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Projectional inequalities and their linear preservers

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Abstract

This paper introduces an inequality on vectors in $\mathbb{R}^n$ which compares vectors in $\mathbb{R}^n$ based on the p-norm of their projections on $\mathbb{R}^k$ $(k \leq n)$. For $p > 0$, we say x is d-projectionally less than or equal to y with respect to p-norm if $\sum_{i=1}^{k} |x_i|^p$ is less than or equal to $\sum_{i=1}^{k} |y_i|^p$ for every $d \leq k \leq n$. For a relation $\sim$ on a set X, we say a map $f : X \to X$ is a preserver of that relation, if $x \sim y$ implies $f(x) \sim f(y)$ for every $x, y \in X$. All the linear maps that preserve d-projectional equality and inequality are characterized in this paper.

1. Introduction

Ordering sets has always been of interest to mathematician. As George Polya says inequalities play an important role in most branches of mathematics and have widely different applications.

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The theory of vector and matrix inequalities and their linear preservers is an interesting subject in matrix theory. There are several ways to define orders on vectors and matrices. For example, you may compare vectors according to their norms or their components. Another way is majorization. Let $\mathbb{R}^n$ be the vector space of all $n \times 1$ real vectors.

Definition 1.1. For $x, y \in \mathbb{R}^n$ we say x is majorized by y, denoted by $x \prec y$, if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow}, \qquad 1 \leq k \leq n,$$

and for $k = n$, these two sums are equal, where $x_i^{\downarrow}$ is the i^{th} largest entry of the vector x. [7]

Majorization inequality has been extended to some other relations such as multivariate majorization [7], row stochastic majorization [6]. Power majorization is a type of majorization which defined as follows. Let x and y be two vectors with non-negative components. We say x is power majorized by y, $x \prec_p y$, if $x_1^p + x_2^p + ... + x_n^p \leq y_1^p + y_2^p + ... + y_n^p$ for all $p \geq 1$, $p \leq 0$ and the inequality will be reversed when $0 \leq p \leq 1$.[4]
In this paper we define a new kind of relation on $\mathbb{R}^n$ called d-projectional (in)equality which is somehow similar to majorization and power majorization. Here, we don't care about the entries of the vector and their order or arrangement anymore. Instead, we consider the p-norm of the projection of vectors. Let e_i, $i = 1, ..., n$ be the standard basis of $\mathbb{R}^n$. For the vectors $x, y \in \mathbb{R}^n$, we compare the p-norm of their projections on the subspace $V = \mathbb{R}^k$ of $\mathbb{R}^n$ generated by $e_1, \cdots, e_k$ for each $k \leq n$.
Characterizing linear maps with special conditions is one of the challenging problems in mathematics. For example in [3] authors have characterized all multiplicative isomorphisms for invertible matrices. One of the most interesting problems is to find the linear maps which preserve that relation on a linear space. More precisely, let $\sim$ be a relation on a linear space $\mathbb{V}$. A linear operator $T : \mathbb{V} \to \mathbb{V}$ is called a linear preserver of $\sim$ if for every $v, w \in \mathbb{V}$

$$v \sim w \Rightarrow Tv \sim Tw.$$

All linear preserving majorization linear maps on $\mathbb{R}^n$ have been characterized in the following theorem[1].

Proposition 1.2. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear preserver of $\prec$. Then T has one of the following forms:*
i) $Tx = \mathrm{tr}(x)\, a$ for some $a \in \mathbb{R}^n$.
ii) $Tx = \alpha\Pi x + \beta \boldsymbol{J} x$ for some $\alpha, \beta \in \mathbb{R}$ and some permutation Π, where $\boldsymbol{J}$ is the matrix with all entries equal to one.

