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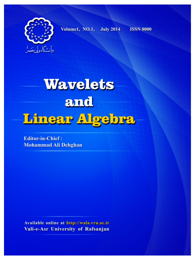

Linear preservers of two-sided matrix majorization

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ABSTRACT

For vectors $X, Y \in \mathbb{R}^n$, it is said that X is left matrix majorized by Y if for some row stochastic matrix R; $X = RY$. The relation $X \sim_\ell Y$, is defined as follows: $X \sim_\ell Y$ if and only if X is left matrix majorized by Y and Y is left matrix majorized by X. A linear operator $T: \mathbb{R}^p \to \mathbb{R}^n$ is said to be a linear preserver of a given relation $\prec$ if $X \prec Y$ on $\mathbb{R}^p$ implies that $TX \prec TY$ on $\mathbb{R}^n$. The linear preservers of $\prec_\ell$ from $\mathbb{R}^p$ to $\mathbb{R}^n$ are characterized before. In this parer we characterize the linear preservers of $\sim_\ell$ from $\mathbb{R}^p$ to $\mathbb{R}^n$, $p \geq 3$. In fact we show that the linear preservers of $\sim_\ell$ from $\mathbb{R}^p$ to $\mathbb{R}^n$ are the same as the linear preservers of $\prec_\ell$ from $\mathbb{R}^p$ to $\mathbb{R}^n$, but for $p = 2$, they are not the same.

1. Introduction

Let M_{nm} be the algebra of all $n \times m$ real matrices, and the usual notation $\mathbb{R}^n$ for $n \times 1$ real vectors. A matrix $R = [r_{ij}] \in M_{nm}$ is called a *row stochastic* matrix if $r_{ij} \geq 0$ and $\Sigma_{k=1}^{m} r_{ik} = 1$ for

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all i, j. For vectors $X, Y \in \mathbb{R}^n$, we say X is left (resp. right) matrix majorized by Y and write $X \prec_\ell Y$ (resp. $X \prec_r Y$) if for some row stochastic matrix R, $X = RY$ (resp. $X = YR$). For more information about right and left matrix majorization and some other majorizations, we refer to [1, 4, 5, 12]. Also for $X, Y \in \mathbb{R}^n$, we write $X \sim_\ell Y$, if $X \prec_\ell Y \prec_\ell X$.

A linear operator $T : \mathbb{R}^p \to \mathbb{R}^n$ is said to be a linear preserver of a given relation $\prec$ if $X \prec Y$ on $\mathbb{R}^p$ implies that $TX \prec TY$ on $\mathbb{R}^n$. The linear preservers of $\prec_\ell$ and $\prec_r$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ are fully characterized in [6] and [7]. For more information about linear preservers of majorization we refer the reader to [1, 2, 3, 11]. In [8], the authors studied the linear preservers of $\prec_\ell$ from $\mathbb{R}^p$ to $\mathbb{R}^n$, where p and n are not necessarily equal, and characterized the structure of these linear preservers of $\prec_\ell$ for $p \le n \le p(p-1)$. In [9], by a geometric approach one can see the characterization of linear preservers of $\prec_\ell$ from $\mathbb{R}^p$ to $\mathbb{R}^n$ without any additional conditions on p and n. In [10], the authors characterized all linear preservers of $\sim_\ell$ from $\mathbb{R}^2$ to $\mathbb{R}^n$. Here we focus on this method and we characterize all linear preservers of $\sim_\ell$ from $\mathbb{R}^p$ to $\mathbb{R}^n$, for $p \ge 3$.

