Wavelets and Linear Algebra 7(1) (2020) 47-56

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Wavelets and Linear Algebra

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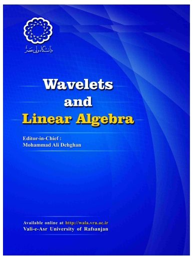

Some properties of controlled K-frames in Hilbert spaces

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Article Info

Article history:
Received 6 January 2020
Accepted 2 April 2020
Available online 7 August 2020
Communicated by Ali Akbar Arefijamaal

Keywords:
Frames, K-frames, Controlled K-frames.

2000 MSC:
42C15.

Abstract

In this paper, we reintroduce the concept of controlled K-frames and then, we show that this definition is equivalent with the concept that has been recently introduced in [21]. Meanwhile, we correct one of the results which was obtained in the mentioned paper. In the sequel, we construct some new controlled K-frames by some operator theory tools. Finally, we provide some conditions under which the sum of two controlled K-frames remains a controlled K-frame.

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http://doi.org/10.22072/wala.2020.119701.1266

1. Introduction and Preliminaries

The notion of frame dates back to Gabor [14] and Duffin and Schaeffer [12]. However, the frame theory had not attracted much attention until the celebrated work by Daubechies, Crossman, and Meyer [10]. A frame for a separable Hilbert space $\mathcal{H}$ is a family of vectors $\{f_i\}_{i\in I}$ in $\mathcal{H}$ so that there are two positive constants A and B satisfying

$$A\|f\|_{\mathcal{H}}^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|_{\mathcal{H}}^2, \qquad (f \in \mathcal{H}), \tag{1.1}$$

the numbers A, B in (1.1) are called frame bounds. Frames have been used as a powerful alternative to Hilbert bases because of their redundancy and flexibility. They are also very important for applications, e.g. in physics [1, 8], signal processing [6, 5, 4], numerical treatment of operator equations [24, 9] and acoustics [2, 20].

Over the years, various extensions of frame theory have been investigated, such as g-frames [25], fusion frames (or frames of subspaces)[7] and else.

Atomic systems for subspaces were first introduced by Feichtinger and Werther in [13] based on examples arising in sampling theory. In [15], Găvruţa introduced K-frames in Hilbert spaces to study atomic decomposition systems, and discussed some properties of them.

Weighted and controlled frames, as one of the newest generalizations of frames, have been introduced to improve the numerical efficiency of iterative algorithms for inverting the frame operator on abstract Hilbert spaces in [3], however they are used earlier in [17] for spherical wavelets. Since then, controlled frames have been generalized to another kinds of frames [16, 18, 19, 22]. In this paper, we reintroduce the concept of controlled K-frames and then we investigate some properties of them. In the sequel, we provide some conditions under which the sum of two controlled K-frame remains a controlled K-frame.

2. Notation and preliminaries

In this section, we collect the basic notation and some preliminary results. Throughout the paper, $\mathcal{H}$ is a separable Hilbert space and $\mathbb{I}$ is an at most countable index set. We denote by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ the set of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. For $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, we use the notations U^* and $R(U)$ to denote respectively the adjoint operator and the range space of U. We define $GL(\mathcal{H}_1, \mathcal{H}_2)$ as the set of all bounded linear operators with a bounded inverse, and similarly for $GL(\mathcal{H})$. A bounded operator T is called positive (respectively non-negative), if $\langle Tf, f\rangle > 0$, for all $0 \neq f \in \mathcal{H}$, (respectively $\langle Tf, f\rangle \geq 0$, for all $f \in \mathcal{H}$). Every non-negative operator is clearly self-adjoint. If $U \in \mathcal{B}(\mathcal{H})$ is non-negative, then there exists a unique non-negative operator V such that $V^2 = U$. This will be denoted by $V = U^{\frac{1}{2}}$. The operator V commutes with every operator that commutes with U. The set of positive operators in $GL(\mathcal{H})$ will be denoted by $GL^+(\mathcal{H})$. Recall that if $U \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ has closed range, then there exists a unique operator $U^\dagger \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, called the pseudo-inverse of U, satisfying $UU^\dagger f = f$, for every $f \in R(U)$.

Later we will need the following important result from operator theory.