Some special kinds of majorization preserving linear maps are characterized in [2] and [5]. In this paper we characterize linear maps that preserve the d-projectional (in)equality. In this paper $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map with the matrix representation $A = (a_{ij})_{i,j=1}^n$ with respect to the standard basis of $\mathbb{R}^n$, and A^r and A_r are the r^{th} row and column of A, respectively. Also by e we denote a vector in $\mathbb{R}^n$ with all entries equal to one.

2. *d*-Projectional equality and its linear preservers

Let's start with the definition of d-projectional equality.

Definition 2.1. Let $p \in \mathbb{R}$, $p > 0$ and $x, y \in \mathbb{R}^n$. We say x is d-projectionally equal to y with respect to p-norm, and denoted by symbol $x =_p^d$, if $\sum_{i=1}^k |x_i|^p = \sum_{i=1}^k |y_i|^p$ for every $d \le k \le n$.

If $d = 1$, we use the notation $x =_p y$. It is easy to see that $e_i =_p^d e_j$ for each $1 \le i, j \le d$. For $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$ we use the notation $\mathbf{|}x\mathbf{|}$ for the vector $(|x_1|, \cdots, |x_n|)^T$, where $|x_i|$ is the absolute value of x_i.
We have the following lemmas.

Lemma 2.2. *Let $x, y \in \mathbb{R}^n$ and $p > 0$. $x =_p y$ if and only if $|x| = |y|$.*

Proof. It is direct result of definition. □

Considering $A = (a_{ij})$ to be the matrix presentation of T, A^k to be the k^{th} row and A_r be the r^{th} of A, we have the following lemmas and theorems.

Lemma 2.3. *Let p be a positive real number and $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear preserver of d-projectional equality. Then for each $i \ge d$, the i^{th} row of the matrix representation of T has at most one nonzero entry.*

Proof. Consider $A = (a_{ij})$ as the matrix presentation of T. Let A^k be the first row with $k \ge d$ and at least two nonzero entries like a_{kt} and a_{ks}. We can choose nonzero numbers α and β such that $a_{kt}\alpha + a_{ks}\beta = 0$. Consider the vectors $x = (x_1, \cdots, x_n)^T$ and $y = (y_1, \cdots, y_n)^T$, where

$$x_i = \begin{cases} -\alpha & \text{if } i = t \\ \beta & \text{if } i = s \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad y_i = \begin{cases} \alpha & \text{if } i = t \\ \beta & \text{if } i = s \\ 0 & \text{otherwise} \end{cases}.$$

We have $Tx = (x'_1, \cdots, x'_n)^T$ and $Ty = (y'_1, \cdots, y'_n)^T$, where $x'_i = a_{is}\beta - a_{it}\alpha$ and $y'_i = a_{is}\beta + a_{it}\alpha$ for $i = 1, \cdots, n$. Since for each $d < i < k$ the i^{th} row has at most one nonzero entry, $|x'_i| = |y'_i|$ for all $d < i < k$.
It is easy to see that $x =_p^d y$, but

$$\begin{aligned} \sum_1^k |y'_i|^p &= \sum_1^{k-1} |y'_i|^p + |a_{ks}\beta + a_{kt}\alpha|^p \\ &= \sum_1^{k-1} |y'_i|^p \\ &< \sum_1^{k-1} |y'_i|^p + |a_{ks}\beta - a_{kt}\alpha|^p \\ &= \sum_1^{k} |x'_i|^p \end{aligned}$$

which is a contradiction. □

Lemma 2.4. *Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear map such that $\|x\|_p = \|y\|_p$ implies $\|Tx\|_p = \|Ty\|_p$. Then $A = rV$ for some isometry matrix V and real number r.*

Proof. If $T = 0$, then $r = 0$. Suppose $T \neq 0$ and consider a unit vector x_0 such that $Tx_0 \neq 0$. Since $\|x\|_p = \|y\|_p$ implies $\|Tx\|_p = \|Ty\|_p$, then $\|Ay\|_p = \|Ax_0\|_p = r$ for every unit vector y . We show that for each x, $\|\frac{1}{r}Ax\|_p = \|x\|_p$. Let $x \neq 0$. Then $\|\frac{1}{r}Ax\|_p = \|x\|_p\|\frac{1}{r}A\frac{x}{\|x\|_p}\|_p = \|x\|_p$. Hence $\frac{1}{r}A$ is an isometry. □

The following theorem is our main theorem in this section.

