, \gamma_1, \gamma_2 \in \mathcal{G}$ and $\xi \in \mathcal{H}$.

Obviously, covariant birepresentations of the dynamical system $(\mathcal{A}, \{e\}, id)$ are in one-to-one correspondence with Hilbert bimodules on $\mathcal{A}$, where $\{e\}$ is the trivial group with one element. Covariant birepresentations of the dynamical system $(\mathbb{C}, \mathcal{G}, id)$ correspond to commuting unitary representations of $\mathcal{G}$. Note that if $(\mathcal{H}, \pi_1, \pi_2)$ is a covariant birepresentation of $(\mathcal{A}, \mathcal{G}, \alpha)$, then so is the triple $(\mathcal{H}, \pi_2, \pi_1)$.

Let $(\mathcal{A}, \mathcal{G}, \alpha)$ be a dynamical system and $\mathcal{B} \subset \mathcal{A}$ an α-invariant C^*-subalgebra containing the identity element of $\mathcal{A}$. Let $(\mathcal{H}, \pi_1, \pi_2)$ be a covariant birepresentation of $(\mathcal{A}, \mathcal{G}, \alpha)$. We say that $(\mathcal{H}, \pi_1, \pi_2)$ has a *non-zero $(\mathcal{B}, \mathcal{G})$-central vector* if there exists a non-zero vector η in $\mathcal{H}$ such that

$$b \cdot \eta = \eta \cdot b, \qquad \pi_1(\gamma)\pi_2(\gamma)(\eta) = \eta,$$

for all $\gamma \in \mathcal{G}$ and $b \in \mathcal{B}$.

If $(\mathcal{A}, \mathcal{G}, \alpha)$ is a dynamical system and $\mathcal{G}$ is a discrete group, then covariant birepresentations with non-zero central vectors are in one-to-one correspondence with α-invariant tracial states of the associated C^*-algebra.

Lemma 3.2. *(i) Let $(\mathcal{H}, \pi_1, \pi_2)$ be a covariant birepresentation of a dynamical system $(\mathcal{A}, \mathcal{G}, \alpha)$ with a non-zero $(\mathcal{A}, \mathcal{G})$-central vector η. Then $\mathcal{A}$ admits an α-invariant tracial state.*

(ii) Let $(\mathcal{A}, \mathcal{G}, \alpha)$ be a dynamical system such that $\mathcal{G}$ is a discrete group. Let $Tr : \mathcal{A} \to \mathbb{C}$ be an α-invariant tracial state on $\mathcal{A}$. Then there exists a covariant birepresentation of $(\mathcal{A}, \mathcal{G}, \alpha)$ with a non-zero $(\mathcal{A}, \mathcal{G})$-central vector.

Proof. (i) Let $\zeta = \frac{\eta}{\|\eta\|}$. Define $Tr : \mathcal{A} \to \mathbb{C}$ by $Tr(a) = \langle a \cdot \zeta, \zeta \rangle$. Then Tr is a tracial state on $\mathcal{A}$, and for all $a \in \mathcal{A}, \gamma \in \mathcal{G}$ we have

$$\begin{aligned} Tr(\alpha_\gamma(a)) &= \langle \pi_1(\gamma)\pi_2(\gamma)(a \cdot \zeta), \zeta \rangle \\ &= \langle a \cdot \zeta, \pi_2(\gamma^{-1})\pi_1(\gamma^{-1})(\zeta) \rangle \\ &= Tr(a). \end{aligned}$$

(ii) First, consider the extension of α-invariant tracial state on $\mathcal{A} \times_\alpha \mathcal{G}$, again denoted by Tr. Setting $N = \{x \in \mathcal{A} \times_\alpha \mathcal{G} \mid Tr(x^* x) = 0\}$, it is easy to check that N is a two-sided ideal of $\mathcal{A} \times_\alpha \mathcal{G}$ and that the map $\langle x + N, y + N \rangle = Tr(y^* x)$ is a well-defined inner product on the quotient space $\mathcal{A} \times_\alpha \mathcal{G}/N$. We denote by $L^2(Tr)$ the Hilbert space completion of $\mathcal{A} \times_\alpha \mathcal{G}/N$. For each $a \in \mathcal{A}$, the mappings $x + N \mapsto ax + N$ and $x + N \mapsto xa + N$ can be extend to bounded operators on $L^2(Tr)$, and $L^2(Tr)$ is a Hilbert bimodule on $\mathcal{A}$. Also, if $\gamma \in \mathcal{G}$, define two operators $\pi_1(\gamma), \pi_2(\gamma) \in \mathcal{L}(L^2(Tr))$ by

$$\pi_1(\gamma)(x + N) = \delta_\gamma x + N, \qquad \pi_2(\gamma)(x + N) = x\delta_{\gamma^{-1}} + N.$$

We obtain two commuting unitary representations π_1, π_2 of $\mathcal{G}$ on $L^2(Tr)$, and $(L^2(Tr), \pi_1, \pi_2)$ is a covariant birepresentation of $(\mathcal{A}, \mathcal{G}, \alpha)$. Moreover, $\eta = \delta_e + N$ is a non-zero $(\mathcal{A}, \mathcal{G})$-central vector.

