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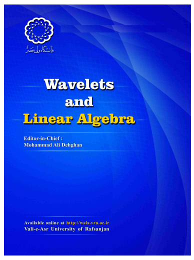

Some results on haar wavelets matrix through linear algebra

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Abstract

Can we characterize the wavelets through linear transformation? the answer for this question is certainly YES. In this paper we have characterized the Haar wavelet matrix by their linear transformation and proved some theorems on properties of Haar wavelet matrix.

1. Introduction

Wavelet theory is a newly emerging area in mathematical research. It has been applied in engineering disciplines, such as signal analysis for wave form representation and segmentations,

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time frequency analysis and fast algorithms for easy implementation. Wavelets were first applied in geophysics to analyze data from seismic surveys, which are used in oil and mineral exploration to get ”pictures” of layering in subsurface rock. In fact, geophysicists rediscovered them; mathematicians had developed them to solve abstract problems some 20 years earlier, but had not anticipated their applications in signal processing. The subject of wavelet appeared in the mid 1980s, influenced by ideas from both pure mathematics (harmonic analysis, functional analysis, approximation theory, fractal sets etc.,) and applied mathematics (signal processing, mathematical physics etc.).

In 1807, Joseph Fourier developed a method for representing a signal with a series of coefficients based on an analysis function. He laid the mathematical basis from which the wavelet theory is developed. The first to mention wavelets was Alfred Haar in 1909 in his Ph. D thesis. In the 1930s, Paul Levy found the scale-varying Haar basis function superior to Fourier basis functions. The word wavelet is due to Morlet and Grossmann in the early 1980s. They used the French word ondelette, meaning small wave. Soon it was transferred to English by translating onde into wave, giving wavelet. In 1986, Stephane Mallat and Yves Meyer developed a multiresolution analysis using wavelets. They mentioned the scaling function of wavelets for the first time; it allowed researchers and mathematicians to construct their own family of wavelets using the derived criteria. Around 1998, Ingrid Daubechies used the theory of multiresolution wavelet analysis to construct their own family of wavelets. These set of wavelets are orthonormal basis functions have become the cornerstone of wavelet applications today. Theoretical treatment of wavelet analysis is covered in the Daubechies article [6]. Since nowadays wavelet theory is a wide field, the selection of material which is included here is naturally subjective. One of the popular families of wavelets is Haar wavelet [13, 14]. Haar wavelet is the lowest member of Daubechies family of wavelets and is convenient for computer implementations due to availability of explicit expression for the Haar scaling and wavelet functions [23]. Due to its simplicity, the Haar wavelet had become an effective tool for solving many problems arising in many branches of science and engineering [24, 25, 26, 27, 28, 29, 30, 31]. Haar functions have been used since 1910. It was introduced by the Hungarian mathematician Alfred Haar [13]. Haar wavelets are made up of pairs of piecewise constant functions and mathematically the simplest orthonormal wavelets with a compact support. Haar wavelets matrix [2, 3, 11, 24] is connected with different areas of science and engineering. In many of these areas, Haar wavelets matrix is necessary. Moreover, the present research is being focused on finding the solutions to differential and integral equations using Haar wavelets matrix. But research towards pure mathematics is rare. So, we are interested in this direction. Hence, we focussed on to study the properties of Haar wavelets matrix. There is some literatures are available on matrices such as, Completely Positive Linear Maps on Complex Matrices [5], Block Diagonalization and Eigenvalues [7], Linear Transformations Which Preserve Hermitian Matrices [10], Linear Transformations on Matrices [19], Bounds for Ratios of Eigenvalues Using Traces [20], Rhotrix Linear Transformation [21], The inverse of a tridiagonal matrix [22]. Also some literatures on wavelets methods in solving differential equations given in [8, 9, 12, 15, 16, 17, 32, 33] Fortunately no one viewed the Haar matrix through linear transformation. So, we thought of expressing Haar wavelets matrix in terms of linear transformation. Along with this, some properties of Haar wavelets matrix are explored.

