Wavelets and Linear Algebra 1 (2014) 33-42

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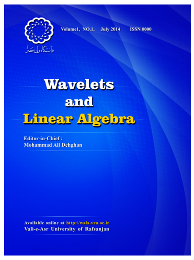

Application of Shannon wavelet for solving boundary value problems of fractional differential equations☆

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Article Info

Article history:
Received 1 October 2013
Accepted 29 April 2014
Available online 1 July 2014
Communicated by Asghar Rahimi

Keywords:
Boundary value problems
Fractional differential equations
Shannon wavelet
Operational matrix

2000 MSC:
65L10
34A08
26A33
65T60

Abstract

Fractional calculus has been used to model physical and engineering processes that are found to be best described by fractional differential equations. Therefore, a reliable and efficient technique as a solution is regarded. This paper develops approximate solutions for boundary value problems of differential equations with non-integer order by using the Shannon wavelet bases. Wavelet bases have different resolution capability for approximating of different functions. Since for Shannon-type wavelets, the scaling function and the mother wavelet are not necessarily absolutely integrable, the partial sums of the wavelet series behave differently and a more stringent condition, such as bounded variation, is needed for convergence of Shannon wavelet series. With nominate Shannon wavelet operational matrices of integration, the solutions are approximated in the form of convergent series with easily computable terms. Also, by applying collocation points the exact solutions of fractional differential equations can be achieved by well-known series solutions. Illustrative examples are presented to demonstrate the applicability and validity of the wavelet base technique. To highlight the convergence, the numerical experiments are solved for different values of bounded series approximation.

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1. Introduction

The fractional calculus is a name for the theory of derivatives and integrals of arbitrary order which unify and generalize the notions of integer-order differentiation and n-fold integration [16]. During the last years, fractional calculus starts to attract much more attention of physicists and mathematicians. In compare with integer order differential equations, fractional differential equations (FDEs) show many advantages over the simulation of natural physical process and dynamic systems with more accurately. For example, the nonlinear oscillation of earthquake can be modeled with fractional derivatives [7], and the fluid dynamic models with fractional derivatives can eliminate the deficiency arising from the assumption of continuum traffic flow [8, 12]. Owing to the increasing applications, a considerable attention has been given to exact and numerical solutions of FDEs.
Analytical solution of FDEs either does not exist or seems hard to find. Therefore, during the last decades, several methods have been used to solve FDEs from numerical points of view, such as Adomian's decomposition method [5, 17, 23], variation iteration method [10, 13, 14, 15], homotopy perturbation method [9, 22], homotopy analysis method [6, 24], spectral methods [4, 19], and other methods. In this paper, we apply the Shannon wavelets bases to approximation the solution of various types of boundary value problems (BVPs) for FDEs.
Wavelet analysis is a relatively new area in mathematic research. It has numerous applications in approximation theory and has been extensively used in the context of numerical approximation in the relevant literature during the last two decades. By the way, mathematicians have employed various types of wavelets. Chen and Wu [2], used Haar wavelet method to solve a class of fractional convection-diffusion equations by variable coefficients. In [20, 21], a CAS wavelet operational matrix of fractional order integration has been derived and is used to solve Fredholm and Volterra integro-differential equations of fractional order. Motivated by the works mentioned above, we derive a Shannon wavelet operational matrix of fractional order integration, and apply it to solve BVPs for FDEs that as follows:

$$D^{\alpha}y(t) = f(t, y(t), D^{\beta}y(t)), \quad 0 \le t \le 1, \tag{1.1}$$

subject to the boundary conditions $y(0) = y_0, y(1) = y_1$, where α, β are positive fractional or integer constants. The method reduces the fractional BVP to algebraic equations system.

The rest of the paper is organized as follows. In Section 2, we introduce some preliminaries of the fractional calculus theory. In Section 3, some relevant properties of the Shannon wavelet bases and function approximation by these bases are presented. Also, operational matrix of integration for Shannon wavelet is obtained. Sections 4,5 are devoted to implementing the Shannon wavelet technique with some numerical experiments.