Theorem 2.5. *Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear preserver of $=_p^d$*
i) If $d > 1$, then $A = A_{11} \oplus A_{22}$ where $A_{11} = rV$ for some $d \times d$ p-norm isometry matrix V or the first d columns of A are zero.
ii) $A_r - A_s =_p^d A_r + A_s$ for all r, s.
iii)$A_r =_p^d A_s$ for all $r, s \leq d$.

Proof. i) For matrix

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right], \tag{2.1}$$

we know $e_1 =_p^d e_2 =_p^d \cdots =_p^d e_d$, hence $Te_1 =_p^d Te_2 =_p^d \cdots =_p^d Te_d$, which is equivalent to

$$\sum_{i=1}^{d} |a_{i1}|^p = \sum_{i=1}^{d} |a_{ij}|^p \qquad \forall j \leq d,$$

and

$$|a_{k1}| = |a_{k2}| = \cdots = |a_{kd}| \qquad \forall k > d.$$

Since each row has at most one nonzero element, $|a_{k1}| = |a_{k2}| = \cdots = |a_{kd}| = 0$ for every $k > d$, i.e. $A_{21} = 0$. If $A_{11} \neq 0$, then for all vectors $x, y \in \mathbb{R}^n$ with $x =_p^d y$, we have $A_{11}(x_1, \cdots, x_d)^T =_p^d A_{11}(y_1, \cdots, y_d)^T$. Hence by Lemma 2.4, $A_{11} = rV$ for some p-norm isometry matrix V.
ii) We know $e_r + e_s =_p^d e_r - e_s$, so $T(e_r + e_s) =_p^d T(e_r - e_s)$, i.e.

$$\sum_{i=1}^{d} |a_{ir} + a_{is}|^p = \sum_{i=1}^{d} |a_{it} - a_{is}|^p$$

and

$$|a_{ir} + a_{is}|^p = |a_{ir} - a_{is}|^p, \qquad \forall i > d$$

which implies $A_r - A_s =_p^d A_r + A_s$
iii) Follows from part (i). □

Corollary 2.6. *Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear map and $A = (a_{ij})$ be the matrix presentation of T. Then the following statements are equivalent:*
i) T is a preserver of $=_p$.
ii) Each row of A has at most one nonzero entry.
iii) $A_r - A_s =_p A_r + A_s$ for all $r, s \geq 1$.

Proof. $(i) \Leftrightarrow (ii)$: If T is a preserver of $=_p$, then the result follows from Lemma 2.3. Conversely, assume that each row of A has at most one nonzero entry and let $x = (x_1, \cdots, x_n)^T$ and $y = (y_1, \cdots, y_n)^T$ be such that $x =_p y$. Then $|x_i| = |y_i|$ for each i. Let $Tx = (x'_1, \cdots, x'_n)^T$ and $Ty = (y'_1, \cdots, y'_n)^T$. Then

$$\begin{aligned}\sum_{i=1}^{k} |x'_i|^p &= \sum_{j=1}^{k} |(\sum_{i=1}^{n} a_{ji}x_i)|^p \\ &= \sum_{j=1}^{k} \sum_{i=1}^{n} |a_{ji}|^p |x_j|^p \\ &= \sum_{j=1}^{k} \sum_{i=1}^{n} |a_{ji}|^p |y_j|^p \\ &= \sum_{i=1}^{k} |y'_i|^p.\end{aligned}$$

$(ii) \Leftrightarrow (iii)$: If (ii) holds, then $|a_{ir} + a_{is}| = |a_{ir} - a_{is}|$ for every i. Hence $A_r - A_s =_p A_r + A_s$. Now, suppose that (ii) does not hold and A_k be the first row with at least two nonzero entries a_{kr} and a_{ks}. Hence $\sum_{i=1}^{k} |a_{ir} + a_{is}|^p \neq \sum_{i=1}^{k} |a_{ir} - a_{is}|^p$, which is a contradiction. □

3. Linear preservers of *d*-projectional inequality

In this section we define d-projectional ineqaulity as follows.