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Lemma 2.1. *[3] Let $T \in \mathcal{B}(\mathcal{H})$. The following are equivalent.*

1. *There exist $0 < m \leq M < \infty$ such that $mI \leq T \leq MI$.*
2. *T is positive and there exist $0 < m \leq M < \infty$ such that $m\|f\|^2 \leq \|T^{\frac{1}{2}} f\|^2 \leq M\|f\|^2$.*
3. *T is positive and $T^{\frac{1}{2}} \in GL(\mathcal{H})$.*
4. *There exists a self-adjoint operator $A \in GL(\mathcal{H})$ such that $A^2 = T$.*
5. *$T \in GL^+(\mathcal{H})$.*

The following theorem is a key tool for the proofs of our main results.

Theorem 2.2. *[11](Douglas majorization theorem) Let $T, S \in \mathcal{B}(\mathcal{H})$. The following statements are equivalents:*

1. *$R(T) \subset R(S)$.*
2. *There exists $\lambda > 0$ such that $TT^* \leq \lambda SS^*$.*
3. *There exists $U \in \mathcal{B}(H)$ such that $T = SU$.*

We now introduce the concept of controlled K-frames which are more general than K-frames and controlled frames.

Definition 2.3. Let $\mathcal{H}$ be a separable Hilbert space, $K \in \mathcal{B}(\mathcal{H})$ and $C \in GL(\mathcal{H})$. A sequence $F = \{f_i\}_{i\in\mathbb{I}}$ in $\mathcal{H}$ is called a *C-Controlled K-frame* for $\mathcal{H}$, if there exist constants $A, B > 0$ such that

$$A\|K^* f\|^2 \leq \sum_{i\in\mathbb{I}} \langle f, f_i\rangle\langle Cf_i, f\rangle \leq B\|f\|^2, \quad (f \in \mathcal{H}). \tag{2.1}$$

Clearly, If $C = I$, then $F = \{f_i\}_{i\in\mathbb{I}}$ is a K-frame in $\mathcal{H}$ and for $K = I$, $F = \{f_i\}_{i\in\mathbb{I}}$ is a C-controlled frame in $\mathcal{H}$. For more details see [26, 3].

From the operator theoretic point of view, the relation (2.1) is equivalent to

$$AKK^* \leq S_{CF} \leq BI, \tag{2.2}$$

where S_{CF} is the controlled frame operator which is defined as

$$S_{CF} : \mathcal{H} \to \mathcal{H}, \quad S_{CF} f := \sum_{i\in\mathbb{I}} \langle f, f_i\rangle C f_i, \quad (f \in \mathcal{H}).$$

Obviously, S_{CF} is positive, since for each $f \in \mathcal{H}$,

$$\langle S_{CF} f, f\rangle = \left\langle \sum_{i\in\mathbb{I}} \langle f, f_i\rangle C f_i, f \right\rangle = \sum_{i\in\mathbb{I}} \langle f, f_i\rangle\langle Cf_i, f\rangle \geq A\|K^* f\|^2 \geq 0.$$

Remark 2.4. Let $C \in GL^+(\mathcal{H})$ and $\{f_i\}_{i\in\mathbb{I}}$ be a C-controlled K-frame in $\mathcal{H}$. Then, the associated synthesis operator and analysis operator is defined as follow.

$$T_{CF} : \ell^2(\mathbb{I}) \to \mathcal{H}, \quad T_{CF}\{f_i\}_{i\in\mathbb{I}} := \sum_{i\in\mathbb{I}} c_i C^{\frac{1}{2}} f_i, \tag{2.3}$$

$$T^*_{CF} : \mathcal{H} \to \ell^2(\mathbb{I}), \quad T^*_{CF} f = \{\langle f, C^{\frac{1}{2}} f_i\rangle\}_{i\in\mathbb{I}}. \tag{2.4}$$

Due to the fact that $S_{CF} = CS$ is a positive operator, where S is the classic frame operator, therefore $SC = CS$ and so it is obtained that $T_{CF}T^*_{CF} = S_{CF}$.

It is worthwhile to mention that the notion of C-controlled K-frames is first defined by Nouri et al [21] as follows.