We shall use the following conventions throughout the paper. Let $T : \mathbb{R}^p \to \mathbb{R}^n$ be a nonzero linear operator and let $[T] = [t_{ij}]$ denote the matrix representation of T with respect to the standard bases $\{e_1, e_2, \ldots, e_p\}$ of $\mathbb{R}^p$ and $\{f_1, f_2, \ldots, f_n\}$ of $\mathbb{R}^n$. If $p = 1$, then all linear operators on $\mathbb{R}^1$ are preservers of $\sim_\ell$. Thus, we assume $p \ge 2$. For every i ($i = 1, \ldots, k$) let A_i be $m_i \times p$ matrix. We use the notation $[A_1/A_2/\ldots/A_k]$ to denote the corresponding $(m_1 + m_2 + \cdots + m_k) \times p$ matrix. Put

$$\begin{aligned} \mathbf{a} : &= \max\{t_{ij} \mid 1 \le i \le n, 1 \le j \le p\}, \\ \mathbf{b} : &= \min\{t_{ij} \mid 1 \le i \le n, 1 \le j \le p\}. \end{aligned} \tag{1.1}$$

We also use the notation P for the permutation matrix such that $P(e_i) = e_{i+1}$, $1 \le i \le p-1$, $P(e_p) = e_1$. Let I denote the $p \times p$ identity matrix, and let $r, s \in \mathbb{R}$ be such that $rs < 0$. Define the $p(p-1) \times p$ matrix $\mathcal{P}_p(r,s) = [P_1/P_2/\ldots/P_{p-1}]$ where $P_j = rI + sP^j$, for all j ($1 \le j \le p-1$). It is clear that up to a row permutation the matrices $\mathcal{P}_p(r,s)$ and $\mathcal{P}_p(s,r)$ are equal. Also define $\mathcal{P}_p(r,0) := rI$, $\mathcal{P}_p(0,s) := sI$ and $\mathcal{P}_p(0,0)$ as $1 \times p$ zero matrix.

Throughout the paper, for a given vector $x \in \mathbb{R}^n$, $\max x$ and $\min x$ denote the maximum and minimum values of components of x, respectively.

Let $T : \mathbb{R}^2 \to \mathbb{R}^n$ be a linear operator and let $[T] = [T_1/\ldots/T_n]$, where $T_i = [t_{i1}, t_{i2}]$, for every i ($1 \le i \le n$). Let

$$\Delta := \mathrm{Conv}(\{(t_{i1}, t_{i2}), (t_{i2}, t_{i1}), 1 \le i \le n\}) \subseteq \mathbb{R}^2, \tag{1.2}$$

where $\mathrm{Conv}(A)$ denotes the convex hull of a set A. Also, let $C(T)$ denote the set of all corners of Δ.

Now, we recall the characterization of linear preservers of $\prec_\ell$ from $\mathbb{R}^p$ to $\mathbb{R}^n$, beginning with $p = 2$.

Theorem 1.1. [9, *Theorem* 3.3] *Let $T : \mathbb{R}^2 \to \mathbb{R}^n$ be a linear operator. Then, T is a linear preserver of $\prec_\ell$ if and only if $\mathcal{P}_2(x,y)$ is a submatrix of $[T]$ and $xy \le 0$ for all $(x,y) \in C(T)$.*

For the case $p \ge 3$ we first need some definitions.

Definition 1.2. Let $T : \mathbb{R}^p \to \mathbb{R}^n$ be a linear operator. We denote by P_i (resp. N_i) the sum of the nonnegative (resp. *nonpositive*) entries in the i^{th} row of $[T]$. If all the entries in the i^{th} row are positive (resp. *negative*), we define $N_i = 0$ (resp. $P_i = 0$).

Definition 1.3. Let $T : \mathbb{R}^p \to \mathbb{R}^n$ be a linear operator. Define

$$\begin{aligned} \Delta : &= \text{Conv}(\{(P_i, N_i), (N_i, P_i) : 1 \le i \le n\}), \\ E(T) : &= \{(P_i, N_i) : (P_i, N_i) \text{ is a corner of } \Delta\}, \end{aligned}$$

where P_i, N_i are as in Definition 1.2.

Theorem 1.4. [9, *Theorem* 4.6] *Let* $T: \mathbb{R}^p \to \mathbb{R}^n$, $p \ge 3$, *and let* $E(T)$ *be as in Definition 1.3. Then* T *preserves* $\prec_\ell$ *if and only if* $\mathcal{P}_p(\alpha,\beta)$ *is a submatrix of* $[T]$ *for all* $(\alpha,\beta) \in E(T)$.

