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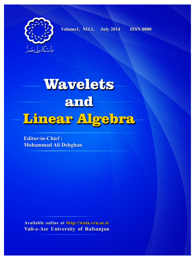

Legendre wavelets method for numerical solution of time-fractional heat equation

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Abstract

In this paper, we develop an efficient Legendre wavelets collocation method for well known time-fractional heat equation. In the proposed method, we apply operational matrix of fractional integration to obtain numerical solution of the inhomogeneous time-fractional heat equation with lateral heat loss and Dirichlet boundary conditions. The power of this manageable method is confirmed. Moreover, the use of Legendre wavelets is found to be accurate, simple and fast.

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1. Introduction

Fractional differential equations (FDEs), as generalizations of classical integer order differential equations, are increasingly used to model problems in fluid flow, mechanics viscoelasticity, biology, physics, engineering and other applications. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes [1, 2, 3, 4, 5, 6]. Fractional differentiation and integration operators are also used for extensions of the diffusion and wave equations [7]. The solutions of fractional differential equations are much involved, because in general, there exists no method that yields an exact solution for fractional differential equations. Only approximate solutions can be derived using linearization or perturbation methods.
In this paper, we study the inhomogeneous time-fractional heat equation of order α ($0 < \alpha \leq 1$), with lateral heat loss in a rod of length L, that is defined by:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \bar{k}\frac{\partial^2 u(x,t)}{\partial x^2} - cu(x,t) + g(x,t), \qquad 0 < x < L, \quad t \geq 0, \tag{1.1}$$

where $u(x,t)$ represents the temperature of the rod at the position x at time t, $\frac{\partial^\alpha}{\partial t^\alpha}$ denotes Caputo fractional derivative of order α, that will be described in the next section, $\bar{k}$ is the thermal diffusivity of the material that measures the rod ability to heat conduction, c is a positive constant and $g(x,t)$ is called the heat source. The domain of the solution is a semi-infinite strip of width L that continues indefinitely in time. In a practical computation, the solution is obtained only for a finite time, say $T = t_{max}$. Solution to Equation (1.1) requires specification of initial condition at $t = 0$ and boundary conditions at $x = 0$ and $x = L$. Simple initial and boundary conditions (IBCs) are:

$$u(x,0) = f(x), \qquad 0 \leq x \leq L, \tag{1.2}$$

$$u(0,t) = h_0(t), \qquad t \geq 0, \tag{1.3}$$

$$u(L,t) = h_1(t), \qquad t \geq 0. \tag{1.4}$$

The initial condition in (1.2) describes the initial temperature $u(x,t)$ at time $t = 0$ and the given boundary conditions in (1.3) and (1.4) indicate that the temperature of rod ends are functions of t. Wavelet methods have been applied for solving PDEs from beginning of the early 1990s [8]. In the last two decades this problem has attracted great attention and numerous papers about this topics have been published, for instances see [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. It is worth noting that a suitable family of orthogonal basis functions which can be used to obtain approximate solutions of fractional functional equations is a family of the Legendre polynomials, because the fractional integrals can be easily computed [22]. Therefore, due to the fact that the Legendre wavelets have mutual properties of the Legendre polynomials and properties of wavelets, we believe that these basis functions are suitable for obtaining approximate solutions for fractional functional equations (for more details see [22]).
The aim of the present work is to develop Legendre wavelets collocation method with operational matrix of fractional integration for numerical solution of time-fractional heat equation with Dirichlet boundary conditions, which is fast, mathematically simple and guarantees the necessary

accuracy for a relative small number of grid points. The outline of this article is as follows. In section 2, we describe some basic definition and properties of fractional calculus. In section 3, we describe some properties of Legendre wavelets. In Section 4, the proposed method is used to approximate solution of the problem. In section 5, the numerical examples of applying the method of this article are presented. Finally, a conclusion is drawn in section 6.

2. Fractional calculus

Here, we give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 2.1. A real function $f(t),\ t > 0$, is said to be in the space $C_\mu,\ \mu \in \mathbb{R}$ if there exists a real number $p\ (> \mu)$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0, \infty]$ and it is said to be in the space C_μ^n if $f^{(n)} \in C_\mu,\ n \in \mathbb{N}$.