Definition 2.5. Let $K \in \mathcal{B}(\mathcal{H})$, $C \in GL^+(\mathcal{H})$ and $CK = KC$. A family $\{f_i\}_{i\in\mathbb{I}}$ in $\mathcal{H}$ is called a *C-controlled K-frame* if $\{f_i\}_{i\in\mathbb{I}}$ is a Bessel sequence in $\mathcal{H}$ and there exist two constants $0 < A' \leq B' < \infty$, such that

$$A'\|C^{\frac{1}{2}} K^* f\|^2 \leq \sum_{i\in\mathbb{I}} \langle f, f_i\rangle\langle C f_i, f\rangle \leq B'\|f\|^2, \quad (f \in \mathcal{H}). \tag{2.5}$$

or equivalently

$$A'CKK^* \leq S_{CF} \leq B'I. \tag{2.6}$$

Actually, we can show that under the assumptions of Definition 2.5, this definition is equivalent with Definition 2.3. Obviously, the upper bound satisfies. For the lower bound, suppose (2.2) holds. Since $C \in GL^+(\mathcal{H})$, by Lemma 2.1, there exists positive constant m such that $mC \leq I$. On the other hand, since $CK = KC$, we have $mCKK^* \leq KK^*$ and so $mACKK^* \leq AKK^*$. Hence, by (2.2), it is concluded that

$$mACKK^* \leq AKK^* \leq S_C.$$

The proof is complete by setting $A' = mA$. To obtain the first definition from the second one, assume that (2.6) holds. Since $C \in GL^+(\mathcal{H})$, by Lemma 2.1, there exists positive constant M such that $I \leq MC$. So, we have

$$S \leq M(SC) = MS_C. \tag{2.7}$$

Moreover, by (2.6),

$$A'C^{-1}CKK^* \leq C^{-1}S_C = C^{-1}CS = S. \tag{2.8}$$

Combining (2.7) and (2.8), we conclude

$$A'KK^* \leq MS_C.$$

For completion of the proof, it is enough to put $A = \dfrac{A'}{M}$.

In [21, Proposition 3.8], it was claimed that for $K \in \mathcal{B}(\mathcal{H})$ and $C \in GL(\mathcal{H})$ with $KC = CK$, if $\{f_i\}_{i\in\mathbb{I}}$ is a K-frame for $\mathcal{H}$, then $\{f_i\}_{i\in\mathbb{I}}$ is also a C-controlled K-frame for $\mathcal{H}$. It seems that this is not true in general. See the following example.

Example 2.6. Suppose that $\{f_k\}_{k=1}^3 = \{e_1 + e_2, e_1, \sqrt{3}e_3\}$ is a frame in $\mathcal{H} = \mathbb{C}^3$ with the frame operator

$$S = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Thus,

$$\sigma(S) = \{3, \frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}\} \subset [0, \infty).$$

Moreover, consider operators $K \in \mathcal{B}(\mathcal{H})$ and $C \in GL(\mathcal{H})$ as

$$C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}, \quad K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$CK = KC = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 3 \end{bmatrix}.$$

On the other hand, for $A = 1$, we have

$$S - AKK^* = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Obviously, $S - AKK^*$ is self-adjoint and $\sigma(S - AKK^*) = \{0, 2\} \subset [0, \infty)$. Hence, it is positive and so by [26, Theorem 3.5], $\{f_i\}_{i\in\mathbb{I}}$ is a K-frame in $\mathcal{H}$. Now, [21, Proposition 3.8] implies that $\{f_i\}_{i\in\mathbb{I}}$ is a C-controlled K-frame in $\mathcal{H}$ and so $CS = SC$. But,

$$SC = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 3 & 0 & 9 \end{bmatrix} \neq \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 0 \\ 2 & 1 & 9 \end{bmatrix} = CS,$$

which is a contradiction.

Now, we correct the result of [21, Proposition 3.8] by adding the necessary condition $CS = SC$.

