Wavelets and Linear Algebra 1 (2014) 9-18

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Wavelets and Linear Algebra

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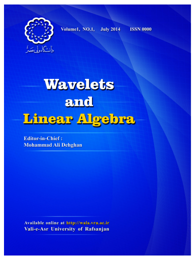

Dilation of a family of g-frames

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Article Info

Article history:
Received 1 November 2013
Accepted 29 April 2014
Available online 1 July 2014
Communicated by Asghar Rahimi

Keywords:
g-Riesz basis
g-frame
disjointness

2000 MSC:
41A58
42C15

Abstract

In this paper, we first discuss about canonical dual of g-frame $\Lambda P = \{\Lambda_i P \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$, where $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-frame for a Hilbert space $\mathcal{H}$ and P is the orthogonal projection from $\mathcal{H}$ onto a closed subspace M. Next, we prove that, if $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\Theta = \{\Theta_i \in B(\mathcal{K}, \mathcal{H}_i) : i \in I\}$ be respective g-frames for non zero Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, and Λ and Θ are unitarily equivalent (similar), then Λ and Θ can not be weakly disjoint. On the other hand, we study dilation property for g-frames and we show that two g-frames for a Hilbert space have dilation property, if they are disjoint, or they are similar, or one of them is similar to a dual g-frame of another one. We also prove that a family of g-frames for a Hilbert space has dilation property, if all the members in that family have the same deficiency.

1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space. A sequence $F = \{f_i\}_{i\in I}$ is called a frame for $\mathcal{H}$, if there exist two positive constants A, B such that

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \quad f \in \mathcal{H}. \tag{1.1}$$

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If $A = B = 1$ in (1.1), then we say that $F = \{f_i\}_{i\in I}$ is a Parseval frame for $\mathcal{H}$. Let $F = \{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$. In this case,

$$T_F : l_2(I) \to \mathcal{H}, \quad T_F(\{c_i\}_{i\in I}) = \sum_{i\in I} c_i f_i$$

is a bounded and onto operator and its adjoint is $T_F^*(f) = \{\langle f, f_i\rangle\}_{i\in I}$, for all $f \in \mathcal{H}$ [6]. The operators T_F, T_F^* and $S_F = T_F T_F^*$ are called the synthesis, analysis and frame operator of $F = \{f_i\}_{i\in I}$, respectively. If $F = \{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$, then S_F is an invertible positive operator and we have

$$f = \sum_{i\in I} \langle f, S_F^{-1} f_i\rangle f_i, \quad f \in \mathcal{H}. \tag{1.2}$$

A sequence $F = \{f_i\}_{i\in I}$ is called a Riesz basis for $\mathcal{H}$, if $\overline{\text{span}}\{f_i\}_{i\in I} = \mathcal{H}$ and there exist two positive constants A, B such that for any finite scalar sequence $\{c_i\}$ we have

$$A \sum_i |c_i|^2 \le \Big\| \sum_i c_i f_i \Big\|^2 \le A \sum_i |c_i|^2.$$

Let $F = \{f_i\}_{i\in I}$ and $G = \{g_i\}_{i\in I}$ be two frames for a Hilbert space $\mathcal{H}$. We say that G is a dual frame for F, if

$$f = \sum_{i\in I} \langle f, g_i\rangle f_i, \quad f \in \mathcal{H}.$$

From (1.2), we conclude that $\tilde{F} = \{S_F^{-1} f_i\}_{i\in I}$ is a dual frame of F, which is called the canonical dual of F. It is proved in [6], each Riesz basis for $\mathcal{H}$ is a frame and has only one dual frame.

The concepts of disjoint frames and strongly disjoint frames introduced by Han and Larson [7], and these notions generalized to frames in Banach spaces by Casazza, Han and Larson [5]. In 2006, more general extension of frames, the so-called g-frames, introduced by Sun [9]. Some properties of g-frames have been investigated in papers [2, 3, 4].

Throughout this paper, $\mathcal{H}$ and $\mathcal{K}$ are separable Hilbert spaces and $\{\mathcal{H}_i\}_{i\in I}$ is a sequence of separable Hilbert spaces.

Definition 1.1. We call a sequence $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$, if there exist two positive constants A and B such that

$$A\|f\|^2 \le \sum_{i\in I} \|\Lambda_i f\|^2 \le B\|f\|^2, \quad f \in \mathcal{H}.$$

A and B are called the lower and upper g-frame bounds, respectively.
We call $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ a tight g-frame if $A = B$ and Parseval g-frame if $A = B = 1$.