□

Let $(\mathcal{H}, \pi_1, \pi_2)$ be a covariant birepresentation of $(\mathcal{A}, \mathcal{G}, \alpha)$. Given a finite subset $\mathcal{F}$ of $\mathcal{A}$, a compact subset Q of $\mathcal{G}$ and $\varepsilon, r > 0$, we say that a unit vector $\xi \in \mathcal{H}$ is $(\mathcal{F}, \varepsilon, Q, r)$-*central* if:

$$\max_{a \in \mathcal{F}} \|a \cdot \xi - \xi \cdot a\| < \varepsilon, \qquad \sup_{\gamma \in Q} \|\pi_1(\gamma)\pi_2(\gamma)(\xi) - \xi\| < r.$$

The covariant birepresentation $(\mathcal{H}, \pi_1, \pi_2)$ *almost has invariant vectors* if it has $(\mathcal{F}, \varepsilon, Q, r)$-central vectors for every finite subset $\mathcal{F}$ of $\mathcal{A}$, compact subset Q of $\mathcal{G}$ and every $\varepsilon, r > 0$.

Definition 3.3. Let $(\mathcal{A}, \mathcal{G}, \alpha)$ be a dynamical system, and $\mathcal{B} \subset \mathcal{A}$ an α-invariant C^*-subalgebra containing the identity element of $\mathcal{A}$. We denote the dynamical system $(\mathcal{A}, \mathcal{G}, \alpha)$ with the α-invariant C^*-subalgebra $\mathcal{B}$, by $((\mathcal{A}, \mathcal{B}), \mathcal{G}, \alpha)$. We say that $((\mathcal{A}, \mathcal{B}), \mathcal{G}, \alpha)$ has *property (T)* if there exist a finite subset $\mathcal{F}$ of $\mathcal{A}$, a compact subset Q of $\mathcal{G}$ and $\varepsilon, r > 0$ such that any covariant birepresentation of $(\mathcal{A}, \mathcal{G}, \alpha)$ with a unit $(\mathcal{F}, \varepsilon, Q, r)$-central possesses non-zero $(\mathcal{B}, \mathcal{G})$-central vectors. Moreover, the dynamical system $(\mathcal{A}, \mathcal{G}, \alpha)$ has *property (T)* if the system $((\mathcal{A}, \mathcal{A}), \mathcal{G}, \alpha)$ has such property.

It is clear that if $(\mathcal{A}, \mathcal{G}, \alpha)$ has property (T), then so has $((\mathcal{A}, \mathcal{B}), \mathcal{G}, \alpha)$. Property (T) of the dynamical system $(\mathcal{A}, \{e\}, id)$ correspond to property (T) of $\mathcal{A}$, and property (T) of the dynamical system $(\mathbb{C}, \mathcal{G}, id)$ correspond to property (T) of $\mathcal{G}$.

Remark 3.4. (i) If α is trivial and $((\mathcal{A},\mathcal{B}),\mathcal{G},\alpha)$ has property (T), then $(\mathcal{A},\mathcal{B})$ has it too, for any $\mathcal{B}$ as above.

(ii) Let $(\mathcal{A},\mathcal{G},\alpha)$ be a dynamical system, and that the C^*-algebra $\mathcal{A}$ admitting an α-invariant character, that is, a non-zero multiplicative linear map $\chi : \mathcal{A} \to \mathbb{C}$ such that $\chi(\alpha_\gamma(a)) = \chi(a)$ for all $\gamma \in \mathcal{G}$ and $a \in \mathcal{A}$. If $(\mathcal{A},\mathcal{G},\alpha)$ has property (T), then $\mathcal{G}$ has property (T).

Let us give an example of a dynamical system which does not have property (T).

Example 3.5. Let X be a smooth vector field on a compact manifold $\mathcal{M}$. Suppose for each point $q \in \mathcal{M}$ there is a unique integral curve $\theta^q : \mathbb{R} \to \mathcal{M}$ of X starting at q, and p be an element in $\mathcal{M}$ such that θ^p is the constant curve $\theta^p(t) \equiv p$. For each $t \in \mathbb{R}$, we can define a map θ_t from $\mathcal{M}$ to itself by sending each point $q \in \mathcal{M}$ to the point obtained by the curve starting at q for time t:

$$\theta_t(q) = \theta^q(t).$$

This defines a family of maps $\theta_t : \mathcal{M} \to \mathcal{M}$ for $t \in \mathbb{R}$. Let $C(\mathcal{M})$ denote the unital C^*-algebra of continuous complex valued functions on $\mathcal{M}$. We obtain a homomorphism $\alpha : \mathbb{R} \to Aut(C(\mathcal{M}))$, defined by

$$\alpha_t(f)(q) = f(\theta_{t^{-1}}(q)),$$

and $(C(\mathcal{M}),\mathbb{R},\alpha)$ is a dynamical system. Define an α-invariant character $\chi : C(\mathcal{M}) \to \mathbb{C}$ by $\chi(f) = f(p)$. We know that $\mathbb{R}$ does not have property (T) (see [2]), it follows from Remark 3.4 that $(C(\mathcal{M}),\mathbb{R},\alpha)$ does not have property (T).