Proposition 2.7. *Let $K \in \mathcal{B}(\mathcal{H})$ and $C \in GL^+(\mathcal{H})$ with $KC = CK$. If $F = \{f_i\}_{i\in\mathbb{I}}$ is a K-frame for $\mathcal{H}$ and $CS = SC$, then $F = \{f_i\}_{i\in\mathbb{I}}$ is a C-controlled K-frame for $\mathcal{H}$.*

Proof. Let $f \in \mathcal{H}$ and $F = \{f_i\}_{i\in\mathbb{I}}$ be a K-frame for $\mathcal{H}$ with bounds A and B. Then

$$\begin{aligned}
A\|C^{\frac{1}{2}}K^*f\|^2 &= A\langle C^{\frac{1}{2}}K^*f, C^{\frac{1}{2}}K^*f\rangle \\
&= A\langle KK^*C^{\frac{1}{2}}f, C^{\frac{1}{2}}f\rangle \\
&\leq \sum_{i\in\mathbb{I}}\langle C^{\frac{1}{2}}f, f_i\rangle\langle f_i, C^{\frac{1}{2}}f\rangle \\
&= \left\langle \sum_{i\in\mathbb{I}}\langle C^{\frac{1}{2}}f, f_i\rangle C^{\frac{1}{2}}f_i, f\right\rangle \\
&= \langle C^{\frac{1}{2}}SC^{\frac{1}{2}}f, f\rangle \\
&= \langle CSf, f\rangle \\
&= \langle S_{CF}f, f\rangle.
\end{aligned}$$

Furthermore, for the upper bound

$$\begin{aligned}
\langle S_{CF}f, f\rangle = \langle CSf, f\rangle = \langle SC^{\frac{1}{2}}f, C^{\frac{1}{2}}f\rangle &\leq B\langle C^{\frac{1}{2}}f, C^{\frac{1}{2}}f\rangle \\
&\leq B\|C^{\frac{1}{2}}f\|^2 \\
&\leq B\|C^{\frac{1}{2}}\|^2\|f\|^2.
\end{aligned}$$

This completes the proof. □

3. Operators and Controlled frames

In this section, we discuss the operator perturbation of controlled K-frames. First, we give a necessary and sufficient condition for a sequence to be a controlled K-frame.

Proposition 3.1. *Suppose that $K \in \mathcal{B}(\mathcal{H})$. A sequence $F = \{f_i\}_{i\in\mathbb{I}}$ is a C-controlled K-frame for $\mathcal{H}$ if and only if $R(K) \subset R(T_{CF})$.*

Proof. Let $F = \{f_i\}_{i\in\mathbb{I}}$ be a C-controlled K-frame for $\mathcal{H}$. For every $f \in \mathcal{H}$,

$$A\langle KK^*f, f\rangle = A\|K^*f\|^2 \leq \sum_{i\in\mathbb{I}}\langle f, f_i\rangle\langle Cf_i, f\rangle = \langle S_Cf, f\rangle = \langle T_{CF}T^*_{CF}f, f\rangle.$$

Hence,

$$KK^* \leq \frac{1}{A}T_{CF}T^*_{CF}.$$

Now, Theorem 2.2 implies that $R(K) \subset R(T_{CF})$. The opposite direction is obtained by a similar argument. □

The next result shows that every C-controlled K-frame is also a C^{-1}-controlled K-frame.

Proposition 3.2. *Let $K \in \mathcal{B}(\mathcal{H})$, $C \in GL^+(\mathcal{H})$ with $CK = KC$ and $F = \{f_i\}_{i\in\mathbb{I}}$ is a C-controlled K-frame for $\mathcal{H}$. Then $F = \{f_i\}_{i\in\mathbb{I}}$ is a C^{-1}-controlled K-frame for $\mathcal{H}$.*

Proof. For each $f \in \mathcal{H}$,

$$\begin{aligned}
A\langle f, C^{-1}KK^*f\rangle = A\|C^{\frac{-1}{2}}K^*f\|^2 &= A\|C^{\frac{1}{2}}C^{-1}K^*f\|^2 \\
&= A\|C^{\frac{1}{2}}K^*C^{-1}f\|^2 \\
&\leq \sum_{i\in\mathbb{I}}\langle C^{-1}f, f_i\rangle\langle Cf_i, C^{-1}f\rangle \\
&= \sum_{i\in\mathbb{I}}\langle f, C^{-1}f_i\rangle\langle f_i, f\rangle \\
&= \left\langle f, \sum_{i\in\mathbb{I}}\langle f, f_i\rangle C^{-1}f_i\right\rangle \\
&= \langle f, S_{C^{-1}F}f\rangle \\
&\leq B\langle f, f\rangle.
\end{aligned}$$

□

In the following proposition, for given an appropriate operator U, we construct a controlled K-frame for the range space $R(U)$.