Definition 2.2. The Riemann-Liouville fractional integration operator of order $\alpha \geq 0$, of a function, $f \in C_\mu,\ \mu \geq -1$, has been defined as [4]:

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \tag{2.1}$$

$$I^0 f(t) = f(t), \tag{2.2}$$

and we have:

$$I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t), \tag{2.3}$$

$$I^\alpha t^\vartheta = \frac{\Gamma(\vartheta+1)}{\Gamma(\alpha+\vartheta+1)} t^{\alpha+\vartheta}, \tag{2.4}$$

where $\alpha,\ \beta \geq 0,\ t > 0$ and $\vartheta > -1$.

Definition 2.3. The fractional derivative of order $\alpha > 0$ in the Riemann-Liouville sense has been defined as [4]:

$$D^\alpha f(t) = \left(\frac{d}{dt}\right)^n I^{n-\alpha} f(t), \qquad (n-1 < \alpha \leq n), \tag{2.5}$$

where n is an integer and $f \in C_1^n$.

The Riemann-Liouville derivatives have certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall now introduce a modified fractional differential operator D_*^α proposed by Caputo [4].

Definition 2.4. The fractional derivative of order $\alpha > 0$ in the Caputo sense has been defined as [4]:

$$D_*^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \qquad (n-1 < \alpha \leq n), \tag{2.6}$$

where n is an integer $t > 0$, and $f \in C_1^n$. Caputos integral operator has an useful property:

$$I^\alpha D_*^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!}, (n-1 < \alpha \leq n). \tag{2.7}$$

where n is an integer $t > 0$, and $f \in C_1^n$.

3. Properties of Legendre wavelets

3.1. Wavelets and Legendre wavelets

Wavelets constitute a family of functions constructed from dilations and translations of a single function called the mother wavelet $\psi(t)$. When the dilation parameter a and the translation parameter b vary continuously we have the following family of continuous wavelets as[24]:

$$\psi_{a,b}(t) = |a|^{-1/2}\psi(\frac{t-b}{a}), \quad a, b \in \mathbb{R}, \quad a \neq 0. \tag{3.1}$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k}, b = nb_0a_0^{-k}, a_0 > 1, b_0 > 0$, and n and k positive integers, we have the following family of discrete wavelets:

$$\psi_{,n}(t) = |a_0|^{k/2}\psi(a_0^k t - nb_0), \tag{3.2}$$

where $\psi_{k,n}(t)$ forms a wavelet basis for $L^2(\mathbb{R})$. In particular, when $a_0 = 2$ and $b_0 = 1$, $\psi_{kn}(t)$ forms an orthonormal basis. That is $(\psi_{k,n}(t), \psi_{l,m}(t)) = \delta_{kl}\delta_{nm}$.
Legendre wavelets $\psi_{n,m}(t) = \psi(k, \hat{n}, m, t)$ have four arguments; $k \in \mathbb{N}, n = 1, 2, \cdots, 2^{k-1}$, and $\hat{n} = 2n - 1$, moreover m is the degree of the Legendre polynomials and t is the normalized time, and are defined on the interval $[0, 1)$ as [24]:

$$\psi_{nm}(t) = \begin{cases} \sqrt{m+\frac{1}{2}}\,2^{\frac{k}{2}}P_m(2^k t - \hat{n}), & \frac{\hat{n}-1}{2^k} \leq t < \frac{\hat{n}}{2^k}, \\ 0, & o.w, \end{cases} \tag{3.3}$$

where $m = 0, 1, \cdots, M-1$, and M is a fixed positive integer. The coefficient $\sqrt{m + 1/2}$ in (3.3) is for orthonormality. Here, $P_m(t)$ are the well-known Legendre polynomials of degree m which are orthogonal with respect to the weight function $w(t) = 1$, on the interval $[-1, 1]$, and satisfy the following recursive formula:

$$p_0(t) = 1, \ p_1(t) = t, \ p_{m+1}(t) = \left(\frac{2m+1}{m+1}\right)tp_m(t) - \left(\frac{m}{m+1}\right)p_{m-1}(t), \ \ m = 1, 2, 3 \cdots .$$

