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Banach Pair Frames

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ABSTRACT

In this article, we consider pair frames in Banach spaces and introduce Banach pair frames. Some various concepts in the frame theory such as frames, Schauder frames, Banach frames and atomic decompositions are considered as special kinds of (Banach) pair frames. Some frame-like inequalities for (Banach) pair frames are presented. The elements that participant in the construction of (Banach) pair frames are characterized. It is shown that a Banach space X has a Banach pair frame with respect to a Banach scalar sequence space ℓ, when it is precisely isomorphic to a complemented subspace of ℓ. It is shown that if we are allowed to choose the scalar sequence space, pair frames and Banach pair frames with respect to the chosen scalar sequence space denote the same concept.

1. Frames, From nonharmonic Fourier series to Banach Pair Frames

In 1952, Duffin and Schaeffer [14], introduced the notion of *frames* in the context of *nonharmonic Fourier series*, although its starting point may go further back to the works of Gabor in

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signal processing in1946 [19]. The importance of Duffin and Schaeffer's results wasn't realized for more than 30 years, when in 1980s' Young [28] formally studied frames in abstract Hilbert spaces. Daubechies, Grossmann and Meyer [13], reintroduced frames again and studied them deeply. The interested readers are referred to [5] and [9] for more basic studies about frames.

Feichtinger and Gröchenig [15] introduced the concept of *atomic decompositions* as an extension of frames from Hilbert spaces to Banach spaces. Gröchenig [20] generalized this notion and presented the definition of *Banach frames* by generalizing the synthesis operator.

The definition of *p-frames* first appeared in an article written by Aldroubi and Sun [1] who studied shift invariant subspaces. Christensen and Stoeva studied properties of p-frames in [11] and [25]. Casazza *et al.* [6] considered such frames where ℓ^p is replaced with a more general *Banach scalar sequence space* X_d.

Omitting Banach scalar sequence space in the definition of atomic decomposition yields another generalization of atomic decompositions, Schauder frames. From another point of view, *Schauder frames* are a direct extension of *Schauder bases*. This concept and some results related to it can be found in [7, 8, 21].

As mentioned above, omitting or generalizing some parts of the frame definition has caused some extensions in the frame theory such as atomic decompositions, Banach frames and Schauder frames. But in these extensions the role of the invertible frame operator is disregarded, by forcing it to be the identity operator. It has caused a gap and prevented these extensions from being direct generalizations of ordinary frames. Pair frames fill the gap between these concepts and frames, as stated in Theorem 3.12, Corollary 3.13, Figures 1-2, Relations (4.2)-(4.5) and Remark 4.10.

Pair frames are introduced by Fereydooni and Safapour [16] in Hilbert spaces setting. It is shown that pair frames are a generalization of some other kinds of frames. In the present paper, we generalize pair frames from Hilbert spaces to Banach spaces and introduce Banach pair frames.

This paper is organized as follows: In section 2, we recall the definitions of frames and Bessel sequences in Hilbert spaces and we give some characterizations of them. Banach pair frames are introduced and studied in section 3. It is shown that Schauder frames, atomic decompositions and Banach frames can be regarded as special kinds of pair frames or Banach pair frames. In section 4, some conditions are considered under which a (Banach) pair Bessel is a (Banach) pair frame. These conditions show how frames and pair frames are mutually related.

In section 5, some characterizations of the elements that participant in the construction of (Banach) pair frames are presented. There are some conditions under which the notions of pair frames and Banach pair frames are the same. It is shown that if we are allowed to choose the scalar sequence space, pair frames and Banach pair frames with respect to the chosen scalar sequence space denote the same concepts.

2. Frames in Hilbert Spaces and Preliminaries

In this section, we recall the definition of frames and some preliminary notations which will be used in this paper.

Throughout this paper, X denotes a separable Banach space and $\mathcal{H}$ is a Hilbert space. The notation $\langle .,. \rangle$ stands for the inner product in Hilbert spaces or the action of the functionals in the dual of a Banach space on itself. The index set of the natural numbers is denoted by $\mathbb{N}$. All norms are denoted by $\|.\|$; the reader can recognize conveniently that to which spaces each norm refers. The set of bounded operators on X is denoted by $\mathcal{B}(X)$. For a bounded linear operator V, $\mathcal{D}(V)$ and $\mathcal{R}(V)$ denote the domain and the range of V, respectively. V is bounded below if for a constant $A > 0$, $A\|f\| \leqslant \|V(f)\|$, $f \in X$. If $0 < p < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$, q is called the exponential conjugate of p.