☆This research was in part supported by a grant from IPM (No. 91650054), and in part by the Research Council of Semnan University.
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2. Preliminaries and basic definitions

In this section, we give some necessary definitions and mathematical preliminaries of the fractional calculus theory [3, 16], which are used further in this paper. There are several definitions of fractional derivative. Most important types of fractional derivatives are the Riemann-Liouville and the Caputo, which can be described as follows:

Definition 2.1. *Riemann-Liouville's definition of the fractional order integration for functions belong to* $L_1[0, a]$ *is as follows:*

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) \mathrm{d}\tau, \qquad 0 \le t \le a. \tag{2.1}$$

Where α*, the order of integration is non-negative and for* $\alpha = 0$*, we set* $I^0 = I$*, the identity operator. Also,* $\Gamma(.)$ *denotes the Gamma function.*

Lemma 2.2. *For* $\alpha, \beta > 0$*,* $f \in C[0, \infty)$*, the Riemann-Liouville fractional order integral has following important properties:*

I) $I^{\alpha}(I^{\beta} f(t)) = I^{\alpha+\beta} f(t)$,

II) $I^{\alpha} t^{\beta} = \dfrac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}$.

Definition 2.3. *The Riemann-Liouville fractional derivative of order* α *is defined as:*

$$D^{\alpha} f(t) = \begin{cases} \dfrac{\mathrm{d}^m f(t)}{\mathrm{d}t^m}, & \alpha = m \in \mathbb{N}, \\ \dfrac{1}{\Gamma(m-\alpha)} \dfrac{\mathrm{d}}{\mathrm{d}t^m} \displaystyle\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-m+1}} \mathrm{d}\tau, & t > 0, 0 < m-1 < \alpha < m, \end{cases} \tag{2.2}$$

and the Caputo's type derivative of this order is defined as:

$$D_*^{\alpha} f(t) = \begin{cases} \dfrac{\mathrm{d}^m f(t)}{\mathrm{d}t^m}, & \alpha = m \in \mathbb{N}, \\ \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} \mathrm{d}\tau, & t > 0, 0 < m-1 < \alpha < m, \end{cases} \tag{2.3}$$

as well as $D^0 = D_*^0 = I$.

Lemma 2.4. *The Riemann-Liouville fractional derivative is the left inverse of the Riemann-Liouville fractional integral of the same order i.e.*

$$D^{\alpha}(I^{\alpha}) f(t) = f(t).$$

Theorem 2.5. *If* $m = \lceil \alpha \rceil$,

I) $I^{\alpha}(D^{\alpha} f(t)) = f(t) - \sum_{i=0}^{m-1} \dfrac{t^{\alpha-i-1}}{\Gamma(\alpha-i)} \lim_{z\to 0^+} D^{m-i-1} I^{m-\alpha} f(z)$,

II) $I^{\alpha}(D_*^{\alpha} f(t)) = f(t) - \sum_{i=0}^{m-1} \dfrac{t^i}{i!} \left[D^i f(t)\right]_{t=0}$.

3. Shannon wavelet operational matrix of the fractional integration

This section is devoted to introduction of Shannon wavelet bases, function approximation with these bases and establish the operational matrix of fractional integration.

3.1. The Shannon wavelet

Wavelets are a family of functions constructed from dilation and translation of a single function called the mother wavelet. The scaling function for the Shannon multiresolution analysis is sinc function that defined on $\mathbb{R}$, and is given below

$$\varphi(t) = sinc(t) = \begin{cases} \dfrac{sin(\pi t)}{\pi t}, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

Theorem 3.1 ([1]). *The function $\varphi(t)$ is a scaling function of a multiresolution analysis and the corresponding mother wavelet is defined by*

$$\psi(t + \frac{1}{2}) = 2\varphi(2t) - \varphi(t).$$

Theorem 3.2 ([1]). *Let j, k be non-negative integers. Then the family*

$$\{\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k)\}_{j,k=0}^{\infty},$$

is an orthonormal bases of $L^2(\mathbb{R})$.

In above theorem, j, k are dilatation and translation parameters, respectively.

3.2. Function approximation

In this section we express the convergence of orthogonal wavelet series when the mother wavelet is of Shannon-type. Also we show how to approximate a reasonable function with these wavelet bases.

Theorem 3.3 ([1]). *Let $y(t) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, if $y(t)$ is of bounded variation on every bounded interval, then the wavelet series*

$$y_j(t) = \sum_k \langle y, \psi_{j,k}\rangle \psi_{j,k}(t),$$

converges to $y(t)$ as $j \to \infty$, at every point of continuity of $y(t)$.