3.2. Function approximation

An arbitrary function $f(x,t)$ defined over $[0, 1)\times[0, 1)$ may be expanded into Legendre wavelets as:

$$f(x,t) = \sum_{n'=1}^{\infty}\sum_{m'=0}^{\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\psi_{n'm'}(x)c_{nm}^{n'm'}\psi_{nm}(t), \tag{3.4}$$

where $c_{nm}^{n'm'} = (\psi_{n'm'}(x), (f(x,t), \psi_{nm}(t)))$, and (,) denotes the inner product.
Usually the infinite series in (3.4) is truncated and written as:

$$f(x,t) \simeq \sum_{n'=1}^{2^{k-1}}\sum_{m'=0}^{M-1}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\psi_{n',m'}(x)c_{n,m}^{n',m'}\psi_{nm}(t) = \Psi(x)^T C\Psi(t), \tag{3.5}$$

which approximates $f(x,t)$ as a finite linear combination of $\psi_{n',m'}(x)$ and $\psi_{nm}(x)$. Here, $\Psi(t)$ is an $m = 2^{k-1}M$ column vector as:

$$\Psi(t) = [\psi_{10}(t), \psi_{11}(t), \ldots, \psi_{1M-1}(t), \psi_{20}(t), \ldots, \psi_{2M-1}(t), \ldots, \psi_{2^{k-1}0}(t), \ldots, \psi_{2^{k-1}M-1}(t)]^T \tag{3.6}$$

and C is an $m \times m$ matrix as:

$$\begin{pmatrix}
c_{10}^{10} & c_{11}^{10} & \ldots & c_{1M-1}^{10} & c_{20}^{10} & \ldots & c_{2M-1}^{10} & \ldots & c_{2^{k-1}0}^{10} & \ldots & c_{2^{k-1}M-1}^{10} \\
c_{10}^{11} & c_{11}^{11} & \ldots & c_{1M-1}^{11} & c_{20}^{11} & \ldots & c_{2M-1}^{11} & \ldots & c_{2^{k-1}0}^{11} & \ldots & c_{2^{k-1}M-1}^{11} \\
\vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \ldots & \vdots & \ldots & \vdots \\
c_{10}^{1M-1} & c_{11}^{1M-1} & \ldots & c_{1M-1}^{1M-1} & c_{20}^{1M-1} & \ldots & c_{2M-1}^{1M-1} & \ldots & c_{2^{k-1}0}^{1M-1} & \ldots & c_{2^{k-1}M-1}^{1M-1} \\
c_{10}^{20} & c_{10}^{21} & \ldots & c_{1M-1}^{20} & c_{20}^{20} & \ldots & c_{2M-1}^{20} & \ldots & c_{2^{k-1}0}^{20} & \ldots & c_{2^{k-1}M-1}^{20} \\
\vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \ldots & \vdots & \ldots & \vdots \\
c_{10}^{2M-1} & c_{11}^{2M-1} & \ldots & c_{1M-1}^{2M-1} & c_{20}^{2M-1} & \ldots & c_{2M-1}^{2M-1} & \ldots & c_{2^{k-1}0}^{2M-1} & \ldots & c_{2^{k-1}M-1}^{2M-1} \\
\vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \ldots & \vdots & \ldots & c_{2^{k-1}M-1}^{2M-1} \\
c_{10}^{2^{k-1}0} & c_{11}^{2^{k-1}0} & \ldots & c_{1M-1}^{2^{k-1}0} & c_{20}^{2^{k-1}0} & \ldots & c_{2M-1}^{2^{k-1}0} & \ldots & c_{2^{k-1}0}^{2^{k-1}0} & \ldots & c_{2^{k-1}M-1}^{2^{k-1}0} \\
\vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \ldots & \vdots & \ldots & \vdots \\
c_{10}^{2^{k-1}M-1} & c_{11}^{2^{k-1}M-1} & \ldots & c_{1M-1}^{2^{k-1}M-1} & c_{20}^{2^{k-1}M-1} & \ldots & c_{2M-1}^{2^{k-1}M-1} & \ldots & c_{2^{k-1}0}^{2^{k-1}M-1} & \ldots & c_{2^{k-1}M-1}^{2^{k-1}M-1}
\end{pmatrix}$$

By taking the collocation points:

$$t_i = \frac{2i-1}{2m}, \qquad i = 1, 2, \ldots, m, \tag{3.7}$$

into (3.6), we define the $m \times m$ Legendre wavelets Φ as:

$$\Phi = [\Psi(\frac{1}{2m}), \Psi(\frac{3}{2m}), \ldots, \Psi(\frac{2m-1}{2m})]. \tag{3.8}$$

We investigate the convergence of the Legendre wavelets expansion in the following Theorems.