If there is no confusion, we use g-frame (g-frame for $\mathcal{H}$) instead of g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$.
Let $\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)$ be given for all $i \in I$. Let us define the set

$$\widehat{\mathcal{H}} = \left\{ \{f_i\}_{i\in I} : f_i \in \mathcal{H}_i, \sum_{i\in I} \|f_i\|^2 < \infty \right\}$$

with the inner product given by $\langle\{f_i\}_{i\in I}, \{g_i\}_{i\in I}\rangle = \sum_{i\in I}\langle f_i, g_i\rangle$. It is easy to show that $\widehat{\mathcal{H}}$ is a Hilbert space with respect to the poitwise operations. It is proved in [8], if $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-Bessel sequence for $\mathcal{H}$, then the operator

$$T_\Lambda : \widehat{\mathcal{H}} \to \mathcal{H}, \quad T_\Lambda(\{f_i\}_{i\in I}) = \sum_{i\in I} \Lambda_i^*(f_i) \tag{1.3}$$

is well defined and bounded and its adjoint is $T_\Lambda^* f = \{\Lambda_i f\}_{i\in I}$ for all $f \in \mathcal{H}$. Also, a sequence $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-frame for $\mathcal{H}$ if and only if the operator T_Λ defined in (1.3) is a bounded and onto operator. We call operators T_Λ and T_Λ^*, the synthesis and analysis operators of Λ, respectively. If $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-frame for $\mathcal{H}$, then

$$S_\Lambda : \mathcal{H} \to \mathcal{H}, \quad S_\Lambda f = \sum_{i\in I} \Lambda_i^* \Lambda_i f$$

is a bounded invertible positive operator [9], and every $f \in \mathcal{H}$ has the following representation

$$f = \sum_{i\in I} S_\Lambda^{-1} \Lambda_i^* \Lambda_i f = \sum_{i\in I} \Lambda_i^* \Lambda_i S_\Lambda^{-1} f. \tag{1.4}$$

S_Λ is called the g-frame operator of Λ. Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g-frame for $\mathcal{H}$ with g-frame bounds A, B and let $\widetilde{\Lambda}_i = \Lambda_i S_\Lambda^{-1}$, for all $i \in I$. Then $\widetilde{\Lambda} = \{\widetilde{\Lambda}_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-frame for $\mathcal{H}$ with bounds $\frac{1}{B}$ and $\frac{1}{A}$ [9].

Definition 1.2. Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\Theta = \{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be two g-frames for $\mathcal{H}$ such that

$$f = \sum_{i\in I} \Theta_i^* \Lambda_i f, \quad f \in \mathcal{H},$$

then Θ is called a dual g-frame of Λ.

By (1.4), $\widetilde{\Lambda} = \{\widetilde{\Lambda}_i\}_{i\in I}$ is a dual g-frame of $\{\Lambda_i\}_{i\in I}$, which is called the canonical dual of $\Lambda = \{\Lambda_i\}_{i\in I}$.

Definition 1.3. A sequence $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is called

(1) a g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$, if there exist two positive constants A and B such that for any finite subset $F \subseteq I$ we have

$$A \sum_{i\in F} \|g_i\|^2 \le \| \sum_{i\in F} \Lambda_i^* g_i \|^2 \le B \sum_{i\in F} \|g_i\|^2, \quad g_i \in \mathcal{H}_i,$$

and $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is g-complete, i.e.,

$$\{f : \Lambda_i f = 0, \forall i \in I\} = \{0\}.$$

(2) a g-orthonormal basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$, if for all $f \in \mathcal{H}$, $\sum_{i\in I} \|\Lambda_i f\|^2 = \|f\|^2$, and

$$\langle \Lambda_i^* g_i, \Lambda_j^* g_j \rangle = \delta_{ij} \langle g_i, g_j \rangle, \quad g_i \in \mathcal{H}_i,\ g_j \in \mathcal{H}_j, \quad i, j \in I.$$

2. Dilation of g-frames

The concepts of disjoint g-frames and strongly disjoint g-frames were introduced in [1]. In this section, we investigate dilation of g-frames and we show that disjoint g-frames for a Hilbert space have dilation property.

Definition 2.1. Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\Theta = \{\Theta_i \in B(\mathcal{K}, \mathcal{H}_i) : i \in I\}$ be g-frames for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Then Λ and Θ are called

(1) disjoint, if $RangeT^*_\Lambda \cap RangeT^*_\Theta = \{0\}$ and $RangeT^*_\Lambda + RangeT^*_\Theta$ is a closed subspace of $\widehat{\mathcal{H}}$.
(2) complementary pair, if $RangeT^*_\Lambda \cap RangeT^*_\Theta = \{0\}$ and

$$RangeT^*_\Lambda + RangeT^*_\Theta = \widehat{\mathcal{H}}.$$

(3) weakly disjoint if $RangeT^*_\Lambda \cap RangeT^*_\Theta = \{0\}$.