Hence, square integrable function $y(t)$ can be expanded into Shannon series as

$$y(t) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_{j,k}\psi_{j,k}(t), \tag{3.1}$$

where $c_{j,k} = \langle y(t), \psi_{j,k}(t)\rangle$. The series expansion $y(t)$ is inclusive infinite terms and will be terminated with finite terms, that is

$$y(t) \approx y_m(t) = \sum_{j=0}^{m-1}\sum_{k=0}^{m-1} c_{j,k}\psi_{j,k}(t). \tag{3.2}$$

In matrix form

$$y(t) \approx \mathbf{C}^T\mathbf{\Psi}(t), \tag{3.3}$$

where $\mathbf{C}$ and $\mathbf{\Psi}(t)$ are $m^2 \times 1$ matrices, given by

$$\mathbf{C} = [c_{0,0}, c_{0,1}, \ldots, c_{0,m-1}, c_{1,0}, c_{1,1}, \ldots, c_{1,m-1}, \ldots, c_{m-1,0}, c_{m-1,1}, \ldots, c_{m-1,m-1}]^T,$$

$$\mathbf{\Psi}(t) = [\psi_{0,0}(t), \psi_{0,1}(t), \ldots, \psi_{0,m-1}(t), \ldots, \psi_{m-1,0}(t), \psi_{m-1,1}(t), \ldots, \psi_{m-1,m-1}(t)]^T.$$

Now, By taking the collocation points as following:

$$t_i = \frac{i}{m^2-1}, \qquad i = mj + k, \qquad j, k = 0, 1, ..., m-1, \tag{3.4}$$

we defined the Shannon matrix $\mathbf{\Psi}_{m^2\times m^2}$ as

$$\mathbf{\Psi}_{m^2\times m^2} = [\mathbf{\Psi}(0), \mathbf{\Psi}\left(\frac{1}{m^2-1}\right), \mathbf{\Psi}\left(\frac{2}{m^2-1}\right), \ldots, \mathbf{\Psi}\left(\frac{m^2-2}{m^2-1}\right), \mathbf{\Psi}(1)]. \tag{3.5}$$

Eventually, for vector $\mathbf{y}_m = [y_m(t_0), y_m(t_1), \cdots, y_m(t_{m^2-1})]$, the Shannon coefficients $c_{j,k}; j, k = 0, 1, \ldots, m-1$ can be determined by

$$\mathbf{C}^T = \mathbf{y}_m\mathbf{\Psi}^{-1}_{m^2\times m^2}. \tag{3.6}$$

For example, when $m = 2$, the Shannon matrix is expressed as

$$\mathbf{\Psi}_{4\times 4} = \begin{bmatrix} 1 & 0.826993 & 0.413497 & 3.89817\times 10^{-17} \\ -0.63662 & 0.699057 & 0.699057 & -0.63662 \\ 0.212207 & 0.372702 & -0.521783 & -0.63662 \\ -0.900316 & 0.988616 & -0.737913 & 0.300105 \end{bmatrix}$$

3.3. Integration of Shannon wavelets

The integration of the Shannon function vector $\mathbf{\Psi}(t)$ can be approximated by Shannon wavelet operational matrix of integration $\mathbf{P}_{m^2\times m^2}$ as follows:

$$\int_0^t \mathbf{\Psi}(\tau)\mathrm{d}\tau \approx \mathbf{P}_{m^2\times m^2}\mathbf{\Psi}(t). \tag{3.7}$$

Our purpose is to derive the Shannon wavelet operational matrix of the fractional integration, namely $\mathbf{P}^{\alpha}_{m^2\times m^2}$. For this point, we introduce the m^2-set of block-pulse functions on $[0, 1)$ as follows

$$b_j(t) = \begin{cases} 1, & j\frac{1}{m^2} \le t < (j+1)\frac{1}{m^2}, \\ 0, & \text{o.w.,} \end{cases}$$

for $j = 0, 1, \ldots, m^2 - 1$. The functions b_js are disjoint and orthogonal,

$$\int_0^1 b_i(t)b_j(t)\mathrm{d}t = \begin{cases} 0, & i \neq j, \\ \dfrac{1}{m^2}, & i = j. \end{cases} \tag{3.8}$$