Definition 2.1. A family $F = \{f_i\}_{i\in\mathbb{N}} \subset \mathcal{H}$ is called a **frame** for $\mathcal{H}$ if there exist positive constants A, B, such that for every $f \in \mathcal{H}$,

$$A \left\|f\right\|^2 \leqslant \sum_{i\in\mathbb{N}} \left|\left\langle f, f_i\right\rangle\right|^2 \leqslant B \left\|f\right\|^2. \tag{2.1}$$

A and B are lower and upper frame bounds, respectively. When the right inequality of (2.1) is satisfied for some $B > 0$, F is called a **Bessel sequence** for $\mathcal{H}$.

The following theorem is a motivation for defining pair frames (see Proposition 3.2).

Theorem 2.2. *[16] Let $F = \{f_i\}_{i\in\mathbb{N}} \subset \mathcal{H}$. For operator*

$$S_F : \mathcal{H} \to \mathcal{H}, \quad S_F(f) := \sum_{i\in\mathbb{N}} \langle f, f_i\rangle f_i,$$

the following statements are equivalent:

1. *$F = \{f_i\}_{i\in\mathbb{N}}$ is a frame for $\mathcal{H}$.*
2. *S_F is a well-defined and invertible operator.*
3. *S_F is a well-defined operator and there exist $\alpha \in (0, \infty)$ such that*

$$\left\|I - \alpha S_F\right\| < 1.$$

Let ℓ be a **Banach scalar sequence space**; i.e a normed vector space of scalar sequences which is also a Banach space with respect to the associated norm. The space ℓ is called a **BK-space** if the coordinate functionals are continuous [6]. Put $\delta_i = \{\delta_{ij}\}_{j\in\mathbb{N}}$ for every $i \in \mathbb{N}$, where δ_{ij} is the Kronecker delta. The set $\{\delta_i\}_{i\in\mathbb{N}}$ is called the set of **canonical vectors**. Additionally, when $\{\delta_i\}_{i\in\mathbb{N}}$ constitutes a basis for a BK-space ℓ, ℓ is said to be a **Schauder sequence space** [4].

For $1 < p < \infty$, ℓ^p spaces have all the above properties. Let c_0 be the space which consists of all the scaler sequences that vanish at infinity and c be the space of convergent sequences. c_0 is a Schauder sequence space but c is not though it contains canonical vectors. Because c is not a separable space the canonical vectors cannot constitute a basis for c. Recall that ℓ^∞ is the space of bounded scaler sequences. By the fact that $c \subset \ell^\infty$, ℓ^∞ is not also a Schauder sequence space.

For the proof of the next lemma we refer to [23, p. 201].

Lemma 2.3. *Let ℓ be a Schauder sequence space. Then the dual of ℓ, ℓ^*, is isometrically isomorphic to the BK-space*

$$\ell^{\circledast} = \left\{ \{\langle \phi, \delta_i \rangle\}_{i\in\mathbb{N}} \;\middle|\; \phi \in \ell^* \right\}.$$

Also, for every linear functional $\phi \in \ell^*$, there is a unique $\{d_i\} \in \ell^{\circledast}$ so that ϕ has the form

$$\left\langle \phi, \{c_i\}_{i\in\mathbb{N}} \right\rangle = \sum_{i\in\mathbb{N}} d_i c_i, \quad \forall \{c_i\}_{i\in\mathbb{N}} \in \ell.$$

The sequence $\{d_i\}_{i\in\mathbb{N}}$ is uniquely determined by $d_i = \langle \phi, \delta_i \rangle$ for all $i \in \mathbb{N}$. Moreover, if ℓ is reflexive, then ℓ^* is also a Schauder sequence space.

In the present paper, ℓ^* is identified with $\ell^{\circledast}$.

Definition 2.4. A sequence $G = \{g_i\}_{i\in\mathbb{N}} \subset \mathrm{X}^*$ is said to be **total** on X, if for every $f \in \mathrm{X}$, the condition

$$\langle f, g_i \rangle = 0, \qquad \forall i \in \mathbb{N},$$

implies $f = 0$.