For $X, Y \in \mathbb{R}^p$, we define $X \sim_\ell Y$, when $X \prec_\ell Y \prec_\ell X$. To obtain our main result (a complete characterization of linear preservers of $\sim_\ell$ from $\mathbb{R}^p$ to $\mathbb{R}^n$), we consider the case $p \ge 3$, and recall the case $p = 2$ which is characterized in [10], as follows.

For $x, y \in R$, define $Q_2(x, y) = \begin{bmatrix} x & y \\ y & x \end{bmatrix}$, if $x \ne y$ and $Q_2(x, x) = [x \ \ x]$.

Theorem 1.5. [8 *and* 10, *Theorem* 2.6] *Let* $T: \mathbb{R}^2 \to \mathbb{R}^n$ *be a linear operator. Then* T *is a linear preserver of* $\sim_\ell$ *if and only if* $Q_2(x, y)$ *is a submatrix of* $[T]$, *for all* $(x, y) \in C(T)$, *where* $C(T)$ *denote the set of all corners of* Δ *as in (1.2).*

Obviously, if $T: \mathbb{R}^p \to \mathbb{R}^n$ is a linear preserver of $\prec_\ell$, then T is a linear preserver of $\sim_\ell$. But by the following example the converse is not true for $p = 2$.

Let $[T] = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$. Then T is a linear preserver of $\sim_\ell$, but by Theorem 1.1, T is not a linear preserver of $\prec_\ell$.

2. Linear Preservers of $\sim_\ell$ on $\mathbb{R}^p$, $p \ge 3$

Lemma 2.1. *Let* $x, y \in \mathbb{R}^n$. *Then the following assertions are true*

(a) $x \prec_\ell y$ *if and only if* $\min y \le \min x \le \max x \le \max y$.

(b) $x \sim_\ell y$ *if and only if* $\min x = \min y$ *and* $\max x = \max y$.

Proof. Let $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$ and let $x \prec_\ell y$. Therefore $x = Ry$ for some $n \times n$ row stochastic matrices R. It follows that $x_i \in \text{Conv}(\{y_1, \dots, y_n\})$ for all i $(1 \le i \le n)$ and hence $\min y \le \min x \le \max x \le \max y$. Since $x \sim_\ell y$ when $x \prec_\ell y \prec_\ell x$, (b) is a consequence of (a). □

Lemma 2.2. *Let* T *be a linear operator on* $\mathbb{R}^p$. *Let* $[T] = [T_1/\dots/T_k]$ *such that* T_i *is a linear preserver of* $\sim_\ell$, *for all* i $(1 \le i \le k)$. *Then* T *is a linear preserver of* $\sim_\ell$.

Proof. Let $x \in \mathbb{R}^p$, $\min T_i x = \alpha_i$ and $\max T_i x = \beta_i$, for all i $(1 \le i \le k)$. Then $\min Tx = \min \alpha_i$ and $\max Tx = \max \beta_i$. Since $Tx = [T_1x/\dots/T_kx]$, for all $x \in \mathbb{R}^p$, T is a linear preserver of $\sim_\ell$. □

Lemma 2.3. *Let* T *be a linear operator on* $\mathbb{R}^p$, *and let* $p \ge 3$. *If* $[T] = \mathcal{P}_p(\alpha,\beta)$, $\beta \le 0 \le \alpha$, *then* T *is a linear preserver of* $\sim_\ell$. *In general, if*

$$[T] = [\mathcal{P}_p(\alpha_1,\beta_1)/\cdots/\mathcal{P}_p(\alpha_k,\beta_k)],\ \beta_i \le 0 \le \alpha_i,\ \forall i = 1,\dots,k$$

then T *is a linear preserver of* $\sim_\ell$.