Proposition 2.2 ([1]). *Two g-frames $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\Theta = \{\Theta_i \in B(\mathcal{K}, \mathcal{H}_i) : i \in I\}$ are disjoint if and only if $\{\Gamma_i \in B(\mathcal{H}\oplus\mathcal{K}, \mathcal{H}_i) : i \in I\}$ is a g-frame for $\mathcal{H}\oplus\mathcal{K}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$, where*

$$\Gamma_i : \mathcal{H}\oplus\mathcal{K} \to \mathcal{H}_i, \quad \Gamma_i(f \oplus g) = \Lambda_i f + \Theta_i g, \tag{2.1}$$

for all $i \in I$.

Proposition 2.3 ([1]). *Two g-frames $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\Theta = \{\Theta_i \in B(\mathcal{K}, \mathcal{H}_i) : i \in I\}$ are complementary pair if and only if $\{\Gamma_i \in B(\mathcal{H}\oplus\mathcal{K}, \mathcal{H}_i) : i \in I\}$ is a g-Riesz basis for $\mathcal{H}\oplus\mathcal{K}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$, where Γ_i is defined by (2.1), for all $i \in I$.*

Proposition 2.4 ([1]). *Two g-frames $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\Theta = \{\Theta_i \in B(\mathcal{K}, \mathcal{H}_i) : i \in I\}$ are weakly disjoint if and only if*

$$\{f \oplus g : \Gamma_i(f \oplus g) = 0, \forall i \in I\} = \{0\},$$

where where Γ_i is defined by (2.1), for all $i \in I$.

Proposition 2.5. *Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\Theta = \{\Theta_i \in B(\mathcal{K}, \mathcal{H}_i) : i \in I\}$ be Parseval g-frames for $\mathcal{H}$ and $\mathcal{K}$, respectively. Then $RangeT^*_\Lambda \oplus RangeT^*_\Theta = \widehat{\mathcal{H}}$ if and only if $\{\Gamma_i\}_{i\in I}$ is a g-orthonormal basis for $\mathcal{H}\oplus\mathcal{K}$, where Γ_i is defined by (2.1), for all $i \in I$.*

Proof. If $\{\Gamma_i\}_{i\in I}$ is a g-orthonormal basis for $\mathcal{H}\oplus\mathcal{K}$ then

$$\begin{aligned} \|f\|^2 + \|g\|^2 &= \sum_{i\in I} \|\Gamma_i(f \oplus g)\|^2 \\ &= \sum_{i\in I} \|\Lambda_i f\|^2 + \sum_{i\in I} \|\Theta_i g\|^2 + 2Re \sum_{i\in I} \langle \Lambda_i f, \Theta_i g\rangle, \end{aligned}$$

and

$$Re \sum_{i\in I} \langle \Lambda_i f, \Theta_i g\rangle = 0, \quad f \in \mathcal{H},\ g \in \mathcal{K}. \tag{2.2}$$

If we replace g by ig in (2.2), then

$$Im\sum_{i\in I}\langle\Lambda_i f,\Theta_i g\rangle = 0,\quad f\in\mathcal{H},\ g\in\mathcal{K}.$$

Therefore $RangeT_\Lambda^* \perp RangeT_\Theta^*$. Since $\Gamma=\{\Gamma_i\in B(\mathcal{H}\oplus\mathcal{K},\mathcal{H}_i): i\in I\}$ is a g-orthonormal basis, T_Γ^* is onto. But $RangeT_\Lambda^* + RangeT_\Theta^* = RangeT_\Gamma^*$, hence $RangeT_\Lambda^* + RangeT_\Theta^* = \widehat{\mathcal{H}}$.
For the converse implication, we have

$$\begin{aligned}\sum_{i\in I}\|\Gamma_i(f\oplus g)\|^2 &= \sum_{i\in I}\|\Lambda_i f+\Theta_i g\|^2 = \sum_{i\in I}\|\Lambda_i f\|^2 + \sum_{i\in I}\|\Theta_i g\|^2\\ &=\|f\|^2+\|g\|^2 = \|f\oplus g\|^2,\end{aligned}$$

for all $f\oplus g\in\mathcal{H}\oplus\mathcal{K}$. If $\{g_i\}_{i\in I}\in\widehat{\mathcal{H}}$, then $\{g_i\}_{i\in I}=\{\Lambda_i f\}_{i\in I}+\{\Theta_i g\}_{i\in I}$ for some $f\in\mathcal{H}$ and for some $g\in\mathcal{K}$. Therefor $g_i=\Lambda_i f+\Theta_i g$, for all $i\in I$. We have

$$\begin{aligned}\Big\|\sum_{i\in I}\Gamma_i^* g_i\Big\|^2 &= \Big\|\sum_{i\in I}(\Lambda_i^* g_i\oplus\Theta_i^* g_i)\Big\|^2 = \Big\|\sum_{i\in I}\Lambda_i^* g_i\Big\|^2 + \Big\|\sum_{i\in I}\Theta_i^* g_i\Big\|^2\\ &=\Big\|\sum_{i\in I}\Lambda_i^*(\Lambda_i f+\Theta_i g)\Big\|^2 + \Big\|\sum_{i\in I}\Theta_i^*(\Lambda_i f+\Theta_i g)\Big\|^2\\ &=\Big\|f+\sum_{i\in I}\Lambda_i^*\Theta_i g\Big\|^2 + \Big\|g+\sum_{i\in I}\Theta_i^*\Lambda_i f\Big\|^2.\end{aligned}$$