The notion of property (T) for a dynamical system $(\mathcal{A},\mathcal{G},\alpha)$ and for C^*-algebras associated to it are related via the correspondence between covariant birepresentations and Hilbert bimodules. More precisely, given a Hilbert $\mathcal{A} \times_\alpha \mathcal{G}$-bimodule $\mathcal{H}$, one can define two commuting unitary representations π_1, π_2 of $\mathcal{G}$ on the same Hilbert space $\mathcal{H}$ by

$$\pi_1(\gamma)(\xi) = \delta_\gamma \cdot \xi, \qquad \pi_2(\gamma)(\xi) = \xi \cdot \delta_{\gamma^{-1}}.$$

Viewing $\mathcal{A}$ as a subalgebra of $\mathcal{A} \times_\alpha \mathcal{G}$, it is simple to see that $(\mathcal{H},\pi_1,\pi_2)$ is a covariant birepresentation of $(\mathcal{A},\mathcal{G},\alpha)$.

Conversely, suppose $(\mathcal{H},\pi_1,\pi_2)$ is a covariant birepresentation of $(\mathcal{A},\mathcal{G},\alpha)$. Take $x \in \mathcal{K}(\mathcal{G},\mathcal{A})$ and define two operators $\pi(x)$ and $\rho(x)$ on $\mathcal{H}$ by

$$\pi(x)\xi = \sum_\gamma x(\gamma) \cdot \pi_1(\gamma)(\xi), \qquad \rho(x)\xi = \sum_\gamma \pi_2(\gamma^{-1})(\xi \cdot x(\gamma)).$$

Since π is obviously norm decreasing, it extends to a representation of $\ell^1(\mathcal{G},\mathcal{A})$, hence to that of $\mathcal{A} \times_\alpha \mathcal{G}$. Similarly, ρ extends to a representation of the opposite algebra of $\mathcal{A} \times_\alpha \mathcal{G}$. Two representations π and ρ are commuting, so that $\mathcal{H}$ is a Hilbert bimodule on $\mathcal{A} \times_\alpha \mathcal{G}$.

Hence, a non-zero $(\mathcal{A},\mathcal{G})$-central vector for a covariant birepresentation of $(\mathcal{A},\mathcal{G},\alpha)$ is a non-zero $\mathcal{A} \times_\alpha \mathcal{G}$-central vector.

By the argument of Remark 15 in [1], we know that every unital C^*-algebra without tracial states has property (T). We will show that a similar fact is true for dynamical systems which the associated C^*-algebra does not admit α-invariant tracial states.

Theorem 3.6. *Let $(\mathcal{A}, \mathcal{G}, \alpha)$ be a dynamical system such that the unital C^*-algebra $\mathcal{A}$ does not admit α-invariant tracial states. Then $(\mathcal{A}, \mathcal{G}, \alpha)$ has property (T).*

Proof. Assume that $(\mathcal{A}, \mathcal{G}, \alpha)$ does not have property (T). Then, there is a covariant birepresentation $(\mathcal{H}, \pi_1, \pi_2)$ almost has invariant vectors. This implies that there is a net of unit vectors $(\xi_i)_{i\in I}$ in $\mathcal{H}$ such that:

$$\lim_i \|a \cdot \xi_i - \xi_i \cdot a\| = 0, \qquad \lim_i \|\pi_1(\gamma)\pi_2(\gamma)(\xi_i) - \xi_i\| = 0,$$

for all $a \in \mathcal{A}, \gamma \in \mathcal{G}$. For each $T \in \mathcal{L}(\mathcal{H})$, let D_T be the closed disc in $\mathbb{C}$ of radius $\|T\|$, and consider the product space

$$X = \prod_{T\in\mathcal{L}(\mathcal{H})} D_T,$$

endowed with the product topology. By Tychonoff's Theorem, X is compact. Since $(\langle T\xi_i, \xi_i\rangle)_{T\in\mathcal{L}(\mathcal{H})}$ is an element of X for all $i \in I$, there exists a subnet $(\xi_j)_{j\in J}$ such that, for all $T \in \mathcal{L}(\mathcal{H})$, the limit

$$\varphi(T) = \lim_j \langle T\xi_j, \xi_j\rangle$$

exists. It is clear that $T \mapsto \varphi(T)$ is a positive linear functional on $\mathcal{L}(\mathcal{H})$ with $\varphi(id_{\mathcal{H}}) = 1$. Moreover, for every $\gamma \in \mathcal{G}$ and $T \in \mathcal{L}(\mathcal{H})$, we have

$$\varphi(\pi_1(\gamma)\pi_2(\gamma)T) = \varphi(T) = \varphi(T\pi_1(\gamma)\pi_2(\gamma)).$$

Then $Tr : \mathcal{A} \to \mathbb{C}$ defined by $Tr(a) = \varphi(\mu(a))$ is an α-invariant tracial state on $\mathcal{A}$, where μ is the representation on $\mathcal{H}$ given by, say, the left action of $\mathcal{A}$. □

Example 3.7. Let $\mathcal{H}$ be an infinite-dimensional Hilbert space and $\mathcal{U}(\mathcal{H})$ be its unitary group. Suppose $\mathcal{B} \subset \mathcal{L}(\mathcal{H})$ is a C^*-subalgebra containing the identity element of $\mathcal{L}(\mathcal{H})$, and that $u \in \mathcal{U}(\mathcal{H})$ is such that $u\mathcal{B}u^* \subset \mathcal{B}$. Then $\varphi(a) = uau^*$ is an automorphism of $\mathcal{L}(\mathcal{H})$. Therefore, we obtain a homomorphism $\alpha : \mathbb{Z} \to Aut(\mathcal{L}(\mathcal{H}))$, defined by $\alpha_n = \varphi^n$, and $(\mathcal{L}(\mathcal{H}), \mathbb{Z}, \alpha)$ is a dynamical system. Using Theorem 3.6, so $((\mathcal{L}(\mathcal{H}), \mathcal{B}), \mathbb{Z}, \alpha)$ has property (T).