Proof. Let $x = (x_1, \dots, x_p)^t, y = (y_1, \dots, y_p)^t \in \mathbb{R}^p$ and $x \sim_\ell y$. By Lemma 2.1, $\min x = \min y$ and $\max x = \max y$. Let $m = \min x = \min y$ and $M = \max x = \max y$. It is easy to show that $\alpha m + \beta M \leq \alpha x_i + \beta x_j$, also $\alpha m + \beta M \leq \alpha y_i + \beta y_j$, $\forall i \neq j \in \{1, \dots, p\}$. Hence $\min Tx = \alpha m + \beta M = \min Ty$. Similarly, $\max Tx = \alpha M + \beta m = \max Ty$. Thus by Lemma 2.1, $Tx \sim_\ell Ty$. The second statement is a consequence of Lemma 2.2. □

Definition 2.4. Let $T : \mathbb{R}^p \to \mathbb{R}^n$ be a linear operator and let $[T] = [T_1 / \dots / T_n]$. Define

$$\Omega := \text{Conv}(\{T_i = (t_{i1}, \dots, t_{ip}), 1 \leq i \leq n\}) \subseteq \mathbb{R}^p.$$

Also, let $C(T)$ be the set of all corners of Ω.

Lemma 2.5. *Let $T : \mathbb{R}^p \to \mathbb{R}^n, p \geq 3$, be a linear preserver of $\sim_\ell$ and $[T] = [T_1 / \dots / T_n]$, where $T_i = (t_{i1}, t_{i2}, \dots, t_{ip})$, for all i $(1 \leq i \leq n)$. Suppose there exists i $(1 \leq i \leq n)$ such that $t_{il} > 0$, for all l $(1 \leq l \leq p)$ or $t_{il} < 0$, for all l $(1 \leq l \leq p)$. Then $T_i \notin C(T)$.*

Proof. Without loss of generality we can assume $[T]$ has no identical rows. Since T is a linear preserver of $\sim_\ell$ if and only if ηT is a linear preserver of $\sim_\ell$ for all $\eta \neq 0$, we can assume that there exists some i such that $T_i \in C(T)$ and $t_{il} > 0$ for all l $(1 \leq l \leq p)$, and reach a contradiction. Since $T_i \in C(T)$, there exists $x = (x_1, \dots, x_p)^t \in \mathbb{R}^p$ such that

$$t_{i1}x_1 + t_{i2}x_2 + \cdots + t_{ip}x_p < t_{j1}x_1 + t_{j2}x_2 + \cdots + t_{jp}x_p, \ \forall j \neq i. \tag{2.1}$$

Let $m = \min x$ and $M = \max x$. Assuming $x_1 = m$ and $x_p = M$, will cause no loss of generality. Also, without loss of generality, we can assume $m \neq M$ and there exists some $1 < k < n$ such that $m < x_k < M$, because by (2.1) we can choose $\varepsilon_0 > 0$ small enough so that for all $0 < \varepsilon \leq \varepsilon_0$,

$$t_{i1}(m - \varepsilon) + t_{i2}x_2 + \cdots + t_{ip}(M + \varepsilon) < t_{j1}(m - \varepsilon) + t_{j2}x_2 + \cdots + t_{jp}(M + \varepsilon), \ \forall j \neq i.$$

Fix $0 < \varepsilon \leq \varepsilon_0$ and define $x_\varepsilon = (m - \varepsilon, x_2, \dots, x_{p-1}, M + \varepsilon)^t \in \mathbb{R}^p$, thus $\min Tx_\varepsilon = t_{i1}(m - \varepsilon) + \cdots + t_{ip}(M + \varepsilon)$ and $\min x_\varepsilon = m - \varepsilon \neq M + \varepsilon = \max x_\varepsilon$. Suppose $m \neq M$ and $m < x_k < M$ for some $1 < k < n$. Choose $\alpha > 0$ small enough such that $m < x_k + \alpha < M$ and

$$t_{i1}x_1 + \cdots + t_{ik}(x_k + \alpha) + \cdots + t_{ip}x_p < t_{j1}x_1 + \cdots + t_{jk}(x_k + \alpha) + \cdots + t_{jp}x_p, \ \forall j \neq i.$$

Define $y = (x_1, \dots, x_k + \alpha, \dots, x_p)^t$. Since $t_{ik} > 0$,

$$\min Tx = t_{i1}x_1 + \cdots + t_{ip}x_p < t_{i1}x_1 + \cdots + t_{ik}(x_k + \alpha) + \cdots + t_{ip}x_p = \min Ty.$$

But $y \sim_\ell x$, a contradiction. □

Let $T : \mathbb{R}^p \to \mathbb{R}^n$ be a linear operator. Without loss of generality, we assume that $[T] = [T^p / T^n / \widetilde{T}]$, where all entries of T^p (resp. T^n) are positive (resp. negative) and each row of $\widetilde{T}$ has nonnegative and nonpositive entries.