If $G = \{g_i\}_{i\in\mathbb{N}}$ is total on the Banach space X,

$$\{0\} = \left\{ f \in \mathrm{X} \;\middle|\; \left\{ \langle f, g_i \rangle \right\}_{i\in\mathbb{N}} = \{0\} \right\} = \left\{ f \in \mathrm{X} \;\middle|\; \left\langle f, \overline{span}\{g_i\}_{i\in\mathbb{N}} \right\rangle \right\} = \{0\}$$

Furthermore, if X is reflexive

$$\{0\} = \left\{ f \in \mathrm{X}^{**} \;\middle|\; \{\langle f, \overline{span}\{g_i\}_{i\in\mathbb{N}} \rangle\} = \{0\} \right\}$$

So $\mathrm{X}^* = \overline{span}\{g_i\}_{i\in\mathbb{N}}$. A subspace ℓ' of ℓ is said to be complemented in ℓ if there exists a closed subspace ℓ'' in ℓ such that $\ell = \ell' \oplus \ell''$.

3. Pair Frames in Banach spaces

Now, we are in a situation to deal with the main subject of the paper.

Definition 3.1. Let $F = \{f_i\}_{i\in\mathbb{N}} \subset \mathrm{X}$ and $G = \{g_i\}_{i\in\mathbb{N}} \subset \mathrm{X}^*$. The pair (G, F) is said to be a **pair Bessel** for X if the operator

$$S_{FG} : \mathrm{X} \to \mathrm{X}, \quad S_{FG}(f) := \sum_{i\in\mathbb{N}} \langle f, g_i \rangle f_i, \tag{3.1}$$

is well-defined, i.e., the series converges for every $f \in \mathrm{X}$. The pair Bessel (G, F) is called a **pair frame** for X when S_{FG} is invertible. If $S_{FG} = I$, the pair Bessel (G, F) is a **Schauder frame**, i.e., for every $f \in \mathrm{X}$, $f = \sum_{i\in\mathbb{N}} \langle f, g_i \rangle f_i$.

The operator of the form S_{FG} is also called the multiplier operator of F and G [2, 26]; and the invertibility of this type of operators is studied in [3, 10, 27]. The reader can refer to [17] for studying the adjoint of pair frames. Pair frames in the context of C^*−Hilbert modules are considered in [24]. The unconditional convergence of series (3.1) is investigated in [17, 18, 26].

If F and G are Bessel sequences and $\|I - S_{FG}\| < 1$, (G, F) is called an **Approximate dual frame** [10]. Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$ and $\alpha_0 > 2$.

However $\left(\{\sqrt{\alpha_0} i e_i\}_{i\in\mathbb{N}}, \{\sqrt{\alpha_0}\frac{1}{i} e_i\}_{i\in\mathbb{N}}\right)$ is a pair frame but is not an approximate dual. The following proposition shows that the category of frames (Bessel sequences) can be embedded in the category of pair frames (Bessels); it can easily be verified by using Theorem 2.2.

Proposition 3.2. *[16] The sequence $F = \{f_i\}_{i\in\mathbb{N}} \subset \mathcal{H}$ is a frame (Bessel squence) for $\mathcal{H}$ if and only if (F, F) is a pair frame (Bessel) for $\mathcal{H}$.*

Let $F = \{f_i\}_{i\in\mathbb{N}} \subset \mathrm{X}$ and $G = \{g_i\}_{i\in\mathbb{N}} \subset \mathrm{X}^*$. The operator

$$U(U_G) : \mathrm{X} \to \mathbb{C}^{\mathbb{N}}, \quad U(f) := \left\{\langle f, g_i\rangle\right\}_{i\in\mathbb{N}},$$

is called the **analysis operator**, where $\mathbb{C}^{\mathbb{N}}$ is the space of all sequences of complex numbers. The operator

$$T_F : \mathcal{D}(T_F) \to \mathrm{X}, \quad T_F(\{c_i\}_{i\in\mathbb{N}}) := \sum_{i\in\mathbb{N}} c_i f_i,$$

is said to be the **synthesis operator**, where

$$\mathcal{D}(T_F) = \left\{\{c_i\}_{i\in\mathbb{N}} \subset \mathbb{C} \,\Big|\, \sum_{i\in\mathbb{N}} c_i f_i \text{ is convergent}\right\}.$$

Definition 3.3. Let $G = \{g_i\} \subset \mathrm{X}^*$ and ℓ be a BK-space. The sequence G is called a **ℓ-Bessel** for X with bound $B > 0$, if for every $f \in \mathrm{X}$,

1. $\left\{\langle f, g_i\rangle\right\}_{i\in\mathbb{N}} \in \ell$,
2. $\left\|\{\langle f, g_i\rangle\}_{i\in\mathbb{N}}\right\| \leqslant B \left\|f\right\|$.