Theorem 3.1 ([23]). *A function $f(x)$ defined on $[0,1)$, with bounded second derivative $|f''(x)| < B$, can be expanded as an infinite sum of the Legendre wavelets and the series converges uniformly to the function $f(x)$, that is $f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(x)$, moreover coefficients c_{nm} is bounded as* $|c_{nm}| < \frac{B\sqrt{12}}{(2n)^{\frac{5}{2}}(2m-3)^2}$.

Theorem 3.2 ([2]). *If the continuous function $f(x,t)$ defined on $[0,1) \times [0,1)$, has bounded mixed fourth partial derivative $|\frac{\partial^4 f(x,t)}{\partial x^2 \partial t^2}| \leq \hat{M}$, then the Legendre wavelets expansion of $f(x,t)$ converges uniformly to it and also*

$$|c_{n,m}^{n',m'}| < \frac{12\hat{M}}{(2n'')^5(2m''-3)^4}, \tag{3.9}$$

where $m'' = min\{m, m'\}$ and $n'' = min\{n, n'\}$.

3.3. The operational matrix of fractional integration

The fractional integration of order α of the vector $\Psi(.)$ defined in (3.6) can be expressed as:

$$(I^{\alpha}\Psi)(.) = P^{\alpha}\Psi(.), \tag{3.10}$$

where P^{α} is the $m \times m$ operational matrix of fractional integration of order α for for Legendre wavelets. It has shown in [24] that the matrix P^{α} can be approximated as:

$$P^{\alpha} \simeq P^{\alpha}_{m\times m} = \Phi P^{\alpha}_{B}\Phi^{-1}, \tag{3.11}$$

where P^{α}_{B} is the operational matrix of fractional integration of order α for the block pulse functions (BPF) [25], which has the following form:

$$P^{\alpha}_{B} = \frac{1}{m^{\alpha}}\frac{1}{\Gamma(\alpha+2)}\begin{pmatrix} 1 & \xi_1 & \xi_2 & \dots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \dots & \xi_{m-2} \\ 0 & 0 & 1 & \dots & \xi_{m-3} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.12}$$

where $\xi_i = (i+1)^{\alpha+1} - 2i^{\alpha+1} + (i-1)^{\alpha+1}$ [25].
Here, we can obtain ΦP^{α}_{B} as follow:

$$\Phi P^{\alpha}_{B} = \begin{pmatrix} B_1 & B_2 & B_3 & \dots & B_{\hat{k}} \\ 0 & B_1 & B_2 & \dots & B_{\hat{k}-1} \\ 0 & 0 & B_1 & \dots & B_{\hat{k}-2} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & B_1 \end{pmatrix}, \tag{3.13}$$

where $B_r,\ r = 1, 2, \dots, \hat{k}$ ($\hat{k} = 2^{k-1}$), are $M \times M$ matrices given by:

$$B_r = (b^{(r)}_{ij})_{M\times M},$$

such that for $r = 1$,

$$b^{(1)}_{ij} = \sum_{l=1}^{j} a_{il}\xi'_{j-l}, \qquad \xi'_0 = 1,$$

and for $r = 2, 3, \dots, M$

$$B_r = (b^{(r)}_{ij})_{M\times M}, \qquad l = 2, 3, \dots, \hat{k},$$

$$b^{(r)}_{ij} = \sum_{l=1}^{M} a_{il}\xi'_{(r-1)M-l+j},$$

where

$$a_{ij} = \Psi_{1(i-1)}(\frac{2j-1}{2m}),$$

and $\xi'_j = \frac{1}{m^{\alpha}\Gamma(\alpha+2)}\xi_j,\ j = 0, 1, \ldots m-1.$
Now from (3.11) and (4.2), we have:

$$P^{\alpha}_{m\times m} = \begin{pmatrix} D_1 & D_2 & D_3 & \ldots & D_{\hat{k}} \\ 0 & D_1 & D_2 & \ldots & D_{\hat{k}-1} \\ 0 & 0 & D_1 & \ldots & D_{\hat{k}-2} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & D_1 \end{pmatrix}, \tag{3.14}$$

where $D_i = B_i A^{-1},\ i = 1, 2, \ldots \hat{k}$ [26].