Corollary 2.6. *Let T and $\widetilde{T}$ be as above, then T is a linear preserver of $\sim_\ell$ if and only if $C(T) = C(\widetilde{T})$ and $\widetilde{T}$ is a linear preserver of $\sim_\ell$.*

Proof. Let T be a linear preserver of $\sim_\ell$. By Lemma 2.5, the rows of T^p and T^n can not be in $C(T)$, hence $C(T) = C(\widetilde{T})$. Thus if $x \in \mathbb{R}^p$, $\min Tx = \min \widetilde{T}x$ and $\max Tx = \max \widetilde{T}x$. Therefore $\widetilde{T}$ is a linear preserver of $\sim_\ell$. Conversely, let $C(T) = C(\widetilde{T})$, and suppose $\widetilde{T}$ is a linear preserver of $\sim_\ell$. Then $\max Tx = \max \widetilde{T}x$ and $\min Tx = \min \widetilde{T}x$, for all $x \in \mathbb{R}^p$. Hence T is a linear preserver of $\sim_\ell$. □

Lemma 2.7. *Let $p \geq 3$ and let $T : \mathbb{R}^p \to \mathbb{R}^n$, be a linear preserver of $\sim_\ell$. Let $[T] = [T_1/\cdots/T_n]$, where $T_i = (t_{i1}, \ldots, t_{ip})^t$, for every i $(1 \leq i \leq n)$. Then $\mathbf{b} \leq \sum_{l=1}^{p} t_{il} \leq \mathbf{a}$, for all i, where $\mathbf{a}$ and $\mathbf{b}$ are as in* (1).

Proof. Without loss of generality we can assume $[T]$ has no identical rows. If $i, j \in \{1, \ldots, p\}$, we have $e_i \sim_\ell e_j$ and so $Te_i \sim_\ell Te_j$ which implies that $\max Te_i = \max Te_j$ and $\min Te_i = \min Te_j$. Hence $\max Te_i = \mathbf{a}$ and $\min Te_i = \mathbf{b}$, for all $i \in \{1, \ldots, n\}$. Since $\sum_{j\in J} e_j \sim_\ell e_1$, for all $J \subset \{1, \ldots, p\}$, $\min T(\sum_{j\in J} e_j) = \min Te_1 = \mathbf{b}$ and $\max T(\sum_{j\in J} e_j) = \max Te_1 = \mathbf{a}$. Therefore if $| J |= p - 1$ then $\mathbf{b} \leq \sum_{j\in J} t_{il} \leq \mathbf{a}$, for all i $(1 \leq i \leq n)$. Which implies, if $\sum_{l=1}^{p} t_{il} > \mathbf{a}$ (resp. $\sum_{l=1}^{p} t_{il} < \mathbf{b}$) for some i $(1 \leq i \leq n)$, then $t_{il} > 0$ (resp. $t_{il} < 0$) for all l $(1 \leq l \leq p)$. Without loss of generality let $t_{il} > 0$ for all l $(1 \leq l \leq p)$ and $\sum_{l=1}^{p} t_{il} > \mathbf{a}$. There exists k $(1 \leq k \leq n)$ such that $t_{kl} > 0$ for all l $(1 \leq l \leq n)$ and $\sum_{l=1}^{p} t_{kl} = \max\{\sum_{l=1}^{p} t_{il} : 1 \leq i \leq n\}$. It is clear that $\sum_{l=1}^{p} t_{kl} > \mathbf{a}$ and $\max Te = \sum_{l=1}^{p} t_{kl}$, where $e = (1, \ldots, 1)^t \in \mathbb{R}^p$. Since $[T]$ has no identical rows, there exists some $\varepsilon > 0$ and some $x \in \mathbb{R}^p$ with $\max x = 1 + \varepsilon$ and $\min x \geq 1$ such that

$$t_{k1}x_1 + \ldots + t_{kp}x_p > t_{i1}x_1 + \ldots + t_{ip}x_p, \ \forall i \neq k.$$

Hence $T_k \in C(T)$, a contradiction. □

Since $T : \mathbb{R}^p \to \mathbb{R}^n$ is a linear preserver of $\sim_\ell$ if and only if ηT is so for all nonzero real number η, we can assume $\mathbf{a} > 0$ and $| \mathbf{b} |\leq \mathbf{a}$.