Definition 3.1. The vector x is d-projectionally less than y with respect to p-norm if $\sum_{i=1}^{k} |x_i|^p \leq \sum_{i=1}^{k} |y_i|^p$ for every $d \leq k \leq n$.

If x is d-projectionally less than y with respect to the p-norm, then we denote it by $x \ll_p^d y$, and if $d = 1$, we just use the notation $x \ll_p y$.

Example 3.2. Consider $x = (1, 2)^t$ and $y = (\sqrt{3}, \sqrt{2})^t$. Then $x \ll_2 y$, but $x \not\ll_3 y$. Also if we consider $x = (\sqrt[3]{16}, 5)^t$ and $y = (4, 3)^t$, then $x \ll_3 y$, but $x \not\ll_2 y$.

We can easily prove the two following lemmas.

Lemma 3.3. *Let p be a positive real number, d be a natural number and $x, y \in \mathbb{R}^n$*
i)$e_{i+1} \ll_p e_i, \forall i \geq 1$.
ii) $e_i \ll_p^d e - e_i, \forall i \geq 2$.
iii)If $y \ll_p^d x$, then $y - y_i e_i \ll_p^d x, \forall i \geq 1$.

Lemma 3.4. *Let $x, y, z \in \mathbb{R}^n$ and p be a positive real number*
i) $x \ll_p^d x$.
ii) If $x \ll_p^d y$ and $y \ll_p^d x$, then $x =_p^d y$.
iii) If $x \ll_p^d y$ and $y \ll_p^d z$, then $x \ll_p^d z$.

The second lemma says that $\ll_p^d$ is a partial order on $\mathbb{R}^n$ with respect to the $d-$ projectional equality.

Theorem 3.5. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map with the matrix presentation A. T preserves $\ll_p$ if and only if the following conditions hold:*
i) Each row of A has at most one nonzero entry.
ii)$A_{i+1} \ll_p A_i$ for each i.

Proof. By Lemma3.3 if $x \ll_p^d y$ and $y \ll_p^d x$, then $x =_p^d y$. Hence if T preserves $\ll_p$, it preserves $=_p$. Consequently by Theorem 2.3 each row of A has at most one nonzero entry. Also we know $e_{i+1} \ll_p e_i$ which implies $Ae_{i+1} \ll_p Ae_i$. Hence $A_{i+1} \ll_p A_i$.
To prove the converse, assume that the conditions hold and let $x = (x_1, \cdots, x_n)^T$ and $y = (y_1, \cdots, y_n)^T$ be in $\mathbb{R}^n$ with $x \ll_p y$. Moreover, let $Tx = (x'_1, \cdots, x'_n)^T$, $Ty = (y'_1, \cdots, y'_n)^T$ and $k(A_j) = \sum_{i=1}^{k} |a_{ij}|^p$. Since each row of A has at most one nonzero entry, the following equation holds:

$$\sum_{j=1}^{k} |x'_j|^p = \sum_{j=1}^{k} |(\sum_{i=1}^{n} a_{ji}x_i)|^p = \sum_{j=1}^{n}\sum_{i=1}^{k} |a_{ij}|^p|x_j|^p = \sum_{j=1}^{n} k(A_j)|x_j|^p. \tag{3.1}$$