Since $\sum_{i\in I}\Lambda_i^*\Theta_i g=0$ and $\sum_{i\in I}\Theta_i^*\Lambda_i f=0$,

$$\begin{aligned}\Big\|\sum_{i\in I}\Gamma_i^* g_i\Big\|^2 &= \|f\|^2+\|g\|^2 = \sum_{i\in I}\|\Lambda_i f\|^2 + \sum_{i\in I}\|\Theta_i g\|^2\\ &= \sum_{i\in I}\|\Lambda_i f+\Theta_i g\|^2 = \sum_{i\in I}\|g_i\|^2.\end{aligned}$$

So

$$\Big\|\sum_{i\in I}\Gamma_i^* g_i\Big\|^2 = \sum_{i\in I}\|g_i\|^2,\quad \{g_i\}_{i\in I}\in\widehat{\mathcal{H}}. \tag{2.3}$$

By (2.3) we have

$$\|\Gamma_i^* g_i\|^2 = \|g_i\|^2;\quad i\in I,\ g_i\in\mathcal{H}_i. \tag{2.4}$$

Again, (2.3) implies that

$$\|\Gamma_i^* g_i+\Gamma_j^* g_j\|^2 = \|g_i\|^2+\|g_j\|^2;\quad i,j\in I,\quad g_i\in\mathcal{H}_i, g_j\in\mathcal{H}_j,$$

or

$$\|\Gamma_i^* g_i\|^2+\|\Gamma_j^* g_j\|^2 + 2Re\langle\Gamma_i^* g_i,\Gamma_j^* g_j\rangle = \|g_i\|^2+\|g_j\|^2;\quad g_i\in\mathcal{H}_i, g_j\in\mathcal{H}_j,$$

for all $i,j\in I$. Therefore, by (2.4)

$$\langle\Gamma_i^* g_i,\Gamma_j^* g_j\rangle = \delta_{ij}\langle g_i,g_j\rangle,\quad g_i\in\mathcal{H}_i,\ g_j\in\mathcal{H}_j,$$

for all $i,j\in I$. □

Let $F = \{f_i\}_{i\in I}$ be a Riesz basis for a Hilbert space $\mathcal{H}$ with unique dual frame $\tilde{F} = \{\tilde{f}_i\}_{i\in I}$. If $M \subset \mathcal{H}$ is a closed subspace of $\mathcal{H}$ and P is the orthogonal projection form $\mathcal{H}$ onto M, then $PF = \{Pf_i\}_{i\in I}$ is a frame for M with dual frame $\widetilde{PF} = \{P\tilde{f}_i\}_{i\in I}$. In general, $P\tilde{F} = \{P\tilde{f}_i\}_{i\in I}$ is not the canonical dual of $PF = \{Pf_i\}_{i\in I}$. But, if P commutes with the frame operator S_F, then $\widetilde{PF} = \{P\tilde{f}_i\}_{i\in I}$ is the canonical dual of $PF = \{Pf_i\}_{i\in I}$ (see [7]). Here, we generalize this result to g-frames.

Proposition 2.6. *Let P be an orthogonal projection from $\mathcal{H}$ onto a closed subspace M and let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$. Then $\Lambda P = \{\Lambda_i P \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-frame for M with respect to $\{\mathcal{H}_i\}_{i\in I}$ and*

$$\forall i \in I, \quad \widetilde{\Lambda_i P} = \widetilde{\Lambda}_i P \Leftrightarrow PS_\Lambda^{-1} = S_\Lambda^{-1} P,$$

where $\widetilde{\Lambda} = \{\widetilde{\Lambda}_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\widetilde{\Lambda P} = \{\widetilde{\Lambda_i P} \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ are canonical dual of Λ and ΛP, respectively.

Proof. Let $f \in M$ and A, B be the g-frame bounds for Λ, then

$$A\|f\|^2 = A\|Pf\|^2 \leq \sum_{i\in I} \|\Lambda_i P f\|^2 \leq B\|Pf\|^2 = B\|f\|^2.$$

