Wavelets and Linear Algebra 1 (2014) 1-7

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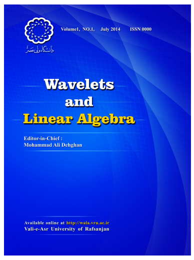

A recursive construction of a class of finite normalized tight frames

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ARTICLE INFO

Article history:
Received 12 March 2013
Accepted 29 April 2014
Available online 1 July 2014
Communicated by Asghar Rahimi

Keywords:
Frames
Tight frames

2000 MSC:
42C15

ABSTRACT

Finite normalized tight frames are interesting because they provide decompositions in applications and some physical interpretations. In this article, we give a recursive method for constructing them.

1. Introduction

Frames were first introduced in 1952 by Duffin and Schaeffer [4] in the context of nonharmonic Fourier series. They are system of functions in Hilbert spaces that provide numerically stable methods for finding overcomplete decompositions of vectors, and such are useful tools in various

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signal processing applications, data compression, wireless communications and so on [6, 10, 11]. Frames in finite dimensional Hilbert spaces have become of interests for many of researches [1, 3]. One of the important subjects in these areas is the way for constructing such frames. Some methods of construction finite tight frames are stated by researchers [7, 12, 5, 9].

In this paper, we provide a recursive method for constructing a finite normalized tight frame (FNTF) for $\mathbb{R}^{n+1}$ from the special FNTF of $\mathbb{R}^n$, and we use it to figure out how we can construct a FNTF for $\mathbb{R}^3$ which is akin to the Nth roots of unity in $\mathbb{C}$.

2. Frames

Letting $\mathbb{K}$ be either the real or complex filed, a sequence of vectores $\{f_i\}_{i\in I}$ in Hilbert space $\mathcal{H}$ over $\mathbb{K}$ is said to be a *frame* (see also [2]) for $\mathcal{H}$ if there exist constants A and B such that $0 < A \leq B < \infty$ and

$$A \parallel f \parallel^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B \parallel f \parallel^2 \quad \forall f \in \mathcal{H} \tag{2.1}$$

The frame $\{f_i\}_{i\in I}$ is said to be *tight*(or A-*tight*) if $A = B$. In this case, A is said to be the *frame constant* and it is a *Parsval frame*, if $A = B = 1$. When the index set I is a finite set, the frame will be called *finite*. A *normalized frame* is the one which elements have the norm one. A normalized frame with the property that there is a constant c so that $|\langle f_i, f_j\rangle| = c$, for all $i \neq j$, is called an *equiangular frame*.

If only the right side of the inequalities (2.1) holds, then $\{f_i\}_{i\in I}$ is called a *Bessel sequence*. If $\mathcal{H}$ is finite dimensional and $\{f_i\}_{i\in I}$ is a normalized Bessel sequence then $\{f_i\}_{i\in I}$ is finite sequence [1]. To each Bessel sequence $\{f_i\}_{i\in I}$, corresponds an operator

$$T : \mathcal{H} \to l^2(I) \quad T(f) = \{\langle f, f_i\rangle\}_{i\in I}$$

called *analysis operator*, where $l^2(I)$ is the space of all complex sequences $\{c_i\}_{i\in I}$ such that $\sum_{i\in I} | c_i |^2 < \infty$. This is well-defined and bounded operator. Its adjoint is the operator

$$T^* : l^2(I) \to \mathcal{H} \quad T^*(\{c_i\}_{i\in I}) = \sum_{i\in I} c_i f_i,$$

called the *synthesis operator*. If $\{f_i\}_{i\in I}$ is a frame with frame bounds A and B, then the operator

$$T^*T : \mathcal{H} \to \mathcal{H}, \quad T^*T(f) = \sum_{i\in I} \langle f, f_i\rangle f_i$$

is called the *frame operator* of the frame $\{f_i\}_{i\in I}$. It is a positive, self-adjoint, bounded and hence invertible operator with the inverse $(T^*T)^{-1}$. In fact, $AI \leq T^*T \leq BI$ and $B^{-1}I \leq (T^*T)^{-1} \leq A^{-1}I$. If $\{g_i\}_{i\in I}$ is another sequence in $\mathcal{H}$ such that each $f \in \mathcal{H}$ can be represented as $f = \sum_{i\in I}\langle f, f_i\rangle g_i$, then $\{g_i\}_{i\in I}$ is called the *dual frame* for $\{f_i\}_{i\in I}$. The most often-used dual frame is the *canonical dual*, namely the pseudoinverse $\{(T^*T)^{-1} f_i\}_{i\in I}$. Note that computing a canonical dual involves the inversion of the frame operator. Having this dual, we get the following reconstruction formula:

$$f = T^*T(T^*T)^{-1}(f) = \sum_{i\in I} \langle f, (T^*T)^{-1} f_i\rangle f_i.$$