Additionally, if for every $f \in \mathrm{X}$,

$$A \left\|f\right\| \leqslant \left\|\left\{\langle f, g_i\rangle\right\}_{i\in\mathbb{N}}\right\|,$$

for some constant $A > 0$, G is said to be an **ℓ-frame** for X. The constants A and B are called lower and upper ℓ-frame bounds, respectively.

If conditions (1) and (2) in the Definition 3.3 are satisfied for $\ell = \ell^p$, $G = \{g_i\}_{i\in\mathbb{N}}$ is called a **p-Bessel sequence** for X, and if additionally, the lower inequality holds for some $A > 0$, it is said to be a **p-frame** for X.

After defining of ℓ-Bessels, we intend to define pair frames (Bessels) with respect to the BK-space ℓ.

Definition 3.4. Let ℓ be a BK-space and $G = \{g_i\}_{i\in\mathbb{N}} \subset \mathrm{X}^*$ be a ℓ-Bessel for X. If there is $F = \{f_i\}_{i\in\mathbb{N}} \subset \mathrm{X}$ such that the pair (G, F) is a pair Bessel for X, i.e., the operator

$$S_{FG} : \mathrm{X} \longrightarrow \mathrm{X}, \quad S_{FG}(f) := \sum_{i\in\mathbb{N}} \langle f, g_i \rangle f_i,$$

is well-defined, then we say that (G, F) is a **pair Bessel for** X **with respect to** ℓ. Furthermore, if the operator S_{FG} is invertible, we call (G, F) a **pair frame for** X **with respect to** ℓ.

Like frames, we use the inverse of S_{FG} to obtain some reconstruction formulas:

$$f = \sum_{i\in\mathbb{N}} \langle f, g_i \rangle S_{FG}^{-1} f_i, \qquad f = \sum_{i\in\mathbb{N}} \langle f, S_{FG}^{-1\,*} g_i \rangle f_i. \tag{3.2}$$

Definition 3.5. Let ℓ be a BK-space, $F = \{f_i\}_{i\in\mathbb{N}} \subset \mathrm{X}$ and $G = \{g_i\}_{i\in\mathbb{N}} \subset \mathrm{X}^*$. The pair (G, F) is an **atomic decomposition for** X **with respect to** ℓ, if there are constants $A, B > 0$ such that for every $f \in \mathrm{X}$,

1. $\left\{\langle f, g_i \rangle\right\}_{i\in\mathbb{N}} \in \ell$,
2. $A \left\|f\right\| \leqslant \left\|\left\{\langle f, g_i \rangle\right\}_{i\in\mathbb{N}}\right\| \leqslant B \left\|f\right\|$,
3. $f = \sum_{i\in\mathbb{N}} \langle f, g_i \rangle f_i$.

By letting $S_{FG} = I$, every atomic decomposition for X with respect to a BK-space ℓ, can be treated as a pair frame with respect to ℓ.

Theorem 3.6. *[6, 17] Let ℓ be a Schauder sequence space, $G = \{g_i\} \subset \mathrm{X}^*$ be a ℓ-Bessel for* X *and $F = \{f_i\} \subset \mathrm{X}$ be a ℓ^*-Bessel for X^*. Then (G, F) is a pair Bessel for* X *with respect to ℓ.*

For exponential conjugates p, q, it is well-known that p-frames and q-frames constitute pair Bessles [11].

Theorem 3.7. *Suppose that ℓ is a Schauder sequence space. $F = \{f_i\}_{i\in\mathbb{N}} \subset \mathrm{X}$ is a ℓ^*-Bessel for X^* if and only if the operator*

$$T : \ell \rightarrow \mathrm{X}, \quad T(\{c_i\}_{i\in\mathbb{N}}) = \sum_{i\in\mathbb{N}} c_i f_i,$$

is well-defined.