If $\widetilde{\Lambda_i P} = \widetilde{\Lambda}_i P$, for all $i \in I$, then $\Lambda_i P S_{\Lambda P}^{-1} = \Lambda_i S_\Lambda^{-1} P$, for all $i \in I$. Therefore, we have $PS_{\Lambda P}^{-1} = S_\Lambda^{-1} P$, and so $PS_{\Lambda P}^{-1} = PS_\Lambda^{-1} P$, which implies that $S_\Lambda^{-1} P = PS_\Lambda^{-1} P$. By taking adjoint we get $PS_\Lambda^{-1} = PS_\Lambda^{-1} P$, and hence $PS_\Lambda^{-1} = S_\Lambda^{-1} P$.
Now we assume that $PS_\Lambda^{-1} = S_\Lambda^{-1} P$ and $f \in M$, then

$$f = \sum_{i\in I} (\Lambda_i P)^* (\widetilde{\Lambda_i P}) f = \sum_{i\in I} P\Lambda_i^* \Lambda_i P S_{\Lambda P}^{-1} f. \tag{2.5}$$

Since $f \in M \subseteq \mathcal{H}$, we can write $f = \sum_{i\in I} \Lambda_i^* \Lambda_i S_\Lambda^{-1} f$ or

$$f = Pf = \sum_{i\in I} P\Lambda_i^* \Lambda_i S_\Lambda^{-1} P f.$$

Now, (2.5) and our assumption imply that

$$\begin{aligned} 0 = &\sum_{i\in I} P\Lambda_i^* \Lambda_i (PS_{\Lambda P}^{-1} - S_\Lambda^{-1} P) f = \sum_{i\in I} P\Lambda_i^* \Lambda_i P (PS_{\Lambda P}^{-1} - S_\Lambda^{-1} P) f \\ = &S_{\Lambda P}(PS_{\Lambda P}^{-1} f - S_\Lambda^{-1} P f), \end{aligned}$$

for all $f \in M$. Therefor $PS_{\Lambda P}^{-1} = S_\Lambda^{-1} P$, and so $\widetilde{\Lambda_i P} = \widetilde{\Lambda}_i P$, for all $i \in I$. □

Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\Theta = \{\Theta_i \in B(\mathcal{K}, \mathcal{H}_i) : i \in I\}$ be g-frames for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. We recall that Λ and Θ are unitarily equivalent (similar), if there exists a unitary (an invertible) operator $U \in B(\mathcal{H}, \mathcal{K})$ such that

$$\Lambda_i = \Theta_i U, \quad i \in I.$$

Proposition 2.7. *Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\Theta = \{\Theta_i \in B(\mathcal{K}, \mathcal{H}_i) : i \in I\}$ be g-frames for non zero Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. If Λ and Θ are unitarily equivalent (similar), then*

$$\overline{span}\{\Gamma_i^*(\mathcal{H}_i)\}_{i\in I} \neq \mathcal{H} \oplus \mathcal{K},$$

where Γ_i is defined by (2.1), for all $i \in I$.

Proof. Let $U \in B(\mathcal{H}, \mathcal{K})$ be a unitary (an invertible) operator such that $\Lambda_i = \Theta_i U$ for any $i \in I$. If $0 \neq g \in \mathcal{K}$, then there exists $f \in \mathcal{H}$ and $Uf = -g$. Then $\Theta_i(Uf + g) = 0$, for all $i \in I$. Hence

$$\{f \oplus g : \Gamma_i(f \oplus g) = 0,\ i \in I\} \neq \{0\},$$

consequently $\overline{span}\{\Gamma_i^*(\mathcal{H}_i)\}_{i\in I} \neq \mathcal{H} \oplus \mathcal{K}$, (see [8]). □

Corollary 2.8. *Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\Theta = \{\Theta_i \in B(\mathcal{K}, \mathcal{H}_i) : i \in I\}$ be respective g-frames for non zero Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. If Λ and Θ are unitarily equivalent (similar), then Λ and Θ can not be weakly disjoint. Moreover, If Λ and Θ are unitarily equivalent (similar), then $\Gamma = \{\Gamma_i \in B(\mathcal{H} \oplus \mathcal{K}, \mathcal{H}_i) : i \in I\}$ is not a g-frame for $\mathcal{H} \oplus \mathcal{K}$, where Γ_i is defined by (2.1), for all $i \in I$.*

Let $\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for $\mathcal{H}_i$, for every $i \in I$. It is proved in [8], $\{E_{ij}\}_{i\in I,\, j\in J_i}$ is an orthonormal basis for $\widehat{\mathcal{H}}$, where

$$(E_{ij})_k = \begin{cases} e_{ij}, & i = k \\ 0, & i \neq k. \end{cases} \tag{2.6}$$

We use the above fact in the rest of this paper.

