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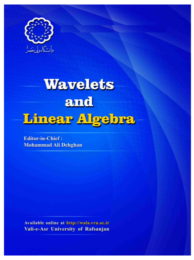

Excess of continuous K-g-frames and some other properties

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Abstract

In this paper, we study the excess of continuous K-g-frames and give some results about this notion. Also, we extend the concept of atomic system to continuous version and study its relations by continuous K-g-frames. Indeed, we give some equivalent characterizations for continuous K-g-frames. As well as, the relationship of a continuous K-g-frame and the range of operator K will be verified. Finally, we study the induced cK-frames by continuous K-g-frames.

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1. Introduction and Preliminaries

The concept of frames, which are a generalization of the bases in Hilbert spaces, were first introduced by Duffin and Schaeffer [7] during their study of nonharmonic Fourier series. After that, Daubechies et al. [5] reintroduced the concept of frames. Now frame theory has been widely used in many fields. An introduction to frame theory and some details and applications can be found in [4].

K-frames in Hilbert spaces were introduced by Gavruta [9] to study atomic decomposition systems, and discussed some properties of them. Actually, K-frames are limited to the range of a bounded linear operator in Hilbert spaces. Afterward, K-g-frames have been introduced in [10] and some properties and characterizations of K-g-frames has been given. A recent progress on continuous frames inspired by the solution of the Kadison-Singer problem is surveyed in [3]. Also, using frame theory techniques, some results concerning atomic decompositions for operators on reproducing kernel Hilbert spaces is given in [8]. The concept of continuous K-g-frames, or briefly c-K-g-frames, is introduced in [2]. In this paper, we investigate some features of these kinds of frames.

Throughout this paper, H is a separable Hilbert space, (Ω, μ) is a measure space, $\{H_\omega\}_{\omega\in\Omega}$ is a family of separable Hilbert spaces and K is a bounded linear operator on H. Furthermore, $B(H, H_\omega)$ shows the set of all bounded linear operators from H into H_ω and $B(H)$ is the algebra of all bounded linear operators on H.

In the following of this section, we review some concepts and results about g-frames and K-g-frames.

Definition 1.1. Let $K \in B(H)$. A sequence $\{f_i\}_{i\in I}$ is called a K-frame for H, if there exist constants $A, B > 0$ such that

$$A\|K^* f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \quad \forall f \in H. \tag{1.1}$$

We call A, B the lower and the upper frame bounds of K-frame $\{f_n\}_{i\in I}$, respectively. If only the right inequality (1.1) is satisfied, $\{f_n\}_{i\in I}$ is called a Bessel sequence. If $K = Id_H$, then it will be an ordinary frame.

Definition 1.2. Assume that $K \in B(H)$ and $\Lambda = \{\Lambda_i \in B(H, H_i) : i \in I\}$. Λ is called a K-g-frame for H with respect to $\{H_i\}_{i\in I}$, if there exist constants $A, B > 0$ such that

$$A\|K^* f\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad \forall f \in H. \tag{1.2}$$

We call the constants A, B, the lower and upper bounds of K-g-frame, respectively.

The space $l^2(\{H_i\}_{i\in I})$ is represented by

$$l^2(\{H_i\}_{i\in I}) = \left\{ \{a_i\}_{i\in I} \mid a_i \in H_i, \ \sum_{i\in I} \|a_i\|^2 < \infty \right\}.$$

Let $\{\Lambda_i \in B(H, H_i) : i \in I\}$ be a K-g-frame for H with respect to $\{H_i\}_{i\in I}$. The synthesis operator $T : l^2(\{H_i\}_{i\in I}) \longrightarrow H$ is defined as follows:

$$T(\{g_i\}_{i\in I}) = \sum_{i\in I} \Lambda_i^* g_i, \quad \forall \{g_i\}_{i\in I} \in l^2(\{H_i\}_{i\in I}).$$

Theorem 1.3. ([11]) The sequence $\Lambda = \{\Lambda_i \in B(H, H_i) : i \in I\}$ is a g-Bessel sequence for H with bound B if and only if the operator

$$T : l^2(\{H_i\}_{i\in I}) \longrightarrow H$$
$$T(\{g_i\}_{i\in I}) = \sum_{i\in I} \Lambda_i^* g_i$$

is a well-defined and bounded operator with $\|T\| \leq \sqrt{B}$.