Because of the condition(ii) and $x \ll_p y$, we have:

$$k(A_n)\sum_{i=1}^{n} |x_i|^p \le k(A_n)\sum_{i=1}^{n} |y_i|^p \tag{3.2}$$

and

$$(k(A_{n-1}) - k(A_n))\sum_{i=1}^{n-1} |x_i|^p \le (k(A_{n-1}) - k(A_n))\sum_{i=1}^{n-1} |y_i|^p. \tag{3.3}$$

By adding (3.2) and (3.3),

$$(k(A_{n-1}))\sum_{i=1}^{n-2} |x_i|^p + k(A_{n-1})|x_{n-1}|^p + k(A_n)|x_n|^p$$
$$\le (k(A_{n-1}))\sum_{i=1}^{n-2} |y_i|^p + k(A_{n-1})|y_{n-1}|^p + k(A_n)|y_n|^p.$$

Repeating the above process, we have

$$\sum_{j=1}^{n} k(A_j)|x_j|^p \le \sum_{j=1}^{n} k(A_j)|y_j|^p$$

which completes the proof. □

Corollary 3.6. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear preserver of $\ll_p$ with the matrix representation A. If A^{r_i} is the r_i^{th} nonzero row of A with $a_{r_i s_i} \neq 0$, then $s_i \le i$.*

Proof. Since T preserves $\ll_p$, it preserves $=_p$. So by Lemma 2.3 each row of A has at most one nonzero entry. Now we complete the proof by induction. Let $A^{r_1}, ..., A^{r_m}$ be nonzero rows of A with $1 \leq r_1 ... \leq r_m \leq n$. For the first nonzero row of A, there exists an s_1 such that $a_{r_1 s_1} \neq 0$. First we show that $s_1 = 1$. If $s_1 > 1$, we may consider the vectors e_{s_1} and $e - e_{s_1}$. We have $Te_{s_1} = (x_1, \cdots, x_n)^T$, where $x_1 = \cdots = x_{s_1 - 1} = 0$ and $x_{s_1} = a_{r_1 s_1}$, and $T(e - e_{s_1}) = (y_1, \cdots, y_n)^T$, where $y_1 = \cdots = y_{s_1} = 0$. Since $s_1 > 1$, by Lemma 3.2 $e_{s_1} \ll_p e - e_{s_1}$, hence $Te_{s_1} \ll_p T(e - e_{s_1})$, which implies $a_{r_1 s_1} = 0$ which is a contradiction. Therefore $s_1 = 1$.

Now suppose that for every i with $i < k < n$, $a_{r_i s_i} \neq 0$ implies $s_i \leq i$. We must show that if $a_{r_k s_k} \neq 0$, then $s_k \leq k$. Suppose that $s_k > k$ and $a_{r_k s_k} \neq 0$. By Lemma 2.3, there is no other nonzero entry in this row. Consider e_{s_k} and $x = (x_1, \cdots, x_n)^T$, where $x_i = 0$ for $1 \leq i \leq k-1$ and $i = s_k$, and $x_i = 1$ otherwise. It is clear that $e_{s_k} \ll_p x$. Based on Lemma 2.3, each nonzero row has at most one nonzero entry, so by the hypothesis of the induction $Te_{s_k} = (y_1, ..., y_n)^T$ where $y_{r_k} = a_{r_k s_k}$ and $y_i = 0$ for any $i \leq r_{k-1}$ and $Tx = (x'_1, \cdots, x'_n)^T$ where $x'_i = 0$ for any $i \leq r_k$. Therefore, Te_{s_k} is not d-projectionally less than Tx, that is contradiction. Hence if $a_{r_k s_k} \neq 0$, then $s_k \leq k$. □

Corollary 3.7. *$T : \mathbb{R}^n \to \mathbb{R}^n$ is a one to one linear preserver of $\ll_p$ if and only if T is diagonal and $a_{nn} \leq \cdots \leq a_{11}$.*

Proof. Follows from Theorem 3.5 and Corollary 3.6. □

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