Let $\mathcal{G}_1 \to \mathcal{G}_2$ be a surjective continuous homomorphism between locally compact groups. It is well-known that if $\mathcal{G}_1$ has property (T), then $\mathcal{G}_2$ has property (T). Similarly, let $\mathcal{A} \to \mathcal{B}$ be a surjective $*$-homomorphism between unital C^*-algebras. If $\mathcal{A}$ has property (T), then so has $\mathcal{B}$. The corresponding statement for dynamical systems is as follows and its proof is straightforward.

Lemma 3.8. *Let $(\mathcal{A}, \mathcal{G}, \alpha)$ and $(\mathcal{B}, \mathcal{G}, \beta)$ be two dynamical systems with actions α and β of a fixed group $\mathcal{G}$ on $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $f : \mathcal{A} \to \mathcal{B}$ be a surjective $*$-homomorphism between $\mathcal{A}$ and $\mathcal{B}$ such that*

$$\beta_\gamma(f(a)) = f(\alpha_\gamma(a)),$$

for all $\gamma \in \mathcal{G}, a \in \mathcal{A}$. If $(\mathcal{A}, \mathcal{G}, \alpha)$ has property (T), then $(\mathcal{B}, \mathcal{G}, \beta)$ has also property (T).

Let $\mathcal{H}$ be a Hilbert bimodule on a C^*-algebra $\mathcal{A}$ and $\mathcal{B} \subset \mathcal{A}$ a C^*-subalgebra containing the identity of $\mathcal{A}$. Let

$$\mathcal{H}^{\mathcal{B}} = \{\eta \in \mathcal{H} \mid b \cdot \eta = \eta \cdot b, \ \forall b \in \mathcal{B}\},$$

and $P^{\mathcal{B}}_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}^{\mathcal{B}}$ be the orthogonal projection from $\mathcal{H}$ over the closed subspace $\mathcal{H}^{\mathcal{B}}$.

Let us recall a notion of strong property (T) in [4]. The pair $(\mathcal{A}, \mathcal{B})$ has *strong property (T)* if for any $r > 0$, there exist a finite subset $\mathcal{F}$ of $\mathcal{A}$ and $\varepsilon > 0$ such that the following property holds: if a Hilbert bimodule $\mathcal{H}$ on $\mathcal{A}$ contains a unit vector $\xi \in \mathcal{H}$ which is $(\mathcal{F}, \varepsilon)$-central, then $\|\xi - P^{\mathcal{B}}_{\mathcal{H}}(\xi)\| < r$. Also, $\mathcal{A}$ has *strong property (T)* if $(\mathcal{A}, \mathcal{A})$ has such property.

By taking $r < \frac{1}{2}$, we see that strong property (T) implies property (T). If $\mathcal{A}$ has no tracial state, then $\mathcal{A}$ has strong property (T), and so does $(\mathcal{A}, \mathcal{B})$ (see [4]).

Also, suppose $(\pi, \mathcal{H})$ is a unitary representation of a locally compact group $\mathcal{G}$ and $\mathcal{N}$ is a closed subgroup of $\mathcal{G}$. Let

$$\mathcal{H}^{\mathcal{N}} = \{\eta \in \mathcal{H} \mid \pi(\gamma)(\eta) = \eta, \ \forall \gamma \in \mathcal{N}\},$$

and $P^{\mathcal{N}}_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}^{\mathcal{N}}$ be the orthogonal projection from $\mathcal{H}$ over the closed subspace $\mathcal{H}^{\mathcal{N}}$.

Theorem 3.9. *Suppose $(\mathcal{A}, \mathcal{B})$ has strong property (T) and $\mathcal{G}$ has property (T). Then $((\mathcal{A}, \mathcal{B}), \mathcal{G}, \alpha)$ has property (T).*

Proof. Since $\mathcal{G}$ has property (T), there exist a compact subset Q of $\mathcal{G}$ and $\varepsilon > 0$ such that for any unitary representation $(\pi, \mathcal{H})$ and unit vector $\xi \in \mathcal{H}$ which is (Q, ε)-invariant, one has a non-zero vector $\eta \in \mathcal{H}$ such that

$$\pi(\gamma)(\eta) = \eta,$$

for all $\gamma \in \mathcal{G}$. Let $h = \min\{\frac{1}{2}, \frac{\varepsilon}{8}\}$. Since $(\mathcal{A}, \mathcal{B})$ has strong property (T), there exist a finite subset $\mathcal{F}$ of $\mathcal{A}$ and $r > 0$ such that for any Hilbert bimodule $\mathcal{H}$ and unit vector $\xi \in \mathcal{H}$ which is $(\mathcal{F}, r)$-central, one has

$$\|\xi - P^{\mathcal{B}}_{\mathcal{H}}(\xi)\| < h.$$

Let $k = \min\{r, \frac{\varepsilon}{4}\}$, and $(\mathcal{H}, \pi_1, \pi_2)$ be a covariant birepresentation of $(\mathcal{A}, \mathcal{G}, \alpha)$ with a unit vector $\xi \in \mathcal{H}$ such that:

$$\max_{a \in \mathcal{F}} \|a \cdot \xi - \xi \cdot a\| < k, \qquad \sup_{\gamma \in Q} \|\pi_1(\gamma)\pi_2(\gamma)(\xi) - \xi\| < k.$$