The adjoint operator of T is called analysis operator of $\Lambda = \{\Lambda_i\}_{i\in I}$ and $T^* : H \longrightarrow l^2(\{H_i\}_{i\in I})$ is given by

$$T^* f = \{\Lambda_i f\}_{i\in I}, \quad \forall f \in H.$$

The frame operator $S : H \longrightarrow H$ of $\Lambda = \{\Lambda_i\}_{i\in I}$ is defined as follows:

$$S f = TT^* f = \sum_{i\in I} \Lambda_i^* \Lambda_i f, \quad \forall f \in H.$$

Definition 1.4. Let $K \in B(H)$ and $f : \Omega \longrightarrow H$ be a weakly measurable mapping. Then f is called a cK-frame for H, if there exist constants $A, B > 0$ such that

$$A\|K^* f\|^2 \leq \int_\Omega |\langle f, f(\omega)\rangle|^2 d\mu(\omega) \leq B\|f\|^2, \quad \forall f \in H. \tag{1.3}$$

We call the constants A, B, the lower and upper bounds of cK-frame f, respectively.

Definition 1.5. ([1]) Assume that

$$\Pi_{\omega\in\Omega} H_\omega = \{f : \Omega \longrightarrow \cup_{\omega\in\Omega} H_\omega : \ f(\omega) \in H_\omega\}.$$

We say that $F \in \Pi_{\omega\in\Omega} H_\omega$ is strongly measurable if F as a mapping of Ω to $\oplus_{\omega\in\Omega} H_\omega$ is measurable.

Now, we review the definition of continuous g-frames.

Definition 1.6. ([1]) A family $\Lambda = \{\Lambda_\omega \in B(H, H_\omega) : \ \omega \in \Omega\}$ is called a continuous g-frame, or simply a cg-frame, for H with respect to $\{H_\omega\}_{\omega\in\Omega}$, if:

(i) for each $f \in H, \{\Lambda_\omega f\}_{\omega\in\Omega}$ is strongly measurable,

(ii) there exist two positive constants A, B such that

$$A\|f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) \leq B\|f\|^2 \ , \ \forall f \in H. \tag{1.4}$$

A and B are called the lower and upper cg-frame bounds, respectively.

Definition 1.7. Consider the set $(\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2}$ which is defined as below:

$$\left\{ F \in \prod_{\omega\in\Omega} H_\omega : \text{ F is strongly measurable and } \int_\Omega \|F(\omega)\|^2 d\mu(\omega) < \infty \right\}.$$

It can be proved that $(\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2}$ is a Hilbert space with the inner product given by

$$\langle F, G\rangle = \int_\Omega \langle F(\omega), G(\omega)\rangle d\mu(\omega).$$

We will denote the norm of $F \in (\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2}$ by $\|F\|_2$, (see [1]).

Proposition 1.8. ([1]) *Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a cg-Bessel family for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ with Bessel bound B. Then the operator*

$$T : (\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2} \longrightarrow H$$

weakly defined by

$$\langle T\varphi, h\rangle = \int_\Omega \langle \Lambda^*_\omega \varphi(\omega), h\rangle d\mu(\omega), \quad \forall \varphi \in (\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2}, \ \forall h \in H, \tag{1.5}$$

is linear and bounded with $\|T\| \leq \sqrt{B}$. Moreover, for each $h \in H$ and $\omega \in \Omega$,

$$T^*(h)(\omega) = \Lambda_\omega h. \tag{1.6}$$

The operators T and T^* are called synthesis and analysis operators of cg-Bessel family $\{\Lambda_\omega\}_{\omega\in\Omega}$, respectively.

Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a cg-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ with frame bounds A, B. The operator $S : H \longrightarrow H$ weakly defined by

$$\langle Sf, g\rangle = \int_\Omega \langle f, \Lambda^*_\omega \Lambda_\omega g\rangle d\mu(\omega), \quad \forall f, g \in H, \tag{1.7}$$

is called the frame operator of $\{\Lambda_\omega\}_{\omega\in\Omega}$. S is a positive and invertible operator.