The Shannon wavelets can be expanded into m^2-set of block-pulse functions as

$$\mathbf{\Psi}(t) = \mathbf{\Psi}_{m^2\times m^2}\mathbf{B}(t), \tag{3.9}$$

where $\mathbf{B}(t) = [b_0(t), b_1(t), \ldots, b_{m^2-1}(t)]^T$.
In Ref.[11], Kilicman and Al Zhour have given the block-pulse operational matrix of the fractional integration as

$$I^\alpha \mathbf{B}(t) \approx \mathbf{F}^\alpha_{m^2\times m^2}\mathbf{B}(t), \tag{3.10}$$

where

$$\mathbf{F}^\alpha_{m^2\times m^2} = (\frac{1}{m^2})^\alpha \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \zeta_1 & \zeta_2 & \ldots & \zeta_{m^2-1} \\ 0 & 1 & \zeta_1 & \ldots & \zeta_{m^2-2} \\ 0 & 0 & 1 & \ldots & \zeta_{m^2-3} \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}, \tag{3.11}$$

with $\zeta_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$, for $k = 1, 2, \ldots, m^2 - 1$.
Next, to determine the Shannon wavelet operational matrix of the fractional integration, we let

$$I^\alpha \mathbf{\Psi}(t) \approx \mathbf{P}^\alpha_{m^2\times m^2}\mathbf{\Psi}(t). \tag{3.12}$$

By using Eqs. (3.9), (3.10) we have

$$\begin{aligned} &\mathbf{P}^\alpha_{m^2\times m^2}\mathbf{\Psi}(t) \approx I^\alpha \mathbf{\Psi}_{m^2\times m^2}\mathbf{B}(t) \approx \mathbf{\Psi}_{m^2\times m^2}\mathbf{F}^\alpha_{m^2\times m^2}\mathbf{B}(t) \\ &\Rightarrow \mathbf{P}^\alpha_{m^2\times m^2} \approx \mathbf{\Psi}_{m^2\times m^2}\mathbf{F}^\alpha_{m^2\times m^2}\mathbf{\Psi}^{-1}_{m^2\times m^2}. \end{aligned} \tag{3.13}$$

For example, when $m = 2$, $\alpha = 1.5$ the Shannon wavelet operational matrix of integration is given by

$$\mathbf{P}^{1.5}_{4\times 4} = \begin{bmatrix} 0.344784 & 0.029417 & -0.648696 & 0.167481 \\ -0.017471 & -0.045238 & -0.152993 & 0.003118 \\ 0.049062 & 0.062126 & -0.048808 & -0.009804 \\ -0.125848 & 0.032364 & 0.105076 & -0.100288 \end{bmatrix}$$

.

4. Implementation of method

In this section we will explain how one can implement our method for solving the Caputo and Riemann-Liouville FDEs. For this purpose, $1 < \alpha \leq 2$ and $0 \leq \beta \leq 1$, have been selected, but this approach is generalizable for different values of α, β.

4.1. Caputo FDEs

Consider the BVP (1.1) with Caputo type FDE given by

$$D_*^\alpha y(t) = f(t, y(t), D_*^\beta y(t)), \quad 0 \le t \le 1, \tag{4.1}$$
$$y(0) = y_0, \qquad y(1) = y_1, \tag{4.2}$$

where $1 < \alpha \le 2, 0 \le \beta \le 1$.
Now, we apply the integral operator I^α to both sides of Eq. (4.1). In view of the Theorem 2.5, immediately obtain that

$$y(t) = I^\alpha f(t, y(t), D_*^\beta y(t)) + c_0 + c_1 t. \tag{4.3}$$

By considering the boundary conditions (4.2), we find c_0, c_1. Then, by substituting c_0, c_1 into Eq. (4.3), we obtain

$$y(t) = I^\alpha f(t, y(t), D_*^\beta y(t)) - tI^\alpha f(1, y(1), D_*^\beta y(1)) + t(y_1 - y_0) + y_0. \tag{4.4}$$

We approximate the solution of above equation by relations (3.3) and (3.12). By this process we achieve to a system of algebraic equations which by solving it, Shannon coefficients and therefore approximate solution of BVP (4.1), (4.2) is obtained.