Corollary 2.8. *Let $T : \mathbb{R}^p \to \mathbb{R}^n$, $p \geq 3$, be a linear preserver of $\sim_\ell$. Then $p \leq n$, moreover, if a row of $[T]$ contains an entry equal to $\mathbf{a}$ (resp. $\mathbf{b}$), then all other entries of that row are nonpositive (resp. nonnegative).*

Proof. As in the proof of Lemma 2.7, $\max Te_i = \mathbf{a}$ and $\min Te_i = \mathbf{b}$, for all $i \in \{1, \ldots, n\}$. Therefore every column of $[T]$ contains at least one entry equal to $\mathbf{a}$ and at least one entry equal to $\mathbf{b}$. Also by Lemma 2.7, since $\mathbf{b} \leq \sum_{l=1}^{p} t_{il} \leq \mathbf{a}$, for all i $(1 \leq i \leq p)$, every row of $[T]$ has at most one entry equal to $\mathbf{a}$ and at most one entry equal to $\mathbf{b}$. Now, $\mathbf{a} > 0$, since every column of $[T]$ has at least one entry equal to $\mathbf{a}$ and every row of $[T]$ has at most one entry equal to $\mathbf{a}$, it follows that $p \leq n$. The second statement is a consequence of Lemma 2.7. □

Theorem 2.9. *Let $T : \mathbb{R}^p \to \mathbb{R}^n$ be a linear preserver of $\sim_\ell$, and let $E(T)$ be as in Definition 1.3. If $(P_r, N_r) \in E(T)$ for some r $(1 \leq r \leq n)$ then there exists $k \in [r] = \{1 \leq i \leq n : P_i = P_r, N_i = N_r\}$ such that $T_k \in C(T)$.*

Proof. Let $(P_r, N_r) \in E(T)$ for some r $(1 \leq r \leq n)$. Then there exist $m \leq M$ such that

$$P_r m + N_r M < P_j m + N_j M, \ j \notin [r]. \tag{2.2}$$

Now we have the following cases for the entries of the r^{th} row of $[T]$.
Case 1. If $t_{rl} > 0$ for all l $(1 \leq l \leq p)$, then $P_r > 0$ and $N_r = 0$, which together with (2.2) implies that $P_r m < P_j m + N_j M,\ j \notin [r]$.
It is easy to check that

$$P_r m < P_j m + N_j M \leq P_j m + N_j m = (P_j + N_j)m,\ j \notin [r]. \tag{2.3}$$

Thus $P_r m < (P_j + N_j)m$ and hence $\min T(me) = P_r m$. By Lemma 2.5, there exists $k \neq r$ such that $P_r m = \min T(me) = \sum_{l=1}^{p} t_{kl} m = P_k m + N_k m$, and $T_k \in C(T)$. By (2.3) $k \in [r]$. Thus there exists $k \in [r]$ such that $T_k \in C(T)$.
Case 2. If $t_{ri} \leq 0$ for some i $(1 \leq i \leq p)$ and $t_{rj} > 0$ for some $j \neq i$, $(1 \leq j \leq p)$. Define $x \in \mathbb{R}^p$ such that $\min x = m$ and $\max x = M$. Then there exists k $(1 \leq k \leq n)$, such that $\min Tx = \sum_{l=1}^{p} t_{kl} x_l$. Hence (2.2) implies

$$P_r m + N_r M \leq P_k m + N_k M \leq \sum_{l=1}^{p} t_{kl} x_l = \min Tx.$$

Define $y \in \mathbb{R}^p$ by $y_l = m$ if $t_{rl} > 0$ and $y_l = M$ if $t_{rl} \leq 0$. Obviously $y \sim_\ell x$. Since T is a linear preserver of $\sim_\ell$, $Ty \sim_\ell Tx$ which implies that

$$P_k m + N_k M \leq \sum_{l=1}^{p} t_{kl} x_l = \min Tx = \min Ty \leq P_r m + N_r M.$$