The organization of the rest of the paper is as follows. In section 2, properties of Haar wavelets, Haar wavelets matrix and some definitions are discussed. Results based on Haar wavelets matrix through liner transformation is presented in section 3. Finally conclusions are drawn in section 4.

2. Preliminaries and basic definitions of Haar Wavelets

The scaling function $h_1(x)$ for the family of the Haar wavelet is defined as

$$h_1(x) = \begin{cases} 1 & \text{for } x\epsilon[0,1) \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

The Haar Wavelet family for $[0,1)$ is defined as,

$$h_i(x) = \begin{cases} 1 & \text{for } x\epsilon[\alpha,\beta) \\ -1 & \text{for } x\epsilon[\beta,\gamma) \\ 0, & \text{otherwise} \end{cases} \tag{2.2}$$

Where $\alpha = \frac{k}{m}$, $\beta = \frac{k+.5}{m}$, $\gamma = \frac{k+1}{m}$, $m = 2^l$, $l = 0,1,\cdots,J$, J is the level of resolution and $k = 0,1,\cdots,m-1$ is translation parameter. Maximum level of resolution is J. The index i in Eq. (2) is calculated using $i = m+k+1$. In case of minimal values $m = 1, k = 0$ then $i = 2$. The maximal value of i is 2^{J+1} . Let us define the collocation points $x_j = \frac{j-.5}{N}, j = 0,1,\cdots,j$. Haar coefficient matrix (Haar matrix) $H = H(i.j) = h_i(x_j)$ whose size is $N \times N$. For instance, if $J = 3 \implies N = 16$, then we have

$$H = H(i,j) = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{bmatrix}$$

In general Haar matrix can be denoted as,

$$H(N \times N) = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1(\frac{n}{2}+1position) & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdots & -1 & -1 & \cdots & -1 & -1 \\ 1 & 1 & 1 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & \cdots & -1 & -1 \\ 1 & 1 & 1 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1(\frac{n}{2}+1position) & -1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & -1 \end{bmatrix}$$

Definition 2.1. Let U and V be two vector spaces over the same field F. A mapping $T : U \longrightarrow V$ is said to be a linear transformation from U into V which associates to each element α of U to a unique element $T(\alpha)$ of V such that $T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$ for all α and β in U and all scalars a, b in F .

Definition 2.2. A scalar λ is called an eigen value of an $n \times n$ matrix A, then there is a nontrivial solution x of $Ax = \lambda x$. Such an x is called an eigenvector corresponding to the eigen value λ.

Definition 2.3. A square matrix A is said to be diagonalizable if it is similar to a diagonal matrix. Therefore, matrix A is diagonalizable if there exists an invertible matrix P and a diagonal matrix D such that $D = PAP^{-1}$.

Definition 2.4. The trace of $n \times n$ matrix A is defined as the sum of the elements on the main diagonal of the matrix A.

3. Results on Linear Transformation of Haar Wavelets Matrix and Its Properties

3.1. Results on Haar wavelet matrix through linear transformation

Theorem 3.1. *Let $n = 2^K$ where $K \epsilon N$, F be a field then a linear map $T : F^n \longrightarrow F^n$ can be represented by a $n \times n$ Haar matrix with respect to the standard basis of F^n iff T is defined as*

$$T(x_1, x_2, \ldots, x_n) = \Bigg(\sum_{i=1}^{n} x_i, \sum_{i=1}^{\frac{n}{2}} x_i - \sum_{i=\frac{n}{2}+1}^{n} x_i, \sum_{i=1}^{\frac{n}{4}} x_i - \sum_{i=\frac{n}{4}+1}^{\frac{n}{2}} x_i, \sum_{i=\frac{n}{2}+1}^{\lfloor \frac{3n+2}{4} \rfloor} x_i - \sum_{i=\lfloor \frac{3n+2}{4} \rfloor+1}^{n} x_i,$$
$$\sum_{i=1}^{\frac{n}{8}} x_i - \sum_{i=\frac{n}{8}+1}^{\frac{n}{4}} x_i, \sum_{i=\frac{n}{4}+1}^{\lfloor \frac{3n+4}{8} \rfloor} x_i - \sum_{i=\lfloor \frac{3n+4}{8} \rfloor+1}^{\frac{n}{2}} x_i, \ldots, x_1 - x_2, \ldots, x_{n-1} - x_n \Bigg).$$