If $\{f_i\}_{i\in I}$ is a tight frame, i.e. $A = B$, then $T^*T = AI$ and hence we have $f = \frac{1}{A}\sum_{i\in I}\langle f, f_i\rangle f_i$ for every $f \in \mathcal{H}$. A finite normalized tight frame with the frame constant A will be called A-FNTF. FNTFs provide Parseval-like decomposition in terms of nonorthogonal vectors of unit norm.

FNTFs with k elements are known to exists for any $\mathbb{C}^n$ where $k \geq n$. The standard example is the *harmonic frame*, which synthesis operator is obtained by extracting any n distinct rows from a suitably scaled $k \times k$ discrete Fourier transform matrix.

At the sequel, we suppose that $\mathcal{H}$ is a finite-dimensional Hilbert space with dimension n over $\mathbb{R}$. According to this, our frame will be the form $\{f_i\}_{i=1}^k$ whenever k is some positive integer. Also we will replace $l^2(I)$ by $\mathbb{R}^k$. In this finite dimensional Hilbert space, every finite sequence $\{f_i\}_{i=1}^k$ is a Bessel sequence. In particular, $\{f_i\}_{i=1}^k$ is a frame if and only if the $span\{f_i\}_{i=1}^k = \mathcal{H}$, which necessitates $n \leq k$.

Any finite sequence $\{f_i\}_{i=1}^k \subset \mathcal{H}$ is necessarily Bessel, guaranteeing the existence of the analysis and synthesis operator in the finite case. In fact, the operators

$$T : \mathcal{H} \to \mathbb{R}^k, \quad T^* : \mathbb{R}^k \to \mathcal{H}, \quad T^*T : \mathcal{H} \to \mathcal{H},$$

from left to right, can be represented as $k \times n, n \times k$ and $n \times n$ matrices, respectively. In the next section we state a method of making FNTF by using the following lemma, which is due to Benedetto and Fickus [1].

Lemma 2.1. *If $\{f_i\}_{i=1}^k$ is an A-FNTF for a n-dimensional Hilbert space $\mathcal{H}$, then $A = \frac{k}{n}$.*

3. Generation of finite tight frames

In this section we provide general methods for constructing FNTFs. In brief, we want to construct $n \times k$ synthesis matrices F which have:

i) columns of unit norm;

ii) orthogonal rows, meaning the frame operator FF^* is diagonal;

iii) rows of constant norm, meaning FF^* is a constant multiple of the identity matrix.

Despite a decade of study, very few general constructions of FNTF are known. Moreover, these known methods unfortunately mainpulate all frame elements simultaneously. In this section, we show that constructing certain examples of FNTFs need not be so difficult. In particular, we provide a new, iterative method for constructing FNTF. The key idea is to iteratively build a matrix F which, at each iteration, exactly satisfies (i), (ii) and (iii).

Benedetto and Fickus [1] have been brought some examples of FNTF for $\mathbb{R}^2$ and $\mathbb{R}^3$ such as, Roots of unity (The N^{th} roots of unity form a FNTF for $\mathbb{R}^2$) and Regular solid (vectors in $\mathbb{R}^3$ pointing to the vertices of regular solid inscribed with in a unit sphere S^2) (see also [9]). Example of regular solids are the tetrahedron, cube, octahedron, dodecahedron, icosahedron and soccer ball.

A question of interest in this area is whether FNTF of a given k elements exist for the Hilbert space $\mathbb{R}^n$. Zimmermann [13] has been answered this by using the Fourier matrix in the complex case. The next theorem state that if there exist FNTF $\{f_i\}_{i=1}^k$ for $\mathbb{R}^n$ which $\sum_{i=1}^k f_i = 0$ then there exist a FNTF with $k+1$elements for $\mathbb{R}^{n+1}$. We shall demonstrate an algorithm to construct FNTF of $n+1$elements for $\mathbb{R}^n$. Our inspiration in this regard are generalization the roots of unity to three dimensions.