Then $\|\xi - P^{\mathcal{B}}_{\mathcal{H}}(\xi)\| < h$ and $\|P^{\mathcal{B}}_{\mathcal{H}}(\xi)\| > \frac{1}{2}$. For all $b \in \mathcal{B}, \gamma \in \mathcal{G}$ and $\zeta \in \mathcal{H}^{\mathcal{B}}$ we have:

$$\begin{aligned} b \cdot \pi_1(\gamma)\pi_2(\gamma)(\zeta) &= \pi_1(\gamma)(\alpha_{\gamma^{-1}}(b) \cdot \pi_2(\gamma)(\zeta)) \\ &= \pi_1(\gamma)\pi_2(\gamma)(\alpha_{\gamma^{-1}}(b) \cdot \zeta) \\ &= \pi_1(\gamma)\pi_2(\gamma)(\zeta \cdot \alpha_{\gamma^{-1}}(b)) \\ &= \pi_1(\gamma)(\pi_2(\gamma)(\zeta) \cdot b) \\ &= \pi_1(\gamma)\pi_2(\gamma)(\zeta) \cdot b. \end{aligned}$$

Hence, $\pi(\gamma) = \pi_1(\gamma)\pi_2(\gamma)$ is a unitary representation of $\mathcal{G}$ on $\mathcal{H}^{\mathcal{B}}$. If we take $\zeta = \frac{P^{\mathcal{B}}_{\mathcal{H}}(\xi)}{\|P^{\mathcal{B}}_{\mathcal{H}}(\xi)\|}$, then we have

$$\sup_{\gamma \in Q} \|\pi(\gamma)(\zeta) - \zeta\| < \frac{k}{\|P^{\mathcal{B}}_{\mathcal{H}}(\xi)\|} + \frac{2h}{\|P^{\mathcal{B}}_{\mathcal{H}}(\xi)\|} < \frac{\varepsilon}{2\|P^{\mathcal{B}}_{\mathcal{H}}(\xi)\|} < \frac{\varepsilon}{2} \times 2 = \varepsilon.$$

Therefore, there exists a non-zero vector $\eta \in \mathcal{H}^{\mathcal{B}}$ such that

$$\pi_1(\gamma)\pi_2(\gamma)(\eta) = \pi(\gamma)(\eta) = \eta,$$

for all $\gamma \in \mathcal{G}$, which implies that η is a non-zero $(\mathcal{B}, \mathcal{G})$-central vector, as required. □

We need the following proposition from [6] to prove the next lemma.

Proposition 3.10. *Let $\mathcal{G}$ be a locally compact and σ-compact group and let $\mathcal{N}$ be a closed subgroup of $\mathcal{G}$. The following properties are equivalent:*

(i) $(\mathcal{G}, \mathcal{N})$ has property (T),

(ii) for every $r > 0$, there exists a pair (Q, ε) of compact subset Q of $\mathcal{G}$ and $\varepsilon > 0$ with the following property: for any unitary representation $(\pi, \mathcal{H})$ of $\mathcal{G}$ which has a (Q, ε)-invariant unit vector ξ, then we have $\|\xi - P^{\mathcal{N}}_{\mathcal{H}}(\xi)\| \leq r$.

Lemma 3.11. *Let $\mathcal{G}$ be a countable discrete group with property (T). Then $(C^*(\mathcal{G}), \mathcal{G}, \alpha)$ has property (T) for any action α of $\mathcal{G}$ on $C^*(\mathcal{G})$.*

Proof. By Theorem 3.9, it suffices to prove that $C^*(\mathcal{G})$ has strong property (T). Let $r > 0$. Since $\mathcal{G}$ has property (T), by Proposition 3.10 there exist a finite subset Q of $\mathcal{G}$ and $\varepsilon > 0$ such that for any unitary representation $(\pi, \mathcal{H})$ and unit vector $\xi \in \mathcal{H}$ which is (Q, ε)-invariant, one has $\|\xi - P^{\mathcal{G}}_{\mathcal{H}}(\xi)\| \leq \frac{r}{2}$. Let $\mathcal{F} = \{\delta_\gamma \mid \gamma \in Q\}$ be the finite subset of $C^*(\mathcal{G})$, and $\mathcal{H}$ a Hilbert bimodule on $C^*(\mathcal{G})$ contains a unit vector $\xi \in \mathcal{H}$ which is $(\mathcal{F}, \varepsilon)$-central. Define a unitary representation $(\pi, \mathcal{H})$ of $\mathcal{G}$ by

$$\pi(\gamma)(\xi) = \delta_\gamma \cdot \xi \cdot \delta_{\gamma^{-1}}.$$

Hence, ξ is (Q, ε)-invariant, and we have

$$\|\xi - P^{C^*(\mathcal{G})}_{\mathcal{H}}(\xi)\| = \|\xi - P^{\mathcal{G}}_{\mathcal{H}}(\xi)\| < r.$$

□

In the following, we show that property (T) of a dynamical system $(\mathcal{A}, \mathcal{G}, \alpha)$ such that $\mathcal{G}$ is a discrete group implies property (T) of its C^*-crossed product.