Definition 1.9. Suppose that (Ω, μ) is a measure space with positive measure μ and $K \in B(H)$. A family $\Lambda = \{\Lambda_\omega \in B(H, H_\omega) : \ \omega \in \Omega\}$, is called a continuous K-g-frame, or simply a c-K-g-frame, for H with respect to $\{H_\omega\}_{\omega\in\Omega}$, if:

(i) for each $f \in H$, $\{\Lambda_\omega f\}_{\omega\in\Omega}$ is strongly measurable,

(ii) there exist constants $0 < A \leq B < \infty$ such that

$$A\|K^* f\|^2 \leq \int_{\Omega} \|\Lambda_\omega f\|^2 d\mu(\omega) \leq B\|f\|^2, \quad \forall f \in H. \tag{1.8}$$

The constants A, B are called lower and upper c-K-g-frame bounds, respectively. If A, B can be chosen such that $A = B$, then $\{\Lambda_\omega\}_{\omega\in\Omega}$ is called a tight c-K-g-frame and if $A = B = 1$, it is called Parseval c-K-g-frame. A family $\{\Lambda_\omega\}_{\omega\in\Omega}$ is called a c-K-g-Bessel family if the right hand inequality in (1.8) holds.

Theorem 1.10. ([2]) Let (Ω, μ) be a measure space, where μ is σ-finite and $K \in B(H)$. Suppose that $\{\Lambda_\omega \in B(H, H_\omega) : \ \omega \in \Omega\}$ is a family of operators such that for each $f \in H$, $\{\Lambda_\omega f\}_{\omega\in\Omega}$ is strongly measurable. Then $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ if and only if the operator

$$T : \Big(\oplus_{\omega\in\Omega} H_\omega, \mu\Big)_{L^2} \longrightarrow H$$

weakly defined by

$$\langle TF, g\rangle = \int_{\Omega} \langle \Lambda_\omega^* F(\omega), g\rangle d\mu(\omega), \quad \forall F \in \Big(\oplus_{\omega\in\Omega} H_\omega, \mu\Big)_{L^2}, \ \ \forall g \in H,$$

is bounded and $R(K) \subseteq R(T)$.

For every closed-ranged operator there exists a right-inverse.

Lemma 1.11. ([4]) *Let H_1 and H_2 be Hilbert spaces and suppose that $U : H_2 \longrightarrow H_1$ is a bounded operator with closed range $R(U)$. Then there exists a bounded operator $U^\dagger : H_1 \longrightarrow H_2$ for which*

$$N(U^\dagger) = R(U)^\perp, \ \ R(U^\dagger) = N(U)^\perp, \ \ UU^\dagger f = f, \quad \forall f \in R(U).$$

The operator $U^\dagger$ is called the pseudo-inverse of U.

Lemma 1.12. ([6]) *Suppose that $L_1 \in B(H_1, H)$, $L_2 \in B(H_2, H)$, where H, H_1, H_2 are Hilbert spaces. Then the following statements are equivalent:*

(1) $R(L_1) \subseteq R(L_2)$,

(2) $L_1 L_1^* \leq \alpha L_2 L_2^*$ *for some* $\alpha \geq 0$,

(3) there exists a bounded operator $Q \in B(H_1, H_2)$ *such that*

$$L_1 = L_2 Q.$$

2. Excess of *c-K-g*-frames

In this section, we investigate the excess of *c-K-g*-frames. We give some conditions on a *c-K-g*-frame such that after an erasure of some elements, it remains still a *c-K-g*-frame. Before that, we need to introduce a new notation which is helpful in the following of this section.

Definition 2.1. For each measurable set $\Delta \subseteq \Omega$, we define the operator

$$T_\Delta : (\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2} \longrightarrow H$$

weakly by

$$\langle T_\Delta \varphi, g\rangle = \int_\Delta \langle \Lambda_\omega^* \varphi(\omega), g\rangle d\mu(\omega), \quad \forall \varphi \in (\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2}, \ \forall g \in H. \tag{2.1}$$

Next theorem states some conditions that under which we can verify the excess of *c-K-g*-frames.