Therefore by (2.2) $P_k m + N_k M = P_r m + N_r M$. Thus $k \in [r]$ and $\min Tx = \sum_{l=1}^{p} t_{kl} x_l$. Hence $T_k \in C(T)$ for some $k \in [r]$.
Case 3. If $t_{rl} < 0$ for all l $(1 \leq l \leq p)$ then $N_r < 0$ and $P_r = 0$, which together with (2.2) implies that $N_r M < P_j m + N_j M,\ j \notin [r]$. Like the Case 1

$$N_r M < P_j m + N_j M \leq P_j M + N_j M = (P_j + N_j)M,\ j \notin [r]. \tag{2.4}$$

Therefore $\min T(Me) = N_r M$. By Lemma 2.5, there exists $k \neq r$ such that $N_r M = \min T(Me) = \sum_{l=1}^{p} t_{kl} M = P_k M + N_k M$, and $T_k \in C(T)$. By (2.4) $k \in [r]$. Thus there exists $k \in [r]$ such that $T_k \in C(T)$.
Case 4. If $t_{ri} < 0$ for some i $(1 \leq i \leq p)$ and $t_{rj} \geq 0$ for some $j \neq i$ $(1 \leq j \leq p)$. We can prove it the same as Case 2 and conclude that there exists $k \in [r]$ such that $T_k \in C(T)$. □

Corollary 2.10. *Let $T : \mathbb{R}^p \to \mathbb{R}^n$ be a linear preserver of $\sim_\ell$. If $1 \leq r \leq n$ and for some $m \leq M$, $P_r m + N_r M < P_j m + N_j M$, for all $j \notin [r]$. Then there exists $k \in [r]$ and $x \in \mathbb{R}^p$ with $\min x = m$ and $\max x = M$ such that $\min Tx = P_r m + N_r M = \sum_{l=1}^{p} t_{kl} x_l$.*

Proof. Consider four Cases of the proof of Theorem 2.9. As we see in the proof of Case 1 $T_k \in C(T)$, $k \neq r$ and $k \in [r]$. By Lemma 2.5 there exists l $(1 \leq l \leq p)$ such that $t_{kl} = 0$. Define $x \in \mathbb{R}^p$ such that $x_l = M$ and $x_i = m$ for all $i \neq l$. There exists j $(1 \leq j \leq p)$ such that $\min Tx = \sum_{l=1}^{p} t_{jl} x_l$. Hence $P_r m + N_r M \leq P_j m + N_j M \leq \sum_{l=1}^{p} t_{jl} x_l \leq \min Tx$. Since $(Tx)_k = P_k m + N_k M = P_r m + N_r M$, $\min Tx = P_r m + N_r M$. By a similar proof, the statement holds in Case 3. As in the proof of Cases 2 and 4 of Theorem 2.9, the statement is immediate. □

Theorem 2.11. *Let $T : \mathbb{R}^p \to \mathbb{R}^n$ $(p \geq 3)$ be a linear operator. T preserves $\sim_\ell$ if and only if $\mathcal{P}_p(P, N)$ is a submatrix of $[T]$ for all $(P, N) \in E(T)$.*

Proof. To prove necessity, let T be a preserver of $\sim_\ell$ and let $(P_r, N_r) \in E(T)$. Then there exists $m \leq M$ such that $P_r m + N_r M < P_j m + N_j M$ for all j, $j \notin [r]$. Choose ε_0 small enough so that for all $0 < \varepsilon < \varepsilon_0$,

$$P_r(m - \varepsilon) + N_r(M + \varepsilon) < P_j(m - \varepsilon) + N_j(M + \varepsilon), \quad \forall j \notin [r].$$