Proposition 2.9. *Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g-frame for Hilbert space $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$. Then there exist a Hilbert space $\mathcal{H} \subset K$ and a g-Riesz basis $\Delta = \{\Delta_i \in B(K, \mathcal{H}_i) : i \in I\}$ for K with respect to $\{\mathcal{H}_i\}_{i\in I}$, such that $\Lambda_i = \Delta_i P_{\mathcal{H}}$ for all $i \in I$, where $P_{\mathcal{H}}$ is the orthogonal projection from K onto $\mathcal{H}$.*

Proof. Let $\Theta_i = \Lambda_i S_\Lambda^{-\frac{1}{2}}$, for all $i \in I$. Then $\Theta = \{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a Parseval g-frames for $\mathcal{H}$ and $RangeT_\Theta^* = RangeT_\Lambda^*$. Let P be the orthogonal projection from $\widehat{\mathcal{H}}$ onto $RangeT_\Theta^*$. We define the operators

$$\varphi_i : P^\perp \widehat{\mathcal{H}} \to \mathcal{H}_i, \quad \varphi_i(g) = \sum_{j\in J_i} \langle g, P^\perp E_{ij}\rangle e_{ij}, \tag{2.7}$$

for all $i \in I$, where E_{ij} is defined by (2.6). Then $\varphi = \{\varphi_i \in B(P^\perp \widehat{\mathcal{H}}, \mathcal{H}_i) : i \in I\}$ is a Parseval g-frame for $P^\perp \widehat{\mathcal{H}}$. In fact

$$\sum_{i\in I} \|\varphi_i g\|^2 = \sum_{i\in I} \Big\| \sum_{j\in J_i} \langle g, P^\perp E_{ij}\rangle e_{ij} \Big\|^2 = \sum_{i\in I} \sum_{j\in J_i} |\langle g, P^\perp E_{ij}\rangle|^2 = \|g\|^2,$$

for all $g \in P^{\perp}\widehat{\mathcal{H}}$. We have

$$\begin{aligned}\sum_{i\in I}\langle\Theta_i f, \varphi_i g\rangle &= \sum_{i\in I}\Big\langle\Theta_i f, \sum_{j\in J_i}\langle g, P^{\perp}E_{ij}\rangle e_{ij}\Big\rangle\\ &= \sum_{i\in I}\sum_{j\in J_i}\langle\Theta_i f, e_{ij}\rangle\overline{\langle g, P^{\perp}E_{ij}\rangle}\\ &=\Big\langle\sum_{i\in I}\sum_{j\in J_i}\langle\Theta_i f, e_{ij}\rangle P^{\perp}E_{ij}, g\Big\rangle\\ &=\langle P^{\perp}T^*_{\Theta}f, g\rangle = \langle 0, g\rangle = 0,\end{aligned}$$

for all $f \in \mathcal{H}$ and $g \in P^{\perp}\widehat{\mathcal{H}}$. So,

$$Range T^*_{\Theta} \perp Range T^*_{\varphi}. \tag{2.8}$$

On the other hand, if $g = \{g_i\}_{i\in I} \in P^{\perp}\widehat{\mathcal{H}}$ then we have

$$\begin{aligned}\varphi_i g &= \sum_{j\in J_i}\langle g, P^{\perp}E_{ij}\rangle e_{ij} = \sum_{j\in J_i}\langle\{g_i\}_{i\in I}, E_{ij}\rangle e_{ij}\\ &= \sum_{j\in J_i}\langle g_i, e_{ij}\rangle e_{ij} = g_i,\end{aligned}$$

so, $g = \{\varphi_i g\}_{i\in I}$. Thus

$$P^{\perp}g = \{\varphi_i(P^{\perp}g)\}_{i\in I};\ \ g = Pg + T^*_{\varphi}(P^{\perp}g), \quad g \in \widehat{\mathcal{H}}.$$

consequently

$$\widehat{\mathcal{H}} = Range T^*_{\Theta} + Range T^*_{\varphi}. \tag{2.9}$$

According to the Proposition 2.5, (2.8) and (2.9) imply that $\{\Gamma_i\}_{i\in I}$ is a g-orthonormal basis for $\mathcal{H} \oplus P^{\perp}\widehat{\mathcal{H}}$, where

$$\Gamma_i : \mathcal{H} \oplus P^{\perp}\widehat{\mathcal{H}} \to \mathcal{H}_i, \quad \Gamma_i(f \oplus g) = \Theta_i f + \varphi_i g. \tag{2.10}$$

We define the operator $F \in B(\mathcal{H}\oplus P^{\perp}\widehat{\mathcal{H}})$ by $F(f\oplus g) = S^{\frac{1}{2}}_{\Lambda} f\oplus g$, then F is invertible. Let $\Delta_i = \Gamma_i F$, for all $i \in I$. In this case, $\{\Delta_i\}_{i\in I}$ is a g-Riesz basis for $K = \mathcal{H}\oplus P^{\perp}\widehat{\mathcal{H}}$ (see [2]). Clearly, $\Delta_i P_{\mathcal{H}} = \Lambda_i$, for all $i \in I$. □

Definition 2.10. Let $\mathcal{F}$ be a family of g-frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$. We say that $\mathcal{F}$ has dilation property, if there is a larger Hilbert space $\mathcal{H} \subset K$ such that for every $\Lambda = \{\Lambda_i\}_{i\in I} \in \mathcal{F}$, there exists a g-Riesz basis $\Gamma = \{\Gamma_i\}_{i\in I}$ for K such that $\Lambda_i = \Gamma_i P_{\mathcal{H}}$, for all $i \in I$, where $P_{\mathcal{H}}$ is orthogonal projection from K onto $\mathcal{H}$.