Theorem 3.12. *Let $(\mathcal{A}, \mathcal{G}, \alpha)$ be a dynamical system, and that $\mathcal{G}$ is a discrete group. If $(\mathcal{A}, \mathcal{G}, \alpha)$ has property (T), then $\mathcal{A} \times_\alpha \mathcal{G}$ has property (T) (and so does $\mathcal{A} \times_{\alpha r} \mathcal{G}$).*

Proof. Since $(\mathcal{A}, \mathcal{G}, \alpha)$ has property (T), there exist a finite subset $\mathcal{F}$ of $\mathcal{A}$, a finite subset Q of $\mathcal{G}$ and $\varepsilon, r > 0$ such that for every covariant birepresentation $(\mathcal{H}, \pi_1, \pi_2)$ of $(\mathcal{A}, \mathcal{G}, \alpha)$ contains a unit vector $\xi \in \mathcal{H}$ which is $(\mathcal{F}, \varepsilon, Q, r)$-central, then $\mathcal{H}$ has a non-zero $(\mathcal{A}, \mathcal{G})$-central vector.

Let $\mathcal{D} = \mathcal{F} \cup \{\delta_\gamma \mid \gamma \in Q\}$ and $\ell = \min\{r, \varepsilon\}$. Let $\mathcal{H}$ be a Hilbert bimodule on $\mathcal{A} \times_\alpha \mathcal{G}$ contains a unit vector $\xi \in \mathcal{H}$ which is $(\mathcal{D}, \ell)$-central. Define two commuting unitary representations $(\pi_1, \mathcal{H})$ and $(\pi_2, \mathcal{H})$ of $\mathcal{G}$ by

$$\pi_1(\gamma)(\xi) = \delta_\gamma \cdot \xi, \qquad \pi_2(\gamma)(\xi) = \xi \cdot \delta_{\gamma^{-1}}.$$

Viewing $\mathcal{A}$ as a subalgebra of $\mathcal{A} \times_\alpha \mathcal{G}$, it is clear that $(\mathcal{H}, \pi_1, \pi_2)$ is a covariant birepresentation of $(\mathcal{A}, \mathcal{G}, \alpha)$, and ξ is a $(\mathcal{F}, \varepsilon, Q, r)$-central. Therefore, there exists a non-zero vector $\eta \in \mathcal{H}$ such that

$$a \cdot \eta = \eta \cdot a, \qquad \delta_\gamma \cdot \eta = \eta \cdot \delta_\gamma,$$

for all $a \in \mathcal{A}$ and $\gamma \in \mathcal{G}$. Then for any $x = \sum_\gamma x(\gamma)\delta_\gamma \in \mathcal{K}(\mathcal{G}, \mathcal{A})$, we have

$$x \cdot \eta = \sum_\gamma x(\gamma)\delta_\gamma \cdot \eta = \sum_\gamma x(\gamma) \cdot \eta \cdot \delta_\gamma = \sum_\gamma \eta \cdot x(\gamma)\delta_\gamma = \eta \cdot x.$$

Since $\mathcal{K}(\mathcal{G}, \mathcal{A})$ is dense in $\ell^1(\mathcal{G}, \mathcal{A})$ and $\ell^1(\mathcal{G}, \mathcal{A})$ is dense in $\mathcal{A} \times_\alpha \mathcal{G}$, we obtain $x \cdot \eta = \eta \cdot x$ for all $x \in \mathcal{A} \times_\alpha \mathcal{G}$. Since $\mathcal{A} \times_{\alpha r} \mathcal{G}$ is a quotient of $\mathcal{A} \times_\alpha \mathcal{G}$, it follows that $\mathcal{A} \times_{\alpha r} \mathcal{G}$ also has property (T). □

Remark 3.13. If $(\mathcal{A}, \mathcal{G}, \alpha)$ is a dynamical system, $\mathcal{G}$ is a discrete group and α trivial, then:

$$\mathcal{A} \times_{\alpha r} \mathcal{G} \cong C_r^*(\mathcal{G}) \otimes_{\min} \mathcal{A}, \qquad \mathcal{A} \times_\alpha \mathcal{G} \cong C^*(\mathcal{G}) \otimes_{\max} \mathcal{A}.$$

By Theorems 3.9 and 3.12 for a discrete group $\mathcal{G}$ with property (T) and a unital C^*-algebra $\mathcal{A}$ with strong property (T), $C_r^*(\mathcal{G}) \otimes_{\min} \mathcal{A}$ and $C^*(\mathcal{G}) \otimes_{\max} \mathcal{A}$ have property (T).

If a locally compact group with property (T) is amenable, then it is compact, a similar fact is true for C^*-algebras with property (T) which are nuclear. A C^*-algebra $\mathcal{A}$ is *nuclear* if, for any C^*-algebra $\mathcal{B}$, there is a unique pre-C^*-norm on $\mathcal{A} \odot \mathcal{B}$. Let Tr be a tracial state on the unital C^*-algebra $\mathcal{A}$. By the *GNS*-construction, Tr defines a Hilbert $\mathcal{A}$-bimodule, denoted by $L^2(Tr)$. In [1], it is shown that if $\mathcal{A}$ is a unital C^*-algebra with property (T) which is nuclear, then for any tracial state Tr on $\mathcal{A}$, the left action of $\mathcal{A}$ on the Hilbert space $L^2(Tr)$ is *completely atomic*, that is, $L^2(Tr)$ decomposes as a direct sum of finite dimensional $\mathcal{A}$-submodules. This implies that if $\mathcal{A}$ is a unital C^*-algebra with property (T), and that there exists a tracial state Tr on $\mathcal{A}$ such that $L^2(Tr)$ is not completely atomic, then $\mathcal{A}$ is not nuclear.