Proof. Since ℓ is a Schauder sequence space with canonical basis $\{\delta_i\}$, the Lemma 2.3 implies that ℓ^* is a BK-space. Assuming $T : \ell \rightarrow \mathrm{X}, \quad T(\{c_i\}_{i\in\mathbb{N}}) = \sum c_i f_i$, is a well-defined operator, then $T(\delta_i) = f_i$, for all $i \in \mathbb{N}$. Boundedness of T implies that the operator

$$T^* : \mathrm{X}^* \rightarrow \ell^*,$$

is well-defined and bounded. For every $g \in X^*$ we have

$$\left\{\left\langle g, f_i\right\rangle\right\}_{i\in\mathbb{N}} = \left\{\left\langle g, T(\delta_i)\right\rangle\right\}_{i\in\mathbb{N}} = \left\{\left\langle T^*(g), \delta_i\right\rangle\right\}_{i\in\mathbb{N}}. \tag{3.3}$$

Since $\phi = T^*(g) \in \ell^*$, the Lemma 2.3 yields that $T^*(g)$ has the form $\{\langle T^*(g), \delta_i\rangle\}_{i\in\mathbb{N}}$ with the same norms. Hence, for every $g \in X^*$,

$$\left\|\left\{\langle g, f_i\rangle\right\}_{i\in\mathbb{N}}\right\| = \left\|\left\{\langle T^*(g), \delta_i\rangle\right\}_{i\in\mathbb{N}}\right\| = \left\|T^*(g)\right\| \leqslant \left\|T^*\right\| \left\|g\right\| = \left\|T\right\| \left\|g\right\|. \tag{3.4}$$

Put B=$\|T\|$. Since the relation (3.4) holds for all $g \in X^*$, then F is a ℓ^*-Bessel for X^* with bound B. For the converse, assume that F is an ℓ^*-Bessel for X^* with bound B. Then the operator

$$U : X^* \to \ell^*, \quad U(g) = \left\{\left\langle g, f_i\right\rangle\right\}_{i\in\mathbb{N}}, \tag{3.5}$$

is well-defined and $\|U\| \leqslant$ B. For $g \in X^*$, put $\phi = U(g) \in \ell^*$. Then Lemma 2.3 yields that $\phi = U(g)$ has the form $\{\langle U(g), \delta_i\rangle\}_{i\in\mathbb{N}}$. Thus,

$$\left\{\left\langle g, f_i\right\rangle\right\}_{i\in\mathbb{N}} = U(g) = \left\{\left\langle U(g), \delta_i\right\rangle\right\}_{i\in\mathbb{N}} = \left\{\left\langle g, U^*(\delta_i)\right\rangle\right\}_{i\in\mathbb{N}}. \tag{3.6}$$

for all $g \in X^*$. Then

$$\left\langle g, f_i\right\rangle = \left\langle g, U^*(\delta_i)\right\rangle, \quad i \in \mathbb{N},$$

for every $g \in X^*$. So

$$U^*(\delta_i) = f_i, \quad i \in \mathbb{N}. \tag{3.7}$$

The operator $U^* : \ell^{**} \to X^{**}$ is bounded on $\ell \subset \ell^{**}$. So $T := U^* |_\ell$ is bounded. Since ℓ is a Schauder sequence space, (3.7) yields that

$$T(\{c_i\}_{i\in\mathbb{N}}) = \sum_{i\in\mathbb{N}} c_i T(\delta_i) = \sum_{i\in\mathbb{N}} c_i U^*(\delta_i) = \sum_{i\in\mathbb{N}} c_i f_i,$$

for every $\{c_i\}_{i\in\mathbb{N}} \in \ell$. Since $T(\ell) \subset \overline{span}\{f_i\}_{i\in\mathbb{N}} \subset X$, $T : \ell \to X$ is a well-defined and bounded operator. □

The notion of pair frame can be generalized by generalizing the synthesis operator T_F.

Definition 3.8. Let ℓ be a BK-space and $G = \{g_i\}_{i\in\mathbb{N}} \subset X^*$ be a ℓ-Bessel for X and $T : \ell \to X$ a bounded operator. We call the pair (G, T) a **Banach pair Bessel for** X **with respect to** ℓ. Define

$$S_{TG} : X \longrightarrow X, \quad S_{TG}(f) := T\Big(\{\langle f, g_i\rangle\}_{i\in\mathbb{N}}\Big).$$

If S_{TG} is invertible, we call (G, T) a **Banach pair frame for** X **with respect to** ℓ.

An example for showing the existence of Banach pair frames is given in the Example 3.9, but, for an interpretation about Banach pair frames see Remark 3.16.