Proposition 3.3. *Let $K \in \mathcal{B}(\mathcal{H})$, $C \in GL(\mathcal{H})$ and $\{f_i\}_{i\in\mathbb{I}}$ is a C-controlled K-frame for $\mathcal{H}$. Moreover, let $U \in \mathcal{B}(\mathcal{H})$ has closed range with $R(K^*) \subset R(U)$, $UC = CU$ and $UK = KU$. Then $\{Uf_i\}_{i\in\mathbb{I}}$ is a C-controlled K-frame for $R(U)$.*

Proof. Since U has closed range, it has the pseudo-inverse $U^\dagger$ such that $UU^\dagger = I_{R(U)}$. Hence, $I_{R(U)} = I^*_{R(U)} = (U^\dagger)^*U^*$. Due to the fact that $R(K^*) \subset R(U)$, so for each $f \in R(U)$,

$$K^*f = (U^\dagger)^*U^*K^*f.$$

Thus, we obtain

$$\|K^*f\| = \|(U^\dagger)^*U^*K^*f\| \leq \|U^\dagger\|\|K^*U^*f\|.$$

Now, for each $f \in R(U)$,

$$\begin{aligned}
A\|U^\dagger\|^{-2}\|K^*f\|^2 = A\|K^*U^*f\|^2 &\leq \sum_{i\in\mathbb{I}}\langle U^*f, f_i\rangle\langle Cf_i, U^*f\rangle \\
&= \sum_{i\in\mathbb{I}}\langle f, Uf_i\rangle\langle UCf_i, f\rangle \\
&= \sum_{i\in\mathbb{I}}\langle f, Uf_i\rangle\langle CUf_i, f\rangle \\
&\leq B\|U\|^2\|f\|^2.
\end{aligned}$$

□

Corollary 3.4. *Let $K \in \mathcal{B}(\mathcal{H})$, $C \in GL(\mathcal{H})$ and $\{f_i\}_{i\in\mathbb{I}}$ is a C-controlled K-frame for $\mathcal{H}$. Moreover, let $U \in \mathcal{B}(\mathcal{H})$ be surjective with $R(K^*) \subset R(U)$, $UC = CU$ and $UK = KU$. Then $\{Uf_i\}_{i\in\mathbb{I}}$ is a C-controlled K-frame for $\mathcal{H}$.*

Proposition 3.5. *Let $K \in \mathcal{B}(\mathcal{H})$, $C \in GL(\mathcal{H})$ and $\{f_i\}_{i\in\mathbb{I}}$ is a C-controlled K-frame for $\mathcal{H}$. Moreover, let $U \in \mathcal{B}(\mathcal{H})$ be a co-isometry with $R(K^*) \subset R(U)$, $UC = CU$ and $UK = KU$. Then $\{Uf_i\}_{i\in\mathbb{I}}$ is a C-controlled K-frame for $\mathcal{H}$.*

Proof. Since U is a co-isometry and $R(K^*) \subset R(U)$, so for every $f \in \mathcal{H}$

$$\|K^*U^*f\| = \|U^*K^*f\| = \|K^*f\|.$$

Hence

$$\begin{aligned} A\|K^*f\|^2 = A\|K^*U^*f\|^2 &\leq \sum_{i\in\mathbb{I}} \langle U^*f, f_i\rangle\langle Cf_i, U^*f\rangle \\ &= \sum_{i\in\mathbb{I}} \langle f, Uf_i\rangle\langle CUf_i, f\rangle \\ &\leq B\|U\|^2\|f\|^2. \end{aligned}$$

□

The next result provides some conditions under which the sum of two controlled K-frames remains a controlled K-frame.