Theorem 2.2. *Suppose that $\{\Lambda_\omega \in B(H, H_\omega) : \ \omega \in \Omega\}$ is a c-K-g-frame for H with bounds A, B. Let $\Delta \subseteq \Omega$ be measurable and $R(K)$ be closed. Then the following statements are equivalent:*

(1) Let

$$W_{\Omega\backslash\Delta} := \overline{R(T_{\Omega\backslash\Delta})} \subseteq R(K)$$

and

$$W_\Delta := \overline{R(T_\Delta)} \subseteq R(K)^\perp.$$

Then $\{\Lambda_\omega \in B(H, H_\omega) : \ \omega \in \Omega\backslash\Delta\}$ is a c-K-g-frame for H with bounds A, B.

(2) Let $W_{\Omega\backslash\Delta} \subseteq R(K)$ and $\|K^\dagger\| < \sqrt{\frac{A}{B}}$. Then $\{\Lambda_\omega \in B(H, H_\omega) : \ \omega \in \Omega\backslash\Delta\}$ is a c-K-g-frame for H with bounds $A - B\|K^\dagger\|^2$ and B, where $K^\dagger$ is the pseudo-inverse of K.

(3) Let $\{0\} \neq W_\Delta \subseteq R(K)$ and $W_{\Omega\backslash\Delta} \perp W_\Delta$. Then $\{\Lambda_\omega \in B(H, H_\omega) : \ \omega \in \Omega\backslash\Delta\}$ is not a c-K-g-frame for H.

Proof. (1). For each $h \in H$,

$$\int_{\Omega\backslash\Delta} \|\Lambda_\omega h\|^2 d\mu(\omega) \le \int_\Omega \|\Lambda_\omega h\|^2 d\mu(\omega) \le B\|h\|^2.$$

So $\{\Lambda_\omega\}_{\omega\in\Omega\backslash\Delta}$ is a *c-K-g*-Bessel family for H. Now, we show that the lower frame condition holds. The assumption $W_\Delta \subseteq R(K)^\perp$ implies that for each $f \in R(K)$,

$$\begin{aligned}\int_\Delta \|\Lambda_\omega f\|^2 d\mu(\omega) &\le \int_\Omega \langle \Lambda_\omega^* \Lambda_\omega f, f\rangle d\mu(\omega)\\ &= \langle T_\Delta(\{\Lambda_\omega f\}_{\omega\in\Delta}), f\rangle = 0.\end{aligned}$$

Hence, for each $f \in R(K)$ and for almost all $\omega \in \Delta$,

$$\Lambda_\omega f = 0. \tag{2.2}$$

Similarly, by assumption $W_{\Omega\backslash\Delta} \subseteq R(K)$, for each $g \in R(K)^\perp$ and for almost all $\omega \in \Omega\backslash\Delta$, we have

$$\Lambda_\omega g = 0. \tag{2.3}$$

By (2.2), for each $f \in R(K)$,

$$\begin{aligned}\int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) &= \int_{\Omega\backslash\Delta} \|\Lambda_\omega f\|^2 d\mu(\omega) + \int_\Delta \|\Lambda_\omega f\|^2 d\mu(\omega)\\ &= \int_{\Omega\backslash\Delta} \|\Lambda_\omega f\|^2 d\mu(\omega).\end{aligned}$$

Since $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K-g-frame for H with bounds A, B, then for each $f \in R(K)$,

$$A\|K^* f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) \leq B\|f\|^2. \tag{2.4}$$

By (2.3) and (2.4), for each $f \in R(K)$, we obtain

$$A\|K^* f\|^2 \leq \int_{\Omega\backslash\Delta} \|\Lambda_\omega f\|^2 d\mu(\omega). \tag{2.5}$$