4.2. Riemann-Liouville FDEs

Consider the BVP (4.1), (4.2) with Riemann-Liouville derivative type. In this case, by according to the Theorem 2.5 and apply the integral operator I^α to both sides of FDE, we obtain

$$y(t) = I^\alpha f(t, y(t), D^\beta y(t)) + c_0 t^{\alpha-1} + c_1 t^{\alpha-2}. \tag{4.5}$$

If $\alpha = 2$, boundary conditions will result

$$c_1 = y_0, \qquad c_0 = y_1 - y_0 - I^\alpha f\left(1, y(1), D^\beta y(1)\right), \tag{4.6}$$

for $1 < \alpha < 2$, we have the above relation with $y_0 = 0$. Similar to subsection 4.1, by using these relationships FDE of Riemann-Liouville type reduces to system of algebraic equations. Then, we solve this system for obtain numerical solution of BVP.

5. Numerical experiments

In this section we give two examples for demonstrate the efficiency of the Shannon wavelet bases to approximate the solution of BVP for both types of FDEs.

Example 1. Consider the following BVP with Caputo derivative,

$$D_*^\alpha y(t) = -y(t) + h(t), \qquad t \in [0, 1]. \tag{5.1}$$
$$y(0) = 0, \qquad y(1) = 1,$$

Table 1: Absolute errors for $m = 2, 3, 4, 5$ of Example 1.

t	$m = 2$	$m = 3$	$m = 4$	$m = 5$
0.1	6.524×10^{-1}	1.168×10^{-2}	2.022447×10^{-2}	1.53841×10^{-3}
0.2	2.02808×10^{-1}	1.0218×10^{-2}	4.79714×10^{-3}	3.34711×10^{-3}
0.3	1.6273×10^{-1}	1.14727×10^{-3}	4.039×10^{-3}	5.55269×10^{-3}
0.4	2.16207×10^{-1}	7.30182×10^{-3}	5.9573×10^{-3}	8.0235×10^{-3}
0.5	1.37637×10^{-2}	1.8914×10^{-3}	4.24281×10^{-3}	1.04288×10^{-2}
0.6	1.6848×10^{-1}	3.9868×10^{-3}	3.74331×10^{-3}	1.22764×10^{-2}
0.7	1.07519×10^{-1}	9.78823×10^{-4}	5.93811×10^{-3}	1.29417×10^{-2}
0.8	1.41586×10^{-1}	8.2192×10^{-4}	8.00243×10^{-3}	1.16871×10^{-2}
0.9	2.76946×10^{-1}	2.96354×10^{-3}	6.34614×10^{-3}	7.67873×10^{-3}

where $1 < \alpha \leq 2$. For $\alpha = \frac{3}{2}$ and $h(t) = t^2 + \frac{2\sqrt{t}}{\Gamma(1.5)}$, the problem has exact solution $y(t) = t^2$. By integration of order α of Eq. (5.1) and incorporate boundary conditions, using the implementation method described in subsection 4.1 results in the following integral representation

$$y(t) = -I^{\alpha}y(t) + tI^{\alpha}y(1) + f(t), \tag{5.2}$$

where $f(t) = I^{\alpha}h(t) - tI^{\alpha}h(1) + t$. Substituting (3.3), (3.12) into Eq. (5.2) and using of collocation points, gives the following algebraic system

$$\mathbf{C}^T\mathbf{\Psi}(t) = -\mathbf{C}^T\mathbf{P}^{\alpha}_{m^2\times m^2}\mathbf{\Psi}(t) + t\mathbf{C}^T\mathbf{P}^{\alpha}_{m^2\times m^2}\mathbf{\Psi}(1) + f(t). \tag{5.3}$$

The numerical and exact solutions for $m = 5$ are shown in Fig. 1. Also, the absolute errors for different values of m is shown in the Table 1.

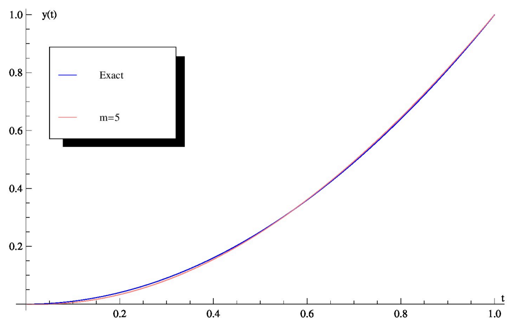

Figure 1: Exact and approximate solutions for $m = 5$ of Example 1.