Proof. Suppose an linear map $T : F^n \longrightarrow F^n$ defined as

$$T(x_1, x_2, \ldots, x_n) = \Big(\sum_{i=1}^{n} x_i, \sum_{i=1}^{\frac{n}{2}} x_i - \sum_{i=\frac{n}{2}+1}^{n} x_i, \sum_{i=1}^{\frac{n}{4}} x_i - \sum_{i=\frac{n}{4}+1}^{\frac{n}{2}} x_i, \sum_{i=\frac{n}{2}+1}^{\lfloor\frac{3n+2}{4}\rfloor} x_i - \sum_{i=\lfloor\frac{3n+2}{4}\rfloor+1}^{n} x_i,$$
$$\sum_{i=1}^{\frac{n}{8}} x_i - \sum_{i=\frac{n}{8}+1}^{\frac{n}{4}} x_i, \sum_{i=\frac{n}{4}+1}^{\lfloor\frac{3n+4}{8}\rfloor} x_i - \sum_{i=\lfloor\frac{3n+4}{8}\rfloor+1}^{\frac{n}{2}} x_i, \ldots, x_1 - x_2, \ldots, x_{i-1} - x_i \Big)$$

Consider the standard basis of F^n is $(1, 0, ..., 0), (0, 1, ..., 0), ..., (0, ..., 0, 1)$ then image of each element of basis is given by,

$$\begin{cases} T(1, 0, \ldots, 0) = (1, 1, 1, 0, 1, 0, \ldots, 0, 1(\frac{n}{2} + 1 \; position), 0, \ldots, 0) \\ T(0, 1, \ldots, 0) = (1, 1, 1, 0, 1, 0, \ldots, 0, -1(\frac{n}{2} + 1 \; position), 0, \ldots, 0) \\ T(0, 0, 1, \ldots, 0) = (1, 1, 1, 0, 1, 0, \ldots, 0, 1(\frac{n}{2} + 2 \; position), 0, \ldots, 0) \\ T(0, 0, 0, 1, \ldots, 0) = (1, 1, 1, 0, 1, 0, \ldots, 0, -1(\frac{n}{2} + 2 \; position), 0, \ldots, 0) \\ \vdots \\ T(0, 0, \ldots, 1(n/2 + 1 \; position), \ldots, 0) = (1, 1, 1, 0, 1, 0, \ldots, 0, 1(\frac{n}{2} + \frac{n}{4} + 1 \; position), 0, \ldots, 0) \\ T(0, 0, \ldots, 1(\frac{n}{2} + 2 \; position), \ldots, 0) = (1, 1, 1, 0, 1, 0, \ldots, 0, -1(\frac{n}{2} + \frac{n}{4} + 1 \; position), 0, \ldots, 0) \\ \vdots \\ T(0, 0, \ldots, 1, 0) = (1, 1, 1, 0, 1, \ldots, 0, 1) \\ T(0, 0, \ldots, 0, 1) = (1, 1, 1, 0, 1, \ldots, 0, -1) \end{cases} \tag{3.1}$$

corresponding matrix of T is,

$$M(T) = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1(\frac{n}{2} + 1 position) & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdots & -1 & -1 & \cdots & -1 & -1 \\ 1 & 1 & 1 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & \cdots & -1 & -1 \\ 1 & 1 & 1 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1(\frac{n}{2} + 1 position) & -1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & -1 \end{pmatrix}$$

here $M(T) = H$, where H is $n \times n$ Haar martix.