If $g \in R(K)^\perp$, then

$$\langle K^* g, h\rangle = \langle g, Kh\rangle = 0, \quad \forall f \in H.$$

That is,

$$K^* g = 0, \quad g \in R(K)^\perp. \tag{2.6}$$

Let $h \in H$, then we can write h as $h = f + g$, where $f \in R(K)$ and $g \in R(K)^\perp$. So by (2.5) and (2.6),

$$\begin{aligned}A\|K^* h\|^2 = A\|K^* f\|^2 &\leq \int_{\Omega\backslash\Delta} \|\Lambda_\omega f\|^2 d\mu(\omega)\\ &= \int_{\Omega\backslash\Delta} \|\Lambda_\omega (f+g)\|^2 d\mu(\omega)\\ &= \int_{\Omega\backslash\Delta} \|\Lambda_\omega h\|^2 d\mu(\omega).\end{aligned}$$

(2). Assume that $h \in H$, then it can be written as $h = f + g$, where $f \in R(K)$ and $g \in R(K)^\perp$. By Lemma 1.11, for each $f \in R(K)$,

$$\|f\| = \|(K^\dagger|_{R(K)})^* K^* f\| \leq \|(K^\dagger|_{R(K)})^*\|\|K^* f\| \leq \|K^\dagger\|\|K^* f\|. \tag{2.7}$$

From (2.3), (2.6) and (2.7), we obtain

$$\begin{aligned}\int_{\Omega\setminus\Delta} \|\Lambda_\omega h\|^2 d\mu(\omega) &= \int_{\Omega\setminus\Delta} \|\Lambda_\omega f\|^2 d\mu(\omega)\\ &= \int_{\Omega} \|\Lambda_\omega f\|^2 d\mu(\omega) - \int_{\Delta} \|\Lambda_\omega f\|^2 d\mu(\omega)\\ &\geq A\|K^* f\| - B\|f\|^2\\ &\geq A\|K^* f\| - B\|K^\dagger\|^2\|K^* f\|^2\\ &= (A - B\|K^\dagger\|^2)\|K^* f\|^2.\end{aligned}$$

(3). Let $0 \neq f \in W_\Delta \subseteq R(K)$. Then there exists a $g \in H$ such that $f = Kg$, so

$$\langle K^* f, g\rangle = \langle K^* Kg, g\rangle = \|Kg\|^2 = \|f\| \neq 0.$$

Hence $K^* f \neq 0$. By $W_{\Omega\setminus\Delta} \perp W_\Delta$, we have

$$\begin{aligned}\int_{\Omega\setminus\Delta} \|\Lambda_\omega f\|^2 d\mu(\omega) &= \int_{\Omega\setminus\Delta} \langle \Lambda_\omega^* \Lambda_\omega f, f\rangle d\mu(\omega)\\ &= \langle T_{\Omega\setminus\Delta}(\{\Lambda_\omega f\}_{\omega\in\Omega\setminus\Delta}), f\rangle = 0.\end{aligned}$$

Therefore, the lower frame condition is not satisfied. □

3. Some properties of c-K-g-frames

In this section, we extend the concept of atomic systems to continuous version. Then we study the properties of c-K-g-frames and their relations with new kinds of atomic systems.

At first, we define atomic cg-systems.

Definition 3.1. Let $K \in B(H)$. A family $\Lambda = \{\Lambda_\omega \in B(H, H_\omega) : \omega \in \Omega\}$ is called an atomic cg-system for K if the following conditions hold:

(i) $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K-g-Bessel family,

(ii) there exists a $C > 0$ such that for each $f \in H$, there exists a $\varphi \in (\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2}$ which satisfies $\|\varphi\|_2 \leq C\|f\|$ and

$$\langle Kf, g\rangle = \int_\Omega \langle \Lambda_\omega^* \varphi(\omega), g\rangle d\mu(\omega), \quad \forall g \in H.$$

Now, we study the relationship between atomic cg-systems and c-K-g-frames.

Theorem 3.2. *Let $K \in B(H)$ and $\{\Lambda_\omega \in B(H, H_\omega) : \omega \in \Omega\}$ be a c-K-g-Bessel family for H. The following statements are equivalent:*

(1) $\{\Lambda_\omega\}_{\omega\in\Omega}$ is an atomic cg-system for K.