4. Implementation of the numerical method

In this section, we use the Legendre wavelets operational matrix of fractional integration for numerical solution of the time-fractional heat equation. Let us consider the inhomogeneous time-fractional heat equation with lateral heat loss as:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \bar{k}\frac{\partial^2 u(x,t)}{\partial x^2} - cu(x,t) + g(x,t), \qquad 0 \le x \le 1, \quad 0 \le t \le 1, \tag{4.1}$$

and inhomogeneous the initial and Dirichlet boundary conditions:

$$u(x, 0) = f(x), \tag{4.2}$$

$$u(0, t) = h_0(t), \tag{4.3}$$

$$u(1, t) = h_1(t), \tag{4.4}$$

where $f(x)$ and $h_i(t)$ are two times continuously differentiable functions on $[0, 1]$. For solving this equation, we assume:

$$\frac{\partial^{\alpha+2} u(x,t)}{\partial t^{\alpha}\partial x^2} \simeq \Psi(x)^T U \Psi(t), \tag{4.5}$$

where $U = (u_{ij})_{m\times m}$ is an unknown matrix which should be found and $\Psi(.)$ is the vector that defined in (3.6). By integrating of order α of (4.5) with respect to t and considering (4.2), we obtain:

$$\frac{\partial^2 u}{\partial x^2} \simeq \Psi(x)^T U P \Psi(t) + f''(x). \tag{4.6}$$

Also by integrating of (4.5) two times with respect to x we get:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} \simeq \Psi(x)^T (P^2)^T U \Psi(t) + \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} \Big|_{x=0} + x\frac{\partial}{\partial x}\left(\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}}\right)\Big|_{x=0}. \tag{4.7}$$

By putting $x = 1$ into (4.7) and considering (4.3) and (4.4), we have:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} \simeq \Psi(x)^T (P^2)^T U \Psi(t) - x\Psi(1)^T (P^2)^T U \Psi(t) + \frac{\partial^{\alpha} h_0(t)}{\partial t^{\alpha}} + x\left(\frac{\partial^{\alpha} h_1(t)}{\partial t^{\alpha}} - \frac{\partial^{\alpha} h_0(t)}{\partial t^{\alpha}}\right). \tag{4.8}$$

Now by integrating of order α of (4.8)with respect to t we get:

$$u(x,t) \simeq \Psi(x)^T (P^2)^T U P^{\alpha} \Psi(t) - x\Psi(1)^T (P^2)^T U P^{\alpha} \Psi(t) + G(x,t), \tag{4.9}$$

where

$$G(x,t) = f(x) + h_0(t) - h_0(0) + x\left(h_1(t) - h_1(0) - h_0(t) + h_0(0)\right). \tag{4.10}$$

Now by replacing (4.6), (4.8), and (4.9) into (4.1) we get:

$$\begin{aligned}\Psi(x)^T \left[(P^2)^T U - \bar{k} U P^{\alpha} + c(P^2)^T U P^{\alpha}\right] \Psi(t) - x\Psi(1)^T (P^2)^T U \Psi(t)\\ - cx\Psi(1)^T (P^2)^T U P^{\alpha} \Psi(t) \simeq H(x,t),\end{aligned} \tag{4.11}$$

where

$$H(x,t) = \frac{\partial^{\alpha} h_0(t)}{\partial t^{\alpha}} + x\left(\frac{\partial^{\alpha} h_1(t)}{\partial t^{\alpha}} - \frac{\partial^{\alpha} h_0(t)}{\partial t^{\alpha}}\right) - \bar{k} f''(x) - cG(x,t) + g(x,t).$$

Equation (4.11) is a linear algebraic equation of m^2 unknown variables u_{ij} $(i, j = 1, 2, \ldots, m)$. Here, by taking the collocation points, expressed in (3.7), for both t and x, equation (4.11) is transformed into a linear system of algebraic equations. By solving this system and determined U, we obtain the numerical solution of this problem by substitute U into (4.9).