Conversely, Let matrix of linear transformation $T : F^n \longrightarrow F^n$ is $n \times n$ Haar matrix with respect to standard basis of F^n then corresponding linear transformationt is ,

$$[Hx]^T = \Big(\sum_{i=1}^{n} x_i, \sum_{i=1}^{\frac{n}{2}} x_i - \sum_{i=\frac{n}{2}+1}^{n} x_i, \sum_{i=1}^{\frac{n}{4}} x_i - \sum_{i=\frac{n}{4}+1}^{\frac{n}{2}} x_i, \sum_{i=\frac{n}{2}+1}^{\lfloor\frac{3n+2}{4}\rfloor} x_i - \sum_{i=\lfloor\frac{3n+2}{4}\rfloor+1}^{n} x_i,$$
$$\sum_{i=1}^{\frac{n}{8}} x_i - \sum_{i=\frac{n}{8}+1}^{\frac{n}{4}} x_i, \sum_{i=\frac{n}{4}+1}^{\lfloor\frac{3n+4}{8}\rfloor} x_i - \sum_{i=\lfloor\frac{3n+4}{8}\rfloor+1}^{\frac{n}{2}} x_i, \ldots, x_1 - x_2, \ldots, x_{n-1} - x_n\Big)$$

Where x^T is $(x_1, x_2, \ldots, x_n)$ be the $1 \times n$ unknown matrix and H is Haar matrix. □

In particularly if we fit $K = 2$ then linear transformation $T : F^4 \longrightarrow F^4$ defined by $T(x_1, x_2, x_3, x_4) = (x_1 + x_2 + x_3 + x_4, x_1 + x_2 - x_3 - x_4, x_1 - x_2, x_3 - x_4)$ and its matrix is given by,

$$M(T) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

where $M(T) = H$ is 4×4 Haar matrix.

3.2. Result on trace of Haar wavelets matrix

Theorem 3.2. *Let H be the $n \times n$ Haar matrix then* $Trace(H) = \begin{cases} 0 & if\ K\ is\ odd \\ K - 1 & if\ K\ is\ even \end{cases}$

Proof. **First part:** If K is even then Haar matrix size is $2^K \times 2^K = n \times n$.
Case 1 : all elements in $(n - (n - 1))^{th}$ row of H are 1's therefore $a_{11} = 1$
Case 2 : in $(n - (n - 2))^{th}$ row beginning $\frac{n}{2}$ elements are1's and remainings are -1's, therefore $a_{22} = 1$.
Case 3 : Here there are two rows are arrenged in the format that only continuous $\frac{n}{2}$ terms are nonzeros rests are zeros, i.e. in $(n - (n - 3))^{th}$ row bigging $\frac{n}{4}$ elements are 1's and $\frac{n}{4} + 1$ to $\frac{n}{2}$ are -1's rests are zeros. $(n - (n - 4))^{th}$ row arranged as $\frac{n}{2} + 1$ to $\lfloor\frac{3n+2}{4}\rfloor$ are 1's and $\lfloor\frac{3n+2}{4}\rfloor + 1$ to n are -1's. Therefore $a_{33} = 1, and\ a_{44} = 0$.
Case 4 : Here there are four rows are arranged in the format that only $\frac{n}{4}$ consecutive terms are nonzeros rests are zeros. In $(n-(n-5))^{th}$ row bigging $\frac{n}{8}$ elements are 1's and $\frac{n}{8}+1$ to $\frac{n}{4}$ are -1's. $(n - (n - 6))^{th}$ row arrenged as $\frac{n}{4} + 1$ to $\lfloor\frac{3n+4}{8}\rfloor$ are 1's and $\lfloor\frac{3n+4}{8}\rfloor + 1$ to $\frac{n}{2}$ are -1's. $(n - (n - 7))^{th}$ row arrenged as $\frac{n}{2} + 1$ to$\lfloor\frac{\frac{n}{2}+1+\frac{3n}{4}}{2}\rfloor$ are 1's and$\lfloor\frac{\frac{n}{2}+1+\frac{3n}{4}}{2}\rfloor + 1$ to $\frac{3n}{4}$ are -1's and in $(n - (n - 8))^{th}$ row has $\frac{3n}{4} + 1$ to $\frac{\frac{3n}{4}+1+n}{2}$ are 1's and $\frac{\frac{3n}{4}+1+n}{2} + 1$ to n are -1's. Therefore any one of $a_{55}, a_{66}, a_{77}, a_{88}$ is 1 others are zeros.
Case 5 : Similarly here we get eight rows arranged in the format that $\frac{n}{8}$ consecutive terms are non zero and remaining is zero, proceed in the above format we get out of eight diagonal element one element is unit and others are zero. Continuing in this way every cases we get only one diagonal element is unit except last case.