(2) $\{\Lambda_\omega U \in B(H, H_\omega) : \ \omega \in \Omega\}$ *is an atomic cg-system for* U^*K*, where* $U \in B(H)$ *is an onto operator.*

(3) $\{\Lambda_\omega\}_{\omega\in\Omega}$ *is a c-K-g-frame for H.*

(4) There exists a c-K-g-Bessel family $\{\Gamma_\omega\}_{\omega\in\Omega}$ *for H such that*

$$\langle Kf, g\rangle = \int_\Omega \langle \Gamma_\omega f, \Lambda_\omega g\rangle d\mu(\omega), \quad \forall f, g \in H. \tag{3.1}$$

(5) There exists a c-K-g-Bessel family $\{\Gamma_\omega\}_{\omega\in\Omega}$ *for H such that*

$$\langle K^* f, g\rangle = \int_\Omega \langle \Lambda_\omega f, \Gamma_\omega g\rangle d\mu(\omega), \quad \forall f, g \in H. \tag{3.2}$$

Proof. (1) $\Rightarrow$ (2). Suppose that $\{\Lambda_\omega\}_{\omega\in\Omega}$ is atomic cg-system for K and U is an onto operator on H. Then obviously $\{\Lambda_\omega U\}_{\omega\in\Omega}$ is a cg-Bessel family for H. Also, there exists a $C > 0$ such that for each $f \in H$, there exists a $\varphi \in (\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2}$ which satisfies $\|\varphi\|_2 \leq C\|f\|$ and

$$\langle Kf, g\rangle = \int_\Omega \langle \Lambda^*_\omega \varphi(\omega), g\rangle d\mu(\omega), \quad \forall g \in H. \tag{3.3}$$

U is onto, so by (3.3), for each $h \in H$, we have

$$\langle Kf, Uh\rangle = \int_\Omega \langle \Lambda^*_\omega \varphi(\omega), Uh\rangle d\mu(\omega) = \int_\Omega \langle (\Lambda_\omega U)^* \varphi(\omega), h\rangle d\mu(\omega).$$

Therefore, $\{\Lambda_\omega U\}_{\omega\in\Omega}$ is an atomic cg-system for U^*K.

(2) $\Rightarrow$ (3). There exists a $C > 0$ such that for each $f \in H$, there exists a $\varphi \in (\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2}$ which $\|\varphi\|_2 \leq C\|f\|$ holds and

$$\langle U^*Kf, g\rangle = \int_\Omega \langle (\Lambda_\omega U)^* \varphi(\omega), g\rangle d\mu(\omega), \quad \forall g \in H.$$

Since U is onto, each $g \in H$ can be written as $g = Uh$, for some $h \in H$. Hence for each $h \in H$,

$$\langle Kf, h\rangle = \int_\Omega \langle \Lambda^*_\omega \varphi(\omega), h\rangle d\mu(\omega) = \langle T\varphi, h\rangle, \quad \forall g \in H.$$

This implies that $R(K) \subseteq R(T)$. By Theorem 1.10, $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K-g-frame for H.

(3) $\Rightarrow$ (4). By Theorem 3.1 in [2], the proof is complete.

(4) $\Leftrightarrow$ (5). By (4), we have for each $f, g \in H$,

$$\langle K^* f, g\rangle = \overline{\langle Kg, f\rangle} = \overline{\int_\Omega \langle \Gamma_\omega g, \Lambda_\omega f\rangle d\mu(\omega)} = \int_\Omega \langle \Lambda_\omega g, \Gamma_\omega f\rangle d\mu(\omega),$$

which implies (5). Similarly, (5) $\Rightarrow$ (4) holds.

(4) $\Rightarrow$ (1). suppose that (4) holds. There exists a cg-Bessel family $\{\Gamma_\omega\}_{\omega\in\Omega}$ for H such that

$$\langle Kf, g\rangle = \int_\Omega \langle \Gamma_\omega f, \Lambda_\omega g\rangle d\mu(\omega), \quad \forall f, g \in H. \tag{3.4}$$

So there exists a $C > 0$ such that

$$\Big(\int_\Omega \|\Gamma_\omega f\|^2 d\mu(\omega)\Big)^{\frac{1}{2}} \leq C\|f\|, \quad \forall f \in H.$$

For a given $f \in H$, put $\varphi = \{\Gamma_\omega f\}_{\omega\in\Omega}$, then $\varphi \in (\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2}$ and by (3.4),

$$\langle Kf, g\rangle = \int_\Omega \langle \Lambda_\omega^* \varphi(\omega), g\rangle d\mu(\omega), \quad \forall f, g \in H.$$

Therefore, $\{\Lambda_\omega\}_{\omega\in\Omega}$ is an atomic cg-system for K. □

In the following, we verify the relationship between c-K-g-frames and $R(K)$. We present the set of all c-K-g-frames for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ by $CG(K)$ and the set of all tight c-K-g-frames for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ by $CGT(K)$.