5. Numerical examples

In this section, some numerical examples of time-fractional heat equation in form (4.1) with the initial and boundary conditions (4.2)–(4.4) with the proposed method are investigated. To show the efficiency of the present method, we report the root mean square error L_2 and maximum error L_{∞} in the case $\alpha = 1$:

$$L_2 = \sqrt{\frac{1}{m} \sum_{i=1}^{m} |u(x_i, t_i) - \tilde{u}(x_i, t_i)|^2}, \tag{5.1}$$

$$L_{\infty} = \max_{1 \le i \le m} |u(x_i, t_i) - \tilde{u}(x_i, t_i)|.$$

Theses examples are considered because either closed form solutions are available for them, and they have also been solved using other numerical schemes. All programs have been performed by Maple 14 and with 15 digits.

Example 1. Consider the heat equation (4.1) with $\bar{k} = 1$, $c = 0$ and $g(x,t) = 0$. The initial and boundary conditions are given by:

$$\begin{cases} u(x,0) = \sin(x), \\ u(0,t) = 0, \\ u(1,t) = \sin(1)e^{-t}. \end{cases}$$

The exact solution of this problem for $\alpha = 1$, is $u(x,t) = \sin(x)e^{-t}$[27]. The space-time graphs of the exact and numerical solutions (in the case $\alpha = 1$), for $m = 16$ $(M = 4, k = 3)$, are presented

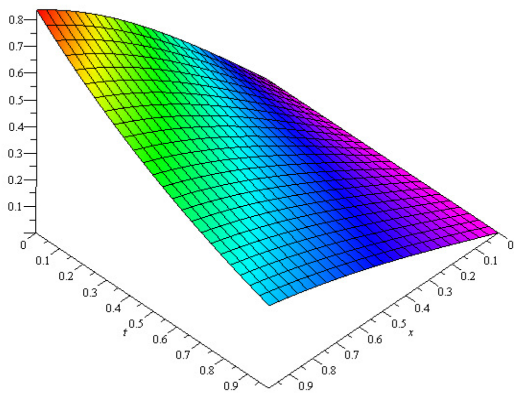

Figure 1: The exact solution for Example 1.

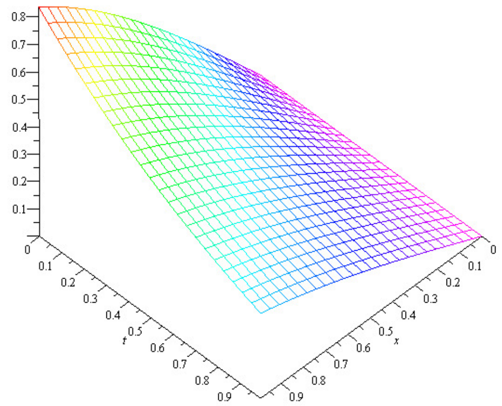

Figure 2: The numerical solution for Example 1.

Table 1: The approximate solutions of Example 1, for some different values of α.

(x_i, t_i)	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 1.0$	Exact solution
$(0.2, 0.2)$	0.16101542	0.16144049	0.16193340	0.16265906	0.16265669
$(0.4, 0.4)$	0.25975634	0.26022073	0.26064947	0.26103811	0.26103492
$(0.6, 0.6)$	0.30971344	0.30995943	0.31005131	0.30988814	0.30988236
$(0.8, 0.8)$	0.32279348	0.32280137	0.32267113	0.32233397	0.32232887
$(1.0, 1.0)$	0.30955987	0.30955987	0.30955988	0.30955987	0.30955987

Table 2: The L_∞ and L_2 errors of Example 1, for some different values of t.

t	$t = 0.1$	$t = 0.3$	$t = 0.5$	$t = 0.7$	$t = 0.9$	$t = 1.0$
L_∞	1.07×10^{-5}	3.45×10^{-6}	5.13×10^{-6}	7.45×10^{-6}	9.47×10^{-6}	1.02×10^{-5}
L_2	4.46×10^{-6}	2.15×10^{-7}	3.18×10^{-6}	4.71×10^{-6}	6.04×10^{-6}	6.55×10^{-6}

in Figs. 1 and 2. Table 1 shows the approximate solutions for some different values of α using the proposed method. The root-mean-square error L_2 and maximum error L_∞ for $0 \le x \le 1$ and $0 \le t \le 1$ are presented in Table 2. As Tables 1 and 2 show, the approximate solutions is very accurate.