Proposition 3.6. *Suppose that $K \in \mathcal{B}(\mathcal{H})$, $C \in GL^+(\mathcal{H})$ with $CK = KC$ and $F = \{f_i\}_{i\in\mathbb{I}}$ and $G = \{g_i\}_{i\in\mathbb{I}}$ are C-controlled K-frames for $\mathcal{H}$ with bounds A, B and A', B', respectively. If $T_F T_G^* = C^{-1}KK^*$, then $\{f_i + g_i\}_{i\in\mathbb{I}}$ is also a C-controlled K-frame for $\mathcal{H}$.*

Proof. First, due the fact that $T_F T_G^* = C^{-1}KK^*$, so for every $f \in \mathcal{H}$,

$$\begin{aligned} \sum_{i\in\mathbb{I}} \langle f, f_i + g_i\rangle\langle C(f_i + g_i), f\rangle &= \sum_{i\in\mathbb{I}} \Big((\langle f, f_i\rangle + \langle f, g_i\rangle)(\langle Cf_i, f\rangle + \langle Cg_i, f\rangle) \Big) \\ &= \sum_{i\in\mathbb{I}} \langle f, f_i\rangle\langle Cf_i, f\rangle + \sum_{i\in\mathbb{I}} \langle f, f_i\rangle\langle Cg_i, f\rangle \\ &+ \sum_{i\in\mathbb{I}} \langle f, g_i\rangle\langle Cf_i, f\rangle + \sum_{i\in\mathbb{I}} \langle f, g_i\rangle\langle Cg_i, f\rangle \\ &= \sum_{i\in\mathbb{I}} \langle f, f_i\rangle\langle Cf_i, f\rangle + \langle C \sum_{i\in\mathbb{I}} \langle f, f_i\rangle g_i, f\rangle \\ &+ \langle C \sum_{i\in\mathbb{I}} \langle f, g_i\rangle f_i, f\rangle + \sum_{i\in\mathbb{I}} \langle f, g_i\rangle\langle Cg_i, f\rangle \\ &= \sum_{i\in\mathbb{I}} \langle f, f_i\rangle\langle Cf_i, f\rangle + 2\langle K^*f, K^*f\rangle + \sum_{i\in\mathbb{I}} \langle f, g_i\rangle\langle Cg_i, f\rangle \\ &= \sum_{i\in\mathbb{I}} \langle f, f_i\rangle\langle Cf_i, f\rangle + 2\|K^*f\|^2 + \sum_{i\in\mathbb{I}} \langle f, g_i\rangle\langle Cg_i, f\rangle. \end{aligned}$$

Hence,

$$(A + A' + 2)\|K^*f\|^2 \leq \sum_{i\in\mathbb{I}} \langle f, f_i + g_i\rangle\langle C(f_i + g_i), f\rangle \leq (B + B' + 2\|K\|^2)\|f\|^2,$$

and the proof is complete. □

As it is mentioned in [21, Proposition 3.7], every controlled K-frame is a K-frame. On the other hand, by [23, Theorem 4.1], if $\{f_i\}_{i\in\mathbb{I}}$ and $\{g_i\}_{i\in\mathbb{I}}$ are two K-frames for $\mathcal{H}$ with synthesis operators T_F and T_G, respectively, and $T_F T_G^*$ and $T_G T_F^*$ are positive operators, then $\{f_i + g_i\}_{i\in\mathbb{I}}$ is a K-frame. From these facts, we can give other conditions under which the sum of two controlled K-frames remains a controlled K-frame.

Proposition 3.7. *Suppose that $K \in \mathcal{B}(\mathcal{H})$, $C \in GL^+(\mathcal{H})$ with $CK = KC$ and $\{f_i\}_{i\in\mathbb{I}}$ and $\{g_i\}_{i\in\mathbb{I}}$ are C-controlled K-frames for $\mathcal{H}$ with synthesis operators T_F and T_G, respectively. Moreover, let $T_F T_G^*$ and $T_G T_F^*$ are positive operators. If $CS = SC$, then $\{f_i+g_i\}_{i\in\mathbb{I}}$ is also a C-controlled K-frame for $\mathcal{H}$, where S is the frame operator associated with $\{f_i + g_i\}_{i\in\mathbb{I}}$.*

4. Acknowledgement

The authors would like to thank the referees for their valuable comments and suggestions which improved the manuscript.

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