Example 3.9. Let $\{w_i\}_{i\in\mathbb{N}}$ be a sequence of positive numbers such that $0 < m := \inf_{i\in\mathbb{N}} w_i \leqslant \sup_{i\in\mathbb{N}} w_i =: M < \infty$. Define

$$\ell^p(w) = \Big\{\alpha = \{\alpha_i\}_{i\in\mathbb{N}} \;\Big|\; \|\alpha\|^p_{\ell^p(w)} = \sum_{i\in\mathbb{N}} w_i|\alpha_i|^p < \infty\Big\},$$

and $T : \ell^p(w) \longrightarrow \ell^p(\mathbb{N})$, $T(\{\alpha_i\}) = \Big\{w_i^{\frac{1}{p}}\alpha_i\Big\}_{i\in\mathbb{N}}$. For $e_i = \frac{1}{w_i^{\frac{1}{p}}}\delta_i, i \in \mathbb{N}$, $\Big(\{e_i\}_{i\in\mathbb{N}}, T\Big)$ is a Banach frame for $\ell^p(\mathbb{N})$ with respect to $\ell^p(w)$: $T\Big(\{\langle\alpha, e_i\rangle\}_{i\in\mathbb{N}}\Big) = \alpha,\ \alpha \in \ell^p(\mathbb{N})$. Now, we give an example of Banach pair frame. For $0 < \epsilon < 1$ let $\{\beta_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ be so that $\sup_{i\in\mathbb{N}} |1 - \beta_i| < \epsilon$ and $T' : \ell^p(w) \longrightarrow \ell^p(\mathbb{N})$ be so that $\|T - T'\| < \epsilon$. Define $S(\cdot) = T'\Big(\{\langle\cdot, \beta_i e_i\rangle\}_{i\in\mathbb{N}}\Big)$. For $\alpha \in \ell^p(\mathbb{N})$,

$$\begin{aligned}\Big\|\Big\{\Big\langle\alpha, (1-\beta_i)e_i\Big\rangle\Big\}_{i\in\mathbb{N}}\Big\|^p &= \sum_{i\in\mathbb{N}}\Big|\Big\langle\alpha, (1-\beta_i)\frac{1}{w_i^{\frac{1}{p}}}\delta_i\Big\rangle\Big|^p \leqslant \Big(\frac{1}{m}\sup_{i\in\mathbb{N}}\Big|1-\beta_i\Big|\Big)^p\Big\|\alpha\Big\|^p \\ &\leq \frac{\epsilon^p}{m^p}\Big\|\alpha\Big\|^p.\end{aligned}$$

Similarly,

$$\begin{aligned}\Big\|\Big\{\Big\langle\alpha, \beta_i e_i\Big\rangle\Big\}_{i\in\mathbb{N}}\Big\|^p = &\sum_{i\in\mathbb{N}}\Big|\Big\langle\alpha, \beta_i\frac{1}{w_i^{\frac{1}{p}}}\delta_i\Big\rangle\Big|^p \leqslant \Big(\frac{1}{m}\sup_{i\in\mathbb{N}}\Big|\beta_i\Big|\Big)^p\Big\|\alpha\Big\|^p \leqslant \Big(\frac{1+\epsilon}{m}\Big)^p\Big\|\alpha\Big\|^p \\ &\leqslant \Big(\frac{2}{m}\Big)^p\Big\|\alpha\Big\|^p.\end{aligned}$$

Therefore

$$\begin{aligned}\Big\|\alpha - S(\alpha)\Big\| &= \Big\|\alpha - T'\Big(\Big\{\Big\langle\cdot, \beta_i e_i\Big\rangle\Big\}_{i\in\mathbb{N}}\Big)\Big\| \leqslant \Big\|T\Big(\{\langle\alpha, e_i\rangle\}_{i\in\mathbb{N}}\Big) - T'\Big(\Big\{\Big\langle\cdot, \beta_i e_i\Big\rangle\Big\}_{i\in\mathbb{N}}\Big)\Big\| \\ &\leqslant \Big\|T\Big\|\ \Big\|\Big\{\Big\langle\alpha, (1-\beta_i)e_i\Big\rangle\Big\}_{i\in\mathbb{N}}\Big\| + \Big\|T - T'\Big\|\ \Big\|\Big\{\Big\langle\alpha, \beta_i e_i\Big\rangle\Big\}_{i\in\mathbb{N}}\Big\| \\ &\leqslant \frac{1}{m}\Big(\Big\|T\Big\| + 2\Big)\,\epsilon\Big\|\alpha\Big\|.\end{aligned}$$

For sufficient small $\epsilon > 0$ we have $\|\alpha - S(\alpha)\| < 1$ and hence S is invertible, i.e., $(\{\beta_i e_i\}_{i\in\mathbb{N}}, T')$ is a Banach pair frame.