Example 2. In this example, we consider the heat equation (4.1) with $\overline{k} = 1$ and coefficient lateral heat $c = 2$ and $g(x,t) = 0$. The initial and boundary conditions are given by:

$$\begin{cases} u(x,0) = \sinh(x), \\ u(0,t) = 0, \\ u(1,t) = \sinh(1)e^{-t}. \end{cases}$$

The exact solution of this problem for $\alpha = 1$, is $u(x,t) = e^{-t}\sinh(x)$[27]. The space-time graphs of the exact and numerical solutions (in the case $\alpha = 1$), for $m = 16$ are presented in Figs. 3

and 4. Table 3 shows the approximate solutions for some different values of α using the proposed method. The root-mean-square error L_2 and maximum error L_∞ for $0 \leq x \leq 1$ and $0 \leq t \leq 1$ are presented in Table 4.

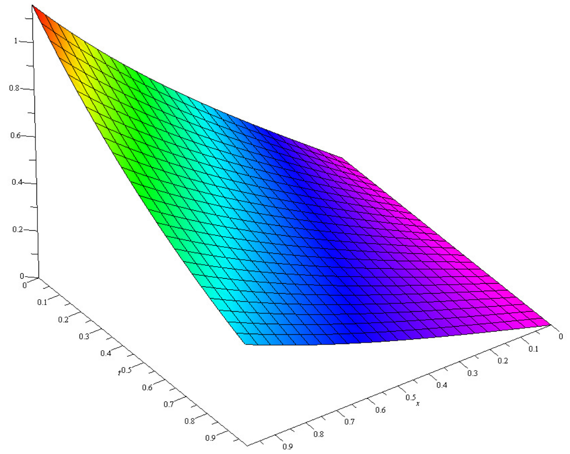

Figure 3: The exact solution for Example 2.

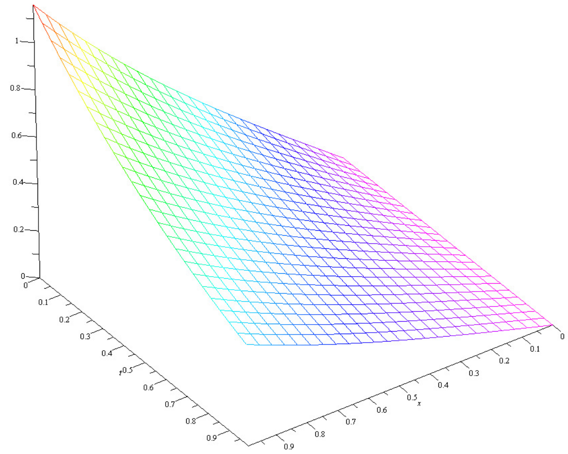

Figure 4: The numerical solution for Example 2.

Table 3: The approximate solutions of Example 2, for some different values of α.

(x_i, t_i)	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 1.0$	Exact solution
(0.2, 0.2)	0.16640171	0.16599091	0.16551057	0.16483997	0.16483997
(0.4, 0.4)	0.27655527	0.27609561	0.27567771	0.27534043	0.27533552
(0.6, 0.6)	0.34956470	0.34931430	0.34922543	0.34941065	0.34940290
(0.8, 0.8)	0.39859803	0.39858529	0.39871604	0.39905789	0.39905174
(1.0, 1.0)	0.43233236	0.43233233	0.43233238	0.43233235	0.43233236

Table 4: The L_∞ and L_2 errors of Example 2, for some different values of t.

t	$t = 0.1$	$t = 0.3$	$t = 0.5$	$t = 0.7$	$t = 0.9$	$t = 1.0$
L_∞	8.44×10^{-6}	8.10×10^{-7}	7.43×10^{-6}	9.18×10^{-6}	1.07×10^{-6}	1.15×10^{-5}
L_2	6.01×10^{-6}	2.59×10^{-7}	4.78×10^{-6}	5.88×10^{-6}	6.89×10^{-6}	7.39×10^{-6}