In the next proposition we provide some sufficient conditions, under which a family of g-frames with two members has dilation property.

Proposition 2.11. *Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\Theta = \{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be two g-frames for Hilbert spaces $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$. If one of the following conditions holds, then $\mathcal{F} = \{\Lambda, \Theta\}$ has the dilation property.*

(1) Λ and Θ are similar.
(2) Λ and Θ are disjoint.
(3) Θ is similar to a dual g-frame of Λ.

Proof. (1) Let $T \in B(\mathcal{H})$ be an invertible operator and $\Theta_i = \Lambda_i T$, for all $i \in I$. By Proposition 2.9, then there exist a Hilbert space $\mathcal{H} \subset K$ ($K = \mathcal{H} \oplus P^{\perp}\widehat{\mathcal{H}}$, where $P_{\mathcal{H}}$ is the orthogonal projection from $\widehat{\mathcal{H}}$ onto $RangeT^*_{\Lambda}$) and a g-Riesz basis $\Gamma = \{\Gamma_i \in B(K, \mathcal{H}_i) : i \in I\}$ for K with $\Lambda_i = \Gamma_i P_{\mathcal{H}}$ for all $i \in I$. Let us define $\Delta_i \in B(K, \mathcal{H}_i)$ by $\Delta_i = \Gamma_i(T \oplus I)$, where

$$T \oplus I : K \to K, \quad (T \oplus I)(f \oplus g) = Tf \oplus g.$$

Since $T \oplus I$ is invertible and $\Gamma = \{\Gamma_i\}_{i\in I}$ is a g-Riesz basis for K, then $\Delta = \{\Delta_i\}_{i\in I}$ is a g-Riesz basis for K and $\Theta_i = \Delta_i P_{\mathcal{H}}$ for all $i \in I$.

(2) Since $\Lambda = \{\Lambda_i\}_{i\in I}$ and $\Theta = \{\Theta_i\}_{i\in I}$ are disjoint, by Proposition 2.2, $\{\psi_i\}_{i\in I}$ and $\{\varphi_i\}_{i\in I}$ are g-frames for $\mathcal{H} \oplus \mathcal{H}$, where for all $i \in I$, $\psi_i, \varphi_i : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H}_i$ defined by

$$\psi_i(f \oplus g) = \Lambda_i f + \Theta_i g, \quad \varphi_i(f \oplus g) = \Theta_i f + \Lambda_i g, \quad f, g \in \mathcal{H}.$$

From the other hand, $\{\psi_i\}_{i\in I}$ and $\{\varphi_i\}_{i\in I}$ are similar. Hence by (1), there exist a Hilbert space $\mathcal{H} \oplus \mathcal{H} \subset K$, and two g-Riesz basis $\Gamma = \{\Gamma_i\}_{i\in I}$ and $\Delta = \{\Delta_i\}_{i\in I}$ for K with respect to $\{\mathcal{H}_i\}_{i\in I}$, such that $\psi_i = \Gamma_i P_{\mathcal{H}\oplus\mathcal{H}}$ and $\varphi_i = \Delta_i P_{\mathcal{H}\oplus\mathcal{H}}$ for all $i \in I$, where $P_{\mathcal{H}\oplus\mathcal{H}}$ is the orthogonal projection from K onto $\mathcal{H} \oplus \mathcal{H}$. If we identify $\mathcal{H}$ by $\mathcal{H} \oplus 0 \oplus 0$ and consider $P_{\mathcal{H}}$ is the orthogonal projection from K onto $\mathcal{H} \oplus 0 \oplus 0$, then $\Lambda_i = \Gamma_i P_{\mathcal{H}}$ and $\Theta_i = \Delta_i P_{\mathcal{H}}$ for all $i \in I$.

(3) Let $\phi = \{\phi_i\}_{i\in I}$ be a dual g-frame for $\Lambda = \{\Lambda_i\}_{i\in I}$ and $T \in B(\mathcal{H})$ be an invertible operator so that $\Theta_i = \phi_i T$, for all $i \in I$. By Theorem 2.9 of [1], there exists a Hilbert space $\mathcal{H} \subset K$ and two g-Riesz basis $\Gamma = \{\Gamma_i\}_{i\in I}$ and $\Delta = \{\Delta_i\}_{i\in I}$ for K with $\Lambda_i = \Gamma_i P_{\mathcal{H}}$ and $\phi_i = \Delta_i P_{\mathcal{H}}$ for all $i \in I$, where $P_{\mathcal{H}}$ is the orthogonal projection from K onto $\mathcal{H}$. Let us define

$$W_i : K \to \mathcal{H}_i, \quad W_i = \Delta_i(T \oplus I), \quad i \in I.$$

Then $W = \{W_i\}_{i\in I}$ is a g-Riesz basis for K with respect to $\{\mathcal{H}_i\}_{i\in I}$, and $\Theta_i = W_i P_{\mathcal{H}}$, for all $i \in I$. □

Definition 2.12. Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g-frame for $\mathcal{H}$. We define the deficiency of Λ to be $\dim(RangeT^*_{\Lambda})^{\perp}$.