Case $K+1$: This is the last case there are $\frac{n}{2}$ rows from $\frac{n}{2}+1$ to n, where each row has two nonzero consecutive entrees as 1 and -1 others are zero, in $(\frac{n}{2}+1)^{th}$ row first position is 1 and second position is -1, in $(\frac{n}{2}+2)^{th}$ row third position is 1 and fourth position is -1, others are zero, in last row $(n-1)$ position is 1 and nth position is -1, in this case we get one of the diagonal element is -1 remains are zero. Totally $(K+1)$ cases arises, from 1 to K cases we get only one diagonal element as 1 and $(K-1)^{th}$ case one diagonal element is -1 remainings are zero. Sum of all diagonal is $K-1$.

Second part: If K is odd then Haar matrix size is $2^K \times 2^K = n \times n$ and H can be represented as,

$$H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C and D are $\frac{n}{2} \times \frac{n}{2}$ matrices and $Trace(H) = Trace(A) + Trace(D)$. here also there are $K+1$ cases arises, if we repeat above explained procedure we get one and only one diagonal element as 1 in each cases from 1 to $\frac{K+1}{2}$, but from $\frac{K+3}{2}$ to $K+1$ is -1 as a diagonal. Hence sum of all diagonals is zero. Hence the proof completes. □

3.3. Results on properties of Haar wavelets matrix

Theorem 3.3. *Let H be the Haar matrix, then rank of H is equal to its size.*

Proof. The linear map $T : F^n \longrightarrow F^n$ is defined by $T(x) = Hx$ for each column vector $x \epsilon F^n$. Rank of the linear map T is equal to the rank of H. Let $T(e_i) = H_i$ is column vector of H and $\{ e_i\}$ is standard basis of F^n for each i. Then image set of $\{ e_i\}$ is generates the image of T also $\{T(e_i)\}$ is linearly independent for each $i = 0, 1, \cdots, n$. Here H is the square matrix therefore row rank is same as column rank. Hence the rank of H is the maximum number of linearly independent columns of H. □

Theorem 3.4. *Let H be Haar matrix then solution set of* $Hx = 0$ *is trivial space.*

Proof. Since the rank of H is equal to its size, therefore determinant of H is nonzero. Hence solution set of $Hx = 0$ is trivial space. □

Remark 3.5. Since all rows of H are linearly independent and mutually orthogonal to each other, hence determinant of H is nonzero and also these rows form vector space whose dimension is equal to size of H

Corollary 3.6. *Haar wavelets matrix is linearly independent.*

Corollary 3.7. *Let H be an Haar wavelets matrix then no linearly independent solution of the equation* $Hx = 0$

Theorem 3.8. *Let H is Haar matrix then algebraic and geometric multiplicity of eigen values of H are same.*

Proof. 1. If $n = 2$ then Haar matrix is

$$H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

then eigen values are $\pm 2^{\frac{1}{2}}$ and corresponding eigen vectors are $[\frac{-1}{1-\sqrt{2}}, 1]^T$, $[\frac{-1}{1+\sqrt{2}}, 1]^T$ respectively. This shows that both algebraic and geometric multiplicities of eigen values $\sqrt{2}$ and $-\sqrt{2}$ are same.