Proposition 3.3. *Suppose that $K_1, K_2 \in B(H)$ are non-zero operators such that $R(K_2) \subseteq R(K_1)$. Then $CG(K_1) \subseteq CG(K_2)$.*

Proof. By Lemma 1.12, there exists $\alpha > 0$ such that

$$\|K_2^* f\|^2 \leq \alpha^2 \|K_1^* f\|^2, \quad \forall f \in H.$$

If $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K_1-g-frame for H with bounds A, B, then

$$\frac{A}{\alpha}\|K_2^* f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) \leq B\|f\|^2, \quad \forall f \in H.$$

□

Proposition 3.4. *If $CGT(K_1) \subseteq CG(K_2)$, then $R(K_2) \subseteq R(K_1)$.*

Proof. If $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a tight c-K_1-g-frame for H with bound A, then

$$A\|K_1^* f\|^2 = \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega), \quad \forall f \in H. \tag{3.5}$$

By inclusion $CGT(K_1) \subseteq CG(K_2)$, $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K_2-g-frame for H. So, there exist constants $C, D > 0$ such that

$$C\|K_2^* f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) \leq D\|f\|^2, \quad \forall f \in H. \tag{3.6}$$

From (3.5) and (3.6), we have

$$\|K_2^* f\|^2 \leq \frac{A}{C}\|K_1^* f\|^2, \quad \forall f \in H.$$

So Lemma 1.12 implies that $R(K_2) \subseteq R(K_1)$. □

Remark 3.5. Consider the family $\{\Lambda_\omega \in B(H, H_\omega) : \omega \in \Omega\}$. By Remark 2.12 in [12], the family $\{u_{\omega,k}\}_{\omega\in\Omega,k\in\mathbb{K}_\omega}$ is called the family induced by $\{\Lambda_\omega\}_{\omega\in\Omega}$ with respect to $\{e_{\omega,k}\}_{\omega\in\Omega,k\in\mathbb{K}_\omega}$, where $\{e_{\omega,k}\}_{\omega\in\Omega,k\in\mathbb{K}_\omega}$ is an orthonormal basis for Hilbert space $\oplus_{\omega\in\Omega}H_\omega$ such that for each $\omega \in \Omega$, $\{e_{\omega,k}\}_{k\in\mathbb{K}_\omega}$ is an orthonormal basis of H_ω. More precisely,

$$u_{\omega,k} = \Lambda^* e_{\omega,k}, \quad \omega \in \Omega,\ k \in \mathbb{K}_\omega. \tag{3.7}$$

Proposition 3.6. *Let $\{\Lambda_\omega \in B(H, H_\omega) : \omega \in \Omega\}$ be a family such that for each $f \in H$, $\{\Lambda_\omega f\}_{\omega\in\Omega}$ is strongly measurable. Then $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K-g-frame for H if and only if $\{u_{\omega,k}\}_{\omega\in\Omega,k\in\mathbb{K}_\omega}$ is a cK-frame for H.*

Proof. By Remark 3.5,

$$\begin{aligned}\int_\Omega \|\Lambda_\omega h\|^2 d\mu(\omega) &= \int_\Omega \sum_{k\in\mathbb{K}_\omega} |\langle h, u_{\omega,k}\rangle|^2 d\mu(\omega)\\ &= \int_\Omega \Big(\int_{\mathbb{K}} |\langle h, u_{\omega,k}\rangle|^2 dl(k)\Big) d\mu(\omega),\end{aligned}$$

where $l : \mathbb{K} \longrightarrow \mathbb{K}$ is the counting measure on $\mathbb{K}$. So the proof is complete. □

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