Example 2. In this example, we consider the BVP for inhomogeneous linear FDE with Riemann-Liouville derivative [18],

$$D^{\alpha}y(t) + ay(t) = g(t), \qquad t \in [0, 1]. \tag{5.4}$$

$$y(0) = 0, \quad y(1) = \frac{1}{\Gamma(\alpha + 2)},$$

Table 2: Absolute errors for $m = 4$ and $\alpha = 1.2, 1.4, 1.6, 1.8, 2$ of Example 2.

t	$\alpha = 1.2$	$\alpha = 1.4$	$\alpha = 1.6$	$\alpha = 1.8$	$\alpha = 2$
0.1	1.04608×10^{-4}	2.81088×10^{-6}	1.11985×10^{-5}	8.9320×10^{-6}	5.75332×10^{-6}
0.2	1.30891×10^{-4}	1.86078×10^{-5}	3.81944×10^{-6}	4.77092×10^{-6}	2.67848×10^{-6}
0.3	1.50119×10^{-4}	3.11931×10^{-5}	1.48163×10^{-6}	2.88336×10^{-6}	1.94724×10^{-6}
0.4	1.67075×10^{-4}	4.62302×10^{-5}	9.52422×10^{-6}	4.72392×10^{-7}	8.8283×10^{-7}
0.5	1.83896×10^{-4}	6.76519×10^{-5}	2.45552×10^{-5}	9.14024×10^{-6}	3.61203×10^{-6}
0.6	1.94832×10^{-4}	9.04528×10^{-5}	4.34921×10^{-5}	2.1600×10^{-5}	1.09074×10^{-5}
0.7	1.91477×10^{-4}	1.06158×10^{-4}	6.0055×10^{-5}	3.38937×10^{-5}	1.87395×10^{-5}
0.8	1.65897×10^{-4}	1.06068×10^{-4}	6.71504×10^{-5}	4.12024×10^{-5}	2.42607×10^{-5}
0.9	1.07803×10^{-4}	7.77512×10^{-5}	5.36694×10^{-5}	3.51453×10^{-5}	2.18447×10^{-5}

where $1 < \alpha \leq 2$, $a \in \mathbb{R}$. For $g(t) = t + \dfrac{at^{\alpha+1}}{\Gamma(\alpha+2)}$, the exact solution of the problem is $y(t) = \dfrac{t^{\alpha+1}}{\Gamma(\alpha+2)}$. With respect to the subsection 4.2, for $1 < \alpha \leq 2$, since $y(0) = 0$ the unique integral representation for Eq. (5.4) is given by

$$y(t) = -aI^{\alpha}y(t) + at^{\alpha-1}I^{\alpha}y(1) + f(t), \tag{5.5}$$

where $f(t) = I^{\alpha}g(t) - t^{\alpha-1}I^{\alpha}g(1) + \dfrac{t^{\alpha-1}}{\Gamma(\alpha+2)}$. As well as, the corresponding algebraic system with Eq. (5.5) is

$$\mathbf{C}^T\mathbf{\Psi}(t) = -a\mathbf{C}^T\mathbf{P}^{\alpha}_{m^2\times m^2}\mathbf{\Psi}(t) + at^{\alpha-1}\mathbf{C}^T\mathbf{P}^{\alpha}_{m^2\times m^2}\mathbf{\Psi}(1) + f(t). \tag{5.6}$$

We solve (5.6) with $a = \dfrac{3}{57}$, $m = 4$ and $\alpha = 1.2, 1.4, 1.6, 1.8, 2$, for calculate Shannon coefficients vector and finally to obtain numerical solution of BVP (5.4). The numerical results are shown in Fig. 2 and Table 2 represents the absolute errors of this example.

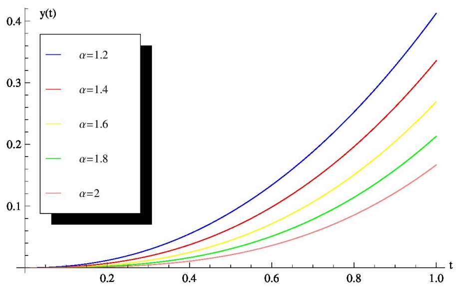

Figure 2: The numerical solutions for different values of α for Example 2.

Acknowledgements

The authors would like to thank the anonymous reviewers for their suggestions to improve the quality of this work.

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