2. If $n = 4$ then Haar matrix is

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

then eigen values are 2, 1.769292, $-.8846 + .5897\ i$, $-.8846 + .5897\ i$. and corresponding eigen vectors are $[1, 1, 0, 0]$,$[1, 1, 1.1303, 0.6388]$, $[1, 1, -1.5651 + 1.0434\ i, 0.6805 - 1.6331\ i]^T$, $[1, 1, -1.5651 - 1.0434\ i, 0.6805 + 1.6331\ i]^T$ respectively. This shows that both algebraic and geometric multiplicities of above eigen values are same.

Continuing in this way for different values of n we get same results. Hence in general above theorem statement is true for all values of n.

□

Lemma 3.9. *Let H be $n \times n$ Haar matrix then zero is not an eigen value of H.*

Proof. Since H is non singular, then $|H| \neq 0$ and product of eigen values of H is $|H|$, therefore no eigen values of H can be zero. □

Corollary 3.10. *Eigen values of Haar matrix are distinct.*

Corollary 3.11. *Haar matrix is diagonisable.*

Corollary 3.12. *Let H be the $n \times n$ Haar matrix then $[H^{-1}]^T =$*

$$\begin{bmatrix}
\frac{1}{2^K} & \frac{1}{2^K} & \frac{1}{2^K} & \frac{1}{2^K} & \cdots & \frac{1}{2^K}(\frac{n}{2}+1position) & \frac{1}{2^K} & \cdots & \frac{1}{2^K} & \frac{1}{2^K} \\
\frac{1}{2^K} & \frac{1}{2^K} & \frac{1}{2^K} & \frac{1}{2^K} & \cdots & -\frac{1}{2^K} & -\frac{1}{2^K} & \cdots & -\frac{1}{2^K} & -\frac{1}{2^K} \\
\frac{1}{2^{K-1}} & \frac{1}{2^{K-1}} & \frac{1}{2^{K-1}} & \frac{1}{2^{K-1}} & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & \frac{1}{2^{K-1}} & \frac{1}{2^{K-1}} & \cdots & -\frac{1}{2^{K-1}} & -\frac{1}{2^{K-1}} \\
\frac{1}{2^{K-2}} & \frac{1}{2^{K-2}} & \frac{1}{2^{K-2}} & \frac{1}{2^{K-2}} & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{1}{2}(\frac{n}{2}+1position) & -\frac{1}{2} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \frac{1}{2} & -\frac{1}{2} & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \frac{1}{2} & -\frac{1}{2}
\end{bmatrix}$$

Theorem 3.13. *Let H be the Haar matrix of linear transformation T. Then linear transformation T is nonsingular if and only if H is invertible.*

Proof. Let assume T is singular $\Longleftrightarrow T(\alpha) = 0$

$$\begin{aligned} &\Longleftrightarrow T(\alpha) = 0.\alpha \\ &\Longleftrightarrow 0 \text{ is an eigen value of } T \\ &\Longleftrightarrow 0 \text{ is eigen value of } H \\ &\Longleftrightarrow det(H) = 0 \\ &\Longleftrightarrow H \text{ is non invertible} \end{aligned}$$

Hence proof is completed. □

Corollary 3.14. *Let H be the Haar matrix. Then Eigen vectors of H are linearly independent.*

Proof. Ths is obvious, If H is $N \times N$ matrix. From the Theorem 3.8 all eigen values of H are distinct. So, corresponding eigen vectors also distinct. the space generated by these eigen vectors is N dimensional Haar linear space. Therfore this set of eigen vectors is basis for N dimensional Haar linear space. Hence eigen vectors of H are linearly independent. □

4. Conclusion

In this paper, we developed an linear transformation to find Haar wavelets matrix. If the size of the Haar wavelets matrix is too big then it is very difficult to represent. Therefore, by using linear transformation we can avoid this difficulty. Also we proved some properties of Haar wavelets matrix such as trace, invertible, eigen values, eigen vectors, algebraic and geometric multiplicities of eigen values of Haar matrix, solution space of Haar matrix and Diagonalization of Haar matrix.

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