In the following theorem we provide a sufficient condition for a family of g-frame $\mathcal{F}$ such that $\mathcal{F}$ has the dilation property.

Theorem 2.13. *Let $\mathcal{F}$ be a family of g-frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$. Then $\mathcal{F}$ has the dilation property if all members of $\mathcal{F}$ have the equal deficiency.*

Proof. Fix a g-frame $\Lambda = \{\Lambda_i\}_{i\in I}$ in $\mathcal{F}$ and let $\Theta = \{\Theta_i\}_{i\in I}$ be any g-frame in $\mathcal{F}$. Let $K = \mathcal{H} \oplus P^{\perp}\widehat{\mathcal{H}}$ and $M = \mathcal{H} \oplus Q^{\perp}\widehat{\mathcal{H}}$, where P and Q are the orthogonal projection from $\widehat{\mathcal{H}}$ onto $RangeT^*_{\Lambda}$ and $RangeT^*_{\Theta}$, respectively. We define

$$\varphi_i : P^{\perp}\widehat{\mathcal{H}} \to \mathcal{H}_i, \quad \varphi_i(g) = \sum_{j\in J_i} \langle g, P^{\perp}E_{ij}\rangle e_{ij},$$

and

$$\psi_i : Q^{\perp}\widehat{\mathcal{H}} \to \mathcal{H}_i, \quad \psi_i(h) = \sum_{j\in J_i} \langle h, Q^{\perp}E_{ij}\rangle e_{ij},$$

for all $i \in I$, where E_{ij} is defined by 2.6. Then $\varphi = \{\varphi_i\}_{i\in I}$ and $\psi = \{\psi_i\}_{i\in I}$ are respective g-frames for $P^{\perp}\widehat{\mathcal{H}}$ and $Q^{\perp}\widehat{\mathcal{H}}$. Now, we consider bounded operators

$$\Gamma_i : K \to \mathcal{H}_i, \quad \Gamma_i(f \oplus g) = \Lambda_i f + \varphi_i g, \tag{2.11}$$

and

$$\Phi_i : M \to \mathcal{H}_i, \quad \Phi_i(f \oplus h) = \Theta_i f + \psi_i h. \tag{2.12}$$

A argument similar to the proof of Proposition 2.9 shows that

$$\widehat{\mathcal{H}} = RangeT^*_{\Lambda} + RangeT^*_{\varphi}, \quad RangeT^*_{\Lambda} \perp RangeT^*_{\varphi}.$$

So by Proposition 2.3, $\Gamma = \{\Gamma_i\}_{i\in I}$ is a g-Riesz basis for K with respect to $\{\mathcal{H}_i\}_{i\in I}$. Similarly, $\Phi = \{\Phi_i\}_{i\in I}$ is a g-Riesz basis for M with respect to $\{\mathcal{H}_i\}_{i\in I}$. Since $\dim(RangeT^*_{\Lambda})^{\perp} = \dim(RangeT^*_{\Theta})^{\perp}$, there is a unitary operator W from $(RangeT^*_{\Lambda})^{\perp}$ onto $(RangeT^*_{\Theta})^{\perp}$. In fact, if $\{x_i\}_{i\in J}$ and $\{y_i\}_{i\in J}$ are orthonormal bases for $(RangeT^*_{\Lambda})^{\perp}$ and $(RangeT^*_{\Theta})^{\perp}$, respectively, then we may consider

$$W : (RangeT^*_{\Lambda})^{\perp} \to (RangeT^*_{\Theta})^{\perp}, \quad Wf = \sum_{i\in J} \langle f, x_i\rangle y_i.$$

It is easy to show that W is a unitary operator. Let us define

$$\Delta_i : K \to \mathcal{H}_i, \quad \Phi_i(f \oplus g) = \Theta_i f + \psi_i W g, \quad i \in I.$$

Since $\Delta_i = \Phi_i F$, for all $i \in I$ and the operator

$$F : K \to M, \quad F(f \oplus g) = f \oplus Wg$$

is invertible, $\Delta = \{\Delta_i\}_{i\in I}$ is a g-Riesz basis for K. Clearly, $\Gamma_i P_{\mathcal{H}} = \Lambda_i$ and $\Delta_i P_{\mathcal{H}} = \Theta_i$ for ever $i \in I$, therefore $\mathcal{F}$ has the dilation property. □

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