From now on, the operators S in all the previous definition will be called **pair frame (Bessel) operator** or briefly **frame (Bessel) operator**. The invertibility of the frame operators of Banach pair frames helps us have the reconstruction formulas

$$f = \Big(S^{-1}T\Big)\Big(\Big\{\Big\langle f, g_i\Big\rangle\Big\}_{i\in\mathbb{N}}\Big), \quad f = T\Big(\Big\{\Big\langle f, S^{-1^*}g_i\Big\rangle\Big\}_{i\in\mathbb{N}}\Big),\ f \in \mathrm{X}. \tag{3.8}$$

Suppose ℓ is a BK-space, with continuous coordinate functionals $\{\eta_i\}_{i\in\mathbb{N}} \subset \ell^*$ and $U : \mathrm{X} \to \ell$ be a bounded operator. Put $G := \{g_i\}_{i\in\mathbb{N}} = \{U^*\eta_i\}_{i\in\mathbb{N}} \subset \mathrm{X}^*$. Then for $f \in \mathrm{X}$,

$$U(f) = \left\{\left\langle U(f), \eta_i\right\rangle\right\}_{i\in\mathbb{N}} = \left\{\left\langle f, U^*\eta_i\right\rangle\right\}_{i\in\mathbb{N}} = \left\{\left\langle f, g_i\right\rangle\right\}_{i\in\mathbb{N}} = U_G(f).$$

Hence $U = U_G$.

In the definition of Banach pair frames and pair frames with respect to a BK-space ℓ, we have assumed that $G = \{g_i\}_{i\in\mathbb{N}}$ is only a ℓ-Bessel, not necessarily a ℓ-frame. In Proposition 3.10 we show that the lower inequality for a ℓ-Bessel G is necessary.

Proposition 3.10. *Let ℓ be a BK-space and $G = \{g_i\}_{i\in\mathbb{N}} \subset \mathrm{X}^*$ be a ℓ-Bessel for X. If*

1. *For the bounded operator $T : \ell \to \mathrm{X}$, (G, T) is a Banach pair frame for X with respect to ℓ, or*

2. *ℓ is a Schauder scalar sequence space and $F = \{f_i\}_{i\in\mathbb{N}}$ is a ℓ^*-Bessel for X^* such that (G, F) is a pair frame for X with respect to ℓ,*

then G is an ℓ-frame for X.

Proof. (1). Since (G, T) is a Banach pair frame, the operator $S = S_{TG}$ is invertible. Hence for every $f \in \mathrm{X}$,

$$\left\|f\right\| = \left\|S^{-1}TU_G(f)\right\| \leqslant \left\|S^{-1}\right\| \left\|T\right\| \left\|U_G(f)\right\|.$$

By assumptions, there is some $B > 0$ such that $\|U_G\| \leqslant B$. By putting $A = \frac{1}{\|S^{-1}\|\|T\|}$, for every $f \in \mathrm{X}$ we get

$$A\left\|f\right\| \leqslant \left\|U_G(f)\right\| \leqslant B\left\|f\right\|,$$

and consequently

$$A\left\|f\right\| \leqslant \left\|\left\{\langle f, g_i\rangle\right\}_{i\in\mathbb{N}}\right\| \leqslant B\left\|f\right\|.$$

(2). Since F is a ℓ-Bessel, Theorem 3.7 implies that the operator $T_F : \ell \to \mathrm{X}$ is well-defined and consequently bounded. Now, use the proof of part (1) with $T = T_F$ and $S = S_{T_F G}$, to establish the claim. □

Letting $S = I$ we see that Banach frames are special kinds of Banach pair frames.

Definition 3.11. Let ℓ be a BK-space, $G = \{g_i\}_{i\in\mathbb{N}} \subset \mathrm{X}^*$ and $T : \ell \to \mathrm{X}$ be a bounded operator. The pair (G, T) is called a **Banach frame for X with respect to ℓ** if there are constants $A, B > 0$ such that for every $f \in \mathrm{X}$,

1. $\left\{\left\langle f, g_i\right\rangle\right\}_{i\in\mathbb{N}} \in \ell$,

2. $A\left\|f\right\| \leqslant \left\|\left\{\left\langle f, g_i\right\rangle\right\}_{i\in\mathbb{N}}\right\| \leqslant B\left\|f\right\|$,

3. $f = T\left(\left\{\left\langle f, g_i\right\rangle\right\}_{i\in\mathbb{N}}\right)$.