Corollary 3.14. *Let $(\mathcal{A}, \mathcal{G}, \alpha)$ be a dynamical system such that $\mathcal{G}$ is a discrete group and $\mathcal{A}$ is nuclear and $\mathcal{G}$ amenable. Suppose that there exists a tracial state Tr of $\mathcal{A} \times_\alpha \mathcal{G}$ such that $L^2(Tr)$ is not completely atomic. Then $(\mathcal{A}, \mathcal{G}, \alpha)$ does not have property (T).*

Proof. Since $\mathcal{G}$ is amenable and $\mathcal{A}$ is nuclear, so $\mathcal{A} \times_\alpha \mathcal{G}$ is nuclear (see [10]). As cited above $\mathcal{A} \times_\alpha \mathcal{G}$ does not have property (T). So by Theorem 3.12, $(\mathcal{A}, \mathcal{G}, \alpha)$ does not have property (T). □

Note that even if $\mathcal{A} \times_\alpha \mathcal{G}$ has strong property (T) and α is trivial, it does not follow that $\mathcal{G}$ has property (T).

Proposition 3.15. *Let $\mathcal{G}$ be a locally compact and σ-compact group and $\mathcal{N}$ a closed subgroup of $\mathcal{G}$. The following properties are equivalent:*

(i) $(\mathcal{G}, \mathcal{N})$ has property (T),

(ii) if a unitary representation $(\pi, \mathcal{H})$ of $\mathcal{G}$ almost has invariant vectors, that is, if it has (Q, ε)-invariant vectors for every compact subset Q of $\mathcal{G}$ and every $\varepsilon > 0$, then $\mathcal{H}$ contains a non-zero finite dimensional subspace which is invariant under $\mathcal{N}$.

We will now use the same technique as in the proof of Theorem 6 in [1] to obtain the following theorem, using the above proposition from [1].

Theorem 3.16. *Let $\mathcal{A}$ be a commutative unital C^*-algebra, and $\mathcal{G}$ a countable discrete group such that there exists a faithful representation of $\mathcal{A}$ to the Hilbert space $\ell^2(\mathcal{G})$. If $C_r^*(\mathcal{G}) \otimes_{\min} \mathcal{A}$ has property (T), then $\mathcal{G}$ has property (T).*

Proof. Viewing $C_r^*(\mathcal{G}) \otimes_{\min} \mathcal{A}$ as $\mathcal{A} \times_{\alpha r} \mathcal{G}$ in dynamical system $(\mathcal{A}, \mathcal{G}, \alpha)$ with α trivial, suppose $\mathcal{A} \times_{\alpha r} \mathcal{G}$ has property (T). Choose a finite subset $\mathcal{F}$ of $\mathcal{A} \times_{\alpha r} \mathcal{G}$ and $\varepsilon > 0$ as in Definition 3.1. We may assume that $\|y\| \leq 1$ for all $y \in \mathcal{F}$. Take an element $\xi_0 \in \ell^2(\mathcal{G})$ such that $\|\xi_0\| = 1$. One can check that there exists a finite subset Q of $\mathcal{G}$ such that:

$$\sum_{\gamma \in \mathcal{G} - Q} \|y(\delta_e \otimes \xi_0)(\gamma)\|^2 < \frac{\varepsilon^2}{9},$$

for all $y \in \mathcal{F}$. Assume that $(\pi, \mathcal{H})$ is a unitary representation of $\mathcal{G}$ almost has invariant vectors. Choose a unit vector $\xi \in \mathcal{H}$ such that is $(Q, \frac{\varepsilon}{3})$-invariant vector. Define a representation μ of $\mathcal{A}$ as well as two unitary representations π_1, π_2 of $\mathcal{G}$ on the Hilbert space tensor product $\ell^2(\mathcal{G}, \ell^2(\mathcal{G})) \otimes \mathcal{H}$ by

$$\mu(a) = \pi_\alpha(a) \otimes id,$$

and,

$$\pi_1(\gamma) = \lambda_\alpha(\gamma) \otimes id, \qquad \pi_2(\gamma) = \mu_\alpha(\gamma) \otimes \pi(\gamma),$$

for all $a \in \mathcal{A}, \gamma \in \mathcal{G}$, where μ_α is a representation of $\mathcal{G}$ on the Hilbert space $\ell^2(\mathcal{G}, \ell^2(\mathcal{G}))$ defined by $\mu_\alpha(\gamma)\widetilde{\xi}(s) = \widetilde{\xi}(s\gamma)$ for all $\gamma, s \in \mathcal{G}$ and $\widetilde{\xi} \in \ell^2(\mathcal{G}, \ell^2(\mathcal{G}))$.

Since (μ, π_1) and (μ, π_2) are covariant representations, are equivalent to multiples of the regular representation $(\pi_\alpha, \lambda_\alpha)$, they extend to commuting representations of $\mathcal{A} \times_{\alpha r} \mathcal{G}$, so that $\ell^2(\mathcal{G}, \ell^2(\mathcal{G})) \otimes \mathcal{H}$ is a Hilbert bimodule on $\mathcal{A} \times_{\alpha r} \mathcal{G}$.