Theorem 3.1. *Suppose* $\{f_i\}_{i=1}^k$ *is a FNTF for* $\mathbb{R}^n$ *which* $\sum_{i=1}^k f_i = 0$. *Then we can construct a FNTF* $\{g_i\}_{i=1}^{k+1}$ *for* $\mathbb{R}^{n+1}$ *from* $\{f_i\}_{i=1}^k$.

Proof. Let $F = \begin{pmatrix} | & | & \dots & | \\ f_1 & f_2 & \dots & f_k \\ | & | & \dots & | \end{pmatrix}$ synthesis matrix corresponding to the frame $\{f_i\}_{i=1}^k$. Put $Y = \begin{pmatrix} y_1 & y_2 & \dots & y_n \end{pmatrix}^t$, $S = \begin{pmatrix} 1 & 1 & \dots & 1 \end{pmatrix}_{1\times k}$, and define an $(n+1)\times(k+1)$ block matrix U by

$$U = \begin{bmatrix} \sqrt{x}F & Y \\ \sqrt{1-x}S & y_{n+1} \end{bmatrix}$$

It is enough to find x, y_{n+1} and the entries of $Y, y_i (1 \le i \le n)$, such that the columns of U form a FNTF for $\mathbb{R}^{n+1}$ with $k+1$ elements i.e. $UU^* = \frac{k+1}{n+1}I$. Since

$$\begin{aligned} UU^* &= \begin{bmatrix} \sqrt{x}F^* & Y \\ \sqrt{1-x}S & y_{n+1} \end{bmatrix}\begin{bmatrix} \sqrt{x}F & \sqrt{1-x}S^t \\ Y^t & y_{n+1} \end{bmatrix} \\ &= \begin{bmatrix} xFF^* + YY^t & \sqrt{x(1-x)}FS^t \\ \sqrt{x(1-x)}SF^* & k(1-x)+y_{n+1}^2 \end{bmatrix} \end{aligned}$$

and $FF^* = \frac{k}{n}I$, we have

$$\begin{aligned} \tfrac{k}{n}x + y_1^2 &= \tfrac{k+1}{n+1} \\ \tfrac{k}{n}x + y_2^2 &= \tfrac{k+1}{n+1} \\ &\vdots \\ \tfrac{k}{n}x + y_n^2 &= \tfrac{k+1}{n+1} \\ k(1-x) + y_{n+1}^2 &= \tfrac{k+1}{n+1} \end{aligned}$$

and $y_i y_j = 0$ if $i \neq j$ and $1 \le i, j \le n$. Hence $x = \frac{n(k+1)}{k(n+1)}, y_i = 0 (1 \le i \le n)$ and $y_{n+1} = 1$ or -1. In brief $\{g_i\}_{i=1}^{k+1}$ where

$$g_i = \sqrt{\frac{n(k+1)}{k(n+1)}} f_i + \sqrt{\frac{k-n}{k(n+1)}} e_{n+1} \qquad (1 \le i \le k)$$

and $g_{k+1} = -e_{n+1}$ form FNTF for $\mathbb{R}^{n+1}$. □

In special case, if $\{f_i\}_{i=1}^{n+1}$ is an equiangular tight frame for $\mathbb{R}^n$ with $\langle f_i, f_j\rangle = \frac{-1}{n}$, then the tight frame that has been constructed by above construction, is also an equiangular [8].

The next corollary provide an algorithm to construct a normalized tight frame with k elements for $\mathbb{R}^3$, by using the $(k-1)$th roots of unity.

Corollary 3.2. *For every positive integer* $k > 3$ *there exists a FNTF with k elements for* $\mathbb{R}^3$.