The following statement shows that pair frames and Banach pair frames are direct extensions of frames. Equivalence of (1) and (2) is proven in the Proposition 3.2.

Proposition 3.12. *Let $F = \{f_i\}_{i\in\mathbb{N}} \subset \mathcal{H}$. The following statements are equivalent:*

1. *F is a frame (Bessel sequence) for $\mathcal{H}$.*
2. *(F, F) is a pair frame (Bessel) for $\mathcal{H}$ with respect to ℓ^2.*
3. *(F, T_F) is a Banach pair frame (Bessel) for $\mathcal{H}$ with respect to ℓ^2.*

In these cases, the frame operators are the same.

Proof. In the Bessel case we know that (F, F) is a pair Bessel if and only if F is a Bessel sequence. F being a Bessel sequence is equivalent to the operator

$$T_F : \ell^2 \to \mathcal{H}, \quad T_F\Big(\{c_i\}_{i\in\mathbb{N}}\Big) = \sum_{i\in\mathbb{N}} c_i f_i,$$

being well-defined. It holds if and only if the operator

$$S_{FF} : \mathcal{H} \to \mathcal{H}, \quad S_{FF}(f) = \sum_{i\in\mathbb{N}} \langle f, f_i\rangle f_i,$$

is well-defined. Or equivalently, (F, F) is a pair frame. S_{FF} is well-defined if and only if the operator

$$S_{T_F F} : \mathcal{H} \to \mathcal{H}, \quad S_{T_F F}(f) = T_F\Big(\Big\{\Big\langle f, f_i\Big\rangle\Big\}_{i\in\mathbb{N}}\Big),$$

is well-defined. Since T_F is defined on the whole of ℓ^2, $S_{T_F F}$ being well-defined yields that (F, T_F) is a Banach pair Bessel. Also, for $f \in \mathcal{H}$,

$$S_F(f) = S_{FF}(f) = \sum_{i\in\mathbb{N}} \langle f, f_i\rangle f_i = T_F\Big(\Big\{\Big\langle f, f_i\Big\rangle\Big\}_{i\in\mathbb{N}}\Big) = T_F U_F(f).$$

Therefore

$$S_F = S_{FF} = S_{T_F F},$$

where S_F, S_{FF} and $S_{T_F F}$ are the frame operators associated to the cases (1),(2) and (3), respectively.

For the frame cases, the invertibility of each of the operators S_F, S_{FF} and $S_{T_F F}$ implies the invertibility of the other ones. □

Atomic decompositions and Banach frames are direct extensions of Parseval frames and not direct extensions of frames:

Corollary 3.13. *Let $F = \{f_i\}_{i\in\mathbb{N}} \subset \mathcal{H}$. The following statements are equivalent:*

1. *F is a Parseval frame for $\mathcal{H}$.*

2. *(F, F) is an atomic decomposition for $\mathcal{H}$ with respect to ℓ^2.*

3. *(F, T_F) is a Banach frame for $\mathcal{H}$ with respect to ℓ^2.*

Proof. We have

$$S_F = S_{FF} = S_{T_F F} = I.$$

Now, Theorem 3.12 gives the assertions. □

Consider equations (3.2) and (3.8). When (G, F) is a pair frame ((G, T)) is a Banach pair frame) for X with respect to a BK-space ℓ with pair frame operator S, then $(G, S^{-1}F), (S^{-1*}G, F)$ are atomic decompositions $(G, S^{-1}T), (S^{-1*}G, T)$ are Banach frames) for X with respect to ℓ.

Figures 1,2 illustrate the inclusion relations between different kinds of frames discussed above.

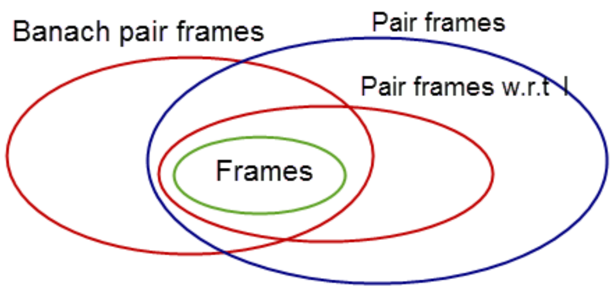

Figure 1: Inclusion relations associated to frames