Let $\widetilde{\eta} = \widetilde{\xi} \otimes \xi$, where $\widetilde{\xi} \in \ell^2(\mathcal{G}, \ell^2(\mathcal{G}))$ is defined by $\widetilde{\xi}(e) = \xi_0$ and $\widetilde{\xi}(\gamma) = 0$ otherwise. For any $y \in \mathcal{F}$, we have

$$\begin{aligned} \|y \cdot \widetilde{\eta} - \widetilde{\eta} \cdot y\|^2 &= \sum_{\gamma \in \mathcal{G}} \|y(\delta_e \otimes \xi_0)(\gamma)\|^2 \|\pi(\gamma)(\xi) - \xi\|^2 \\ &\leq \frac{4\varepsilon^2}{9} + \sum_{\gamma \in Q} \|y(\delta_e \otimes \xi_0)(\gamma)\|^2 \|\pi(\gamma)(\xi) - \xi\|^2 \\ &\leq \frac{4\varepsilon^2}{9} + \frac{\varepsilon^2}{9} \\ &< \varepsilon^2. \end{aligned}$$

Therefore, there exists a non-zero vector η in $\ell^2(\mathcal{G}, \ell^2(\mathcal{G})) \otimes \mathcal{H}$ which is $\mathcal{A} \times_{\alpha r} \mathcal{G}$-central. Viewing η as a non-zero vector in the Hilbert space $\ell^2(\mathcal{G}, \ell^2(\mathcal{G}, \mathcal{H}))$, in particular, we have

$$\eta(\gamma t \gamma^{-1})(s) = \pi(\gamma)(\eta(t)(s)),$$

for all $\gamma, t, s \in \mathcal{G}$. Then $\gamma \mapsto \|\eta(\gamma)\|$ is a non-zero function in $\ell^2(\mathcal{G})$ which is invariant under conjugation by elements of $\mathcal{G}$. Let $t_0 \in \mathcal{G}$ be such that $\eta(t_0) \neq 0$. It follows that $\{\gamma t_0 \gamma^{-1} \mid \gamma \in \mathcal{G}\}$ is a finite subset of $\mathcal{G}$. Let $s_0 \in \mathcal{G}$ be such that $\eta(t_0)(s_0) \neq 0$. Then $\{\eta(\gamma t_0 \gamma^{-1})(s_0) \mid \gamma \in \mathcal{G}\}$ is finite, hence $\{\pi(\gamma)(\eta(t_0)(s_0)) \mid \gamma \in \mathcal{G}\}$ is a finite subset of $\mathcal{H}$ and its linear span defines a non-zero finite dimensional invariant subspace under $\mathcal{G}$. It follows from Proposition 3.15 that $\mathcal{G}$ has property (T). □

Remark 3.17. (i) Let $\mathcal{G}$ be a countable discrete group. Since all finite dimensional C^*-algebras have strong property (T) (see [4]), using Theorems 3.9, 3.12 and 3.16, $\mathcal{G}$ has property (T) if and only if $C_r^*(\mathcal{G})$ has property (T). This is a well-known result of Bekka (see [1]).

(ii) Let $\mathcal{G}$ be a countable discrete abelian group. Since $\mathcal{G}$ is amenable there exists a faithful representation of $C^*(\mathcal{G})$ in the Hilbert space $\ell^2(\mathcal{G})$. In fact the regular representation can be extended to an $*$-isomorphism between the group C^*-algebra $C^*(\mathcal{G})$ and the reduce group C^*-algebra $C_r^*(\mathcal{G})$, and we have $C^*(\mathcal{G}) \cong C_r^*(\mathcal{G})$. Using Lemma 3.11 and Theorems 3.12, 3.16 it follows that $\mathcal{G}$ has property (T) if and only if $C_r^*(\mathcal{G}) \otimes_{\min} C_r^*(\mathcal{G})$ has property (T), by choosing $\mathcal{A} = C^*(\mathcal{G})$ and α trivial in dynamical system $(\mathcal{A}, \mathcal{G}, \alpha)$.

References

[1] M. B. Bekka, *Property (T) for C^*-algebras*, Bull. London Math. Soc. 38 (2006), 857 - 867.
[2] B. Bekka, P. de la Harpe and A. Valette, *Kazhdan's Property (T)*, New Mathematical Monographs 11, Cambridge University Press, Cambridge 2008.
[3] A. Connes and V. Jones, *Property (T) for von Neumann algebras*, Bull. London Math. Soc. 17 (1985), 51 - 62.
[4] Chi-Wai Leung and Chi-Keung Ng, *Property (T) and strong Property (T) for unital C^*-algebras*, J. Func. Anal. 256 (2009), 3055 - 3070.
[5] K. R. Davidson, *C^*-algebras by example*, Fields Inst. Monograph 6, Amer. Math. Soc., Providence 1996.
[6] P. Jolissaint, *Property (T) for pairs of topological groups*, Enseign. Math., 215 (2005), 31 - 45.
[7] D. Kazhdan, *Connection of the dual space of a group with the structure of its closed subgroups*, Funct. Anal. Appl. (1967), 63 - 65.
[8] G. J. Murphy, *C^*-algebras and operator theory*, Academic Press, San Diego 1990.
[9] M. Takesaki, *Theory of operator algebras*, Springer, Berlin 2003.
[10] D. Williams, *Crossed products of C^*-algebras*, Math. Surveys Monogr., 134, Amer. Math. Soc., Providence 2007.