Example 3. Finally, consider the heat equation (4.1) with $\overline{k} = 1$, $c = 0$ and nonhomogeneous term $g(x,t) = (2t + t^2)\sin(x)$. The initial and boundary conditions are given by:

$$\begin{cases} u(x,0) = 0, \\ u(0,t) = 0, \\ u(1,t) = \sin(1)t^2. \end{cases}$$

The exact solution of this problem when $\alpha = 1$, is $u(x,t) = t^2 \sin(x)$[27]. The space-time graphs of the exact and numerical solutions (in the case $\alpha = 1$), for $m = 16$ are presented in Figs.5 and 5. Table 5 shows the approximate solutions for some different values of α using the proposed method. The root-mean-square error L_2 and maximum error L_∞ for $0 \le x \le 1$ and $0 \le t \le 1$ are presented in Table 6.

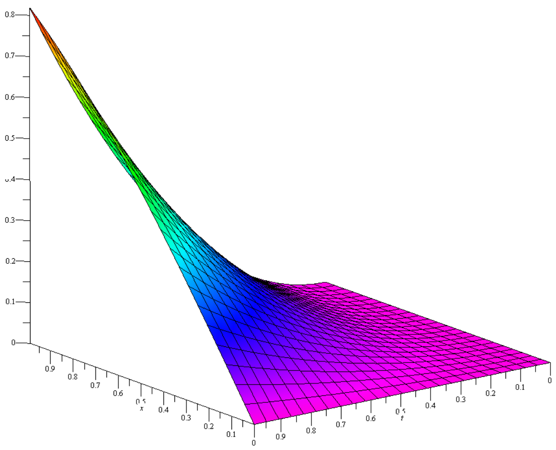

Figure 5: The exact solution for Example 3.

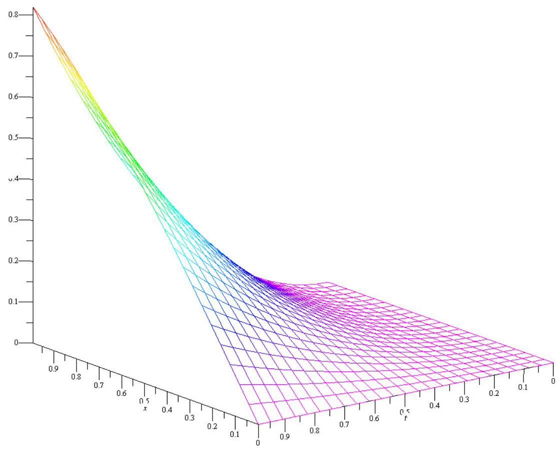

Figure 6: The numerical solution for Example 3.

Table 5: The approximate solutions of Example 3, for some different values of α.

(x_i, t_i)	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 1.0$	Exact solution
$(0.2, 0.2)$	0.01108113	0.01066591	0.01009010	0.00924607	0.00794677
$(0.4, 0.4)$	0.06954557	0.06891681	0.06821181	0.06731800	0.06230693
$(0.6, 0.6)$	0.20851246	0.20815757	0.20793165	0.20788385	0.20327129
$(0.8, 0.8)$	0.45638952	0.45644533	0.45669382	0.45721727	0.45910790
$(1.0, 1.0)$	0.84147098	0.84147098	0.84147098	0.84147098	0.84147098

Table 6: The L_∞ and L_2 errors of Example 3, for some different values of t.

t	$t = 0.1$	$t = 0.3$	$t = 0.5$	$t = 0.7$	$t = 0.9$	$t = 1.0$
L_∞	1.40×10^{-3}	5.78×10^{-3}	5.67×10^{-3}	1.10×10^{-3}	1.63×10^{-3}	2.27×10^{-3}
L_2	1.23×10^{-3}	4.70×10^{-3}	4.89×10^{-3}	1.98×10^{-3}	1.11×10^{-3}	1.98×10^{-3}

6. Conclusion

This paper presented a numerical method by combining Legendre wavelets together with their operational matrix of fractional integration to approximate numerical solutions of well known time-fractional heat equation. In the proposed method, already a small number of grids points guarantees the necessary accuracy. The method is very convenient for solving boundary value problems, since the boundary condition are taken into account automatically. Also the proposed method is very simple in implementation and as the numerical results showed the method is very efficient for numerical solution of mentioned problem and can be used for other kind partial differential equations.

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