Proof. Let

$$F = \begin{bmatrix} 1 & \cos\frac{2\pi}{k-1} & \cos\frac{4\pi}{k-1} & \cdots & \cos\frac{2(k-2)\pi}{k-1} \\ 0 & \sin\frac{2\pi}{k-1} & \sin\frac{4\pi}{k-1} & \cdots & \sin\frac{2(k-2)\pi}{k-1} \end{bmatrix}$$

where the columns of matrix F are $(k-1)$th roots of unity. By Theorem 3.1,

$$U = \begin{bmatrix} \sqrt{\frac{2k}{3(k-1)}}F & 0 \\ \sqrt{\frac{k-3}{3(k-1)}}S & -1 \end{bmatrix}$$

is synthesis matrix of FNTF for $\mathbb{R}^3$. In other words, $\{g_i\}_{i=1}^{k}$ is FNTF for $\mathbb{R}^3$ which

$$g_i = \frac{1}{\sqrt{3k-3}}\left(\sqrt{2k}\cos\left(\frac{2(i-1)\pi}{k-1}\right)e_1 + \sqrt{2k}\sin\left(\frac{2(i-1)\pi}{k-1}\right)e_2 + \sqrt{k-3}e_3\right)$$

for $1 \leq i \leq k-1$ and $g_k = -e_3$. □

In the next theorem we provide an algorithm to construct a normalized tight frame with $n+1$ elements for $\mathbb{R}^n (n \geq 3)$.

Theorem 3.3. *For every positive integer $n \geq 2$, there exists a FNTF with $n+1$ elements for $\mathbb{R}^n$.*

Proof. The third roots of unity consist a FNTF for $\mathbb{R}^2$.

$$U_2 = \begin{bmatrix} 1 & \cos\frac{2\pi}{3} & \cos\frac{4\pi}{3} \\ 0 & \sin\frac{2\pi}{3} & \sin\frac{4\pi}{3} \end{bmatrix} = \begin{bmatrix} 1 & \frac{-1}{2} & \frac{-1}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{-\sqrt{3}}{2} \end{bmatrix}$$

Denote by U_t the $t \times (t+1)$-matrix such that the columns of U_t form a FNTF for $\mathbb{R}^t$. By Theorem 3.1, for every $t > 2$ the following recurrent relations hold:

$U_t[i, j] = \frac{\sqrt{t^2-1}}{t} U_{t-1}[i, j] \quad 1 \leq i \leq t-1, \quad 1 \leq j \leq t$

$U_t[t, j] = \frac{1}{t} \quad 1 \leq j \leq t$

$U_t[i, t+1] = 0 \quad 1 \leq i \leq t-1$

$U_t[t, t+1] = -1$

The goal of constructing matrix U_t is to iteratively create larger FNTFs from U_2, continuing until $t = n$, at which point the matrix U_t is a FNTF with $n+1$ elements for $\mathbb{R}^n$. □

Since the roots of unity (vertices of regular polygon) Form a FNTF for $\mathbb{R}^2$, so we have the following corollary.

Corollary 3.4. *Vertices of a right prism with $(k-1)-$sided regular base and height $1 + \sqrt{\frac{k-3}{3k-3}}$ which are located on sphere S^2 form a FNTF for $\mathbb{R}^3$.*

Example 3.5. The third roots of the unity$\{(1,0), (\frac{-1}{2}, \frac{\sqrt{3}}{2}), (\frac{-1}{2}, \frac{-\sqrt{3}}{2})\}$ form a FNTF for $\mathbb{R}^2$. So

$$\{(\frac{\sqrt{8}}{3}, 0, \frac{1}{3}), (-\frac{\sqrt{2}}{3}, \sqrt{\frac{2}{3}}, \frac{1}{3}), (-\frac{\sqrt{2}}{3}, -\sqrt{\frac{2}{3}}, \frac{1}{3}), (0, 0, -1)\}$$

form a FNTF for $\mathbb{R}^3$. Also the columns of the matrix

$$\begin{bmatrix} \frac{\sqrt{30}}{6} & -\frac{\sqrt{30}}{12} & -\frac{\sqrt{30}}{12} & 0 & 0 \\ 0 & \frac{\sqrt{10}}{4} & -\frac{\sqrt{10}}{4} & 0 & 0 \\ \frac{\sqrt{15}}{12} & \frac{\sqrt{15}}{12} & \frac{\sqrt{15}}{12} & -\frac{\sqrt{15}}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -1 \end{bmatrix}$$