Let $0 < \varepsilon < \varepsilon_0$, be fixed and let $x_\varepsilon = (x_1, \ldots, x_p)^t \in \mathbb{R}^p$ with $\min x_\varepsilon = m - \varepsilon$ and $\max x_\varepsilon = M + \varepsilon$. By Corollary 2.10 there exists $k \in [r]$ such that $\min T(x_\varepsilon) = \sum_{l=1}^{p} t_{kl} x_l = P_r(m - \varepsilon) + N_r(M + \varepsilon)$. Fix $i \neq j \in \{1, \ldots, p\}$ and define $y_\varepsilon = (y_1, \ldots, y_p)^t \in \mathbb{R}^p$ such that $y_i = m - \varepsilon$, $y_j = M + \varepsilon$ and $y_l = \alpha_l$, $m - \varepsilon < \alpha_l < M + \varepsilon$, $l \neq i, j$. Since $x_\varepsilon \sim_\ell y_\varepsilon$, $Tx_\varepsilon \sim_\ell Ty_\varepsilon$, there exists $s \in [r]$ such that $\min T(y_\varepsilon) = t_{si}(m - \varepsilon) + t_{sj}(M + \varepsilon) + \sum_{l \neq i,j} t_{sl}\alpha_l = P_r(m - \varepsilon) + N_r(M + \varepsilon) = \min T(x_\varepsilon)$. Since $0 < \varepsilon < \varepsilon_0$ and $m - \varepsilon \leq \alpha_l \leq M + \varepsilon$, $l \neq i, j$ are arbitrary and the number of rows of $[T]$ is finite, there exists $q \in [r]$ such that $t_{qi} = P_r$ and $t_{qj} = N_r$ and $t_{ql} = 0$, for all l, $l \neq i, j$. Also, since $i, j \in \{1, \ldots, n\}$ are arbitrary, $[T]$ has $\mathcal{P}_p(P_r, N_r)$ as a submatrix. For sufficiency, let $E(T) = \{(P_{i_1}, N_{i_1}), \ldots, (P_{i_k}, N_{i_k})\}$. Then up to a row permutation $[T] = [\widehat{T}/Q]$, where $\widehat{T}$ is the operator on $\mathbb{R}^p$ such that $\widehat{T} = [\mathcal{P}_p(P_{i_1}, N_{i_1})/ \ldots /\mathcal{P}_p(P_{i_k}, N_{i_k})]$. By Lemma 2.3, $\widehat{T}$ is a linear preserver of $\sim_\ell$. To prove T is a linear preserver of $\sim_\ell$, we are going to show that $\min Tx = \min \widehat{T}x$ and $\max Tx = \max \widehat{T}x$, for all $x \in \mathbb{R}^p$. Let $x \in \mathbb{R}^p$, obviously $\min Tx \leq \min \widehat{T}x$, it is enough to prove $\min \widehat{T}x \leq \min Tx$. If $\min T(x) = \sum_{l=1}^{p} t_{il} x_l$, for some i, $T_i \in Q$. Let $m = \min x$ and $M = \max x$, therefore $P_i m + N_i M \leq \sum_{l=1}^{p} t_{il} x_l$. There is $1 \leq r \leq n$ such that $(P_r, N_r) \in E(T)$ and $P_r m + N_r M \leq P_i m + N_i M$, because $(P_i, N_i) \in \Delta$ and Δ is convex. By Corollary 2.10 $\min Tx = P_r m + N_r M$. Then $\min \widehat{T}x \leq \min Tx$ and hence $\min \widehat{T}x = \min Tx$. Similarly $\max \widehat{T}x = \max Tx$ and therefore T is a linear preserver of $\sim_\ell$. □

Corollary 2.12. *Let $T : \mathbb{R}^p \to \mathbb{R}^n$, $p \geq 3$ be a linear operator then the following assertions are equivalent*

(a) *T preserves $\prec_\ell$,*

(b) *T preserves $\sim_\ell$,*

(c) *$\mathcal{P}_p(P, N)$ is a submatrix of $[T]$ for all $(P, N) \in E(T)$.*

Problem. [7] characterizes all linear preservers of $\prec_r$ from M_n to M_n, and also from $\mathbb{R}^n$ to $\mathbb{R}^n$. It would be nice to characterize all linear preservers of $\prec_r$ (or $\sim_r$) from $\mathbb{R}^p$ to $\mathbb{R}^n$.

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