form a FNTF for $\mathbb{R}^4$ and the columns of the matrix

$$\begin{bmatrix} \frac{2\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} & 0 & 0 & 0 \\ 0 & \frac{\sqrt{15}}{5} & -\frac{\sqrt{15}}{5} & 0 & 0 & 0 \\ \frac{\sqrt{10}}{10} & \frac{\sqrt{10}}{10} & \frac{\sqrt{10}}{10} & \frac{3\sqrt{10}}{10} & 0 & 0 \\ \frac{\sqrt{6}}{10} & \frac{\sqrt{6}}{10} & \frac{\sqrt{6}}{10} & \frac{\sqrt{6}}{10} & -\frac{\sqrt{24}}{5} & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & -1 \end{bmatrix}$$

form a FNTF for $\mathbb{R}^5$, and so on for every $\mathbb{R}^n$ we can construct a frame with $n + 1$ elements.

Theorem 3.6. *$\mathbb{R}^n$ has a FNTF with $k = 2^n$ elements.*

Proof. We prove this theorem by induction. For $n = 2$, the fourth roots of unity form a FNTF with $k = 4$ elements for $\mathbb{R}^2$. Suppose $k = 2^n$ and $\{f_i\}_{i=1}^k$ is a FNTF for $\mathbb{R}^n$ and let $F = \begin{pmatrix} | & | & \dots & | \\ f_1 & f_2 & \dots & f_k \\ | & | & \dots & | \end{pmatrix}$ be synthesis matrix corresponding to $\{f_i\}_{i=1}^k$. Put $S = \begin{pmatrix} 1 & 1 & \dots & 1 \end{pmatrix}_{1\times k}$ and define $(n + 1) \times 2k$ matrix U by

$$U = \begin{bmatrix} \sqrt{x}F & \sqrt{x}F \\ \sqrt{1-x}S & -\sqrt{1-x}S \end{bmatrix}.$$

We want to find x such that the columns of U make a FNTF for $\mathbb{R}^{n+1}$. Hence

$$UU^* = \begin{bmatrix} \sqrt{x}F & \sqrt{x}F \\ \sqrt{1-x}S & -\sqrt{1-x}S \end{bmatrix}\begin{bmatrix} \sqrt{x}F^* & \sqrt{1-x}S^t \\ \sqrt{x}F^* & -\sqrt{1-x}S^t \end{bmatrix}$$

$$= \begin{bmatrix} 2xFF^* & 0 \\ 0 & 2k(1-x) \end{bmatrix}$$

By lemma (2.1) we have $FF^* = \frac{k}{n}I$ and $UU^* = \frac{2k}{n+1}I$. Hence, we have $2x(\frac{k}{n}) = \frac{2k}{n+1}$ and $2k(1 - x) = \frac{2k}{n+1}$, and so x must be equal to $\frac{n}{n+1}$. □

Example 3.7. The fourth roots of the unity, $\{(1, 0), (0, 1), (-1, 0), (0, -1)\}$ form a FNTF for $\mathbb{R}^2$. By the above theorem the columns of the matrix

$$\begin{bmatrix} \frac{\sqrt{6}}{3} & 0 & 0 & -\frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{3} & 0 & 0 & -\frac{\sqrt{6}}{3} \\ 0 & \frac{\sqrt{6}}{3} & -\frac{\sqrt{6}}{3} & 0 & 0 & \frac{\sqrt{6}}{3} & -\frac{\sqrt{6}}{3} & 0 \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} \end{bmatrix}$$

form a FNTF with 2^3 elements for $\mathbb{R}^3$.

Theorem 3.8. *For $\mathbb{R}^n$, there exists a FNTF with $2n$ elements.*

Proof. Suppose $\{e_k\}_{k=1}^n$ is the canonical orthonormal basis of $\mathbb{R}^n$. For $1 \leq i \leq n$ let $f_i = -f_{i+n} = e_i$ and define

$$F = \begin{pmatrix} | & | & \dots & | \\ f_1 & f_2 & \dots & f_{2n} \\ | & | & \dots & | \end{pmatrix}.$$

It is clear that the columns of F are FNTF for $\mathbb{R}^n$ with $A = 2$. □

Acknowledgements: We are grateful to the editor and the anonymous referee for their valuable comments that improved the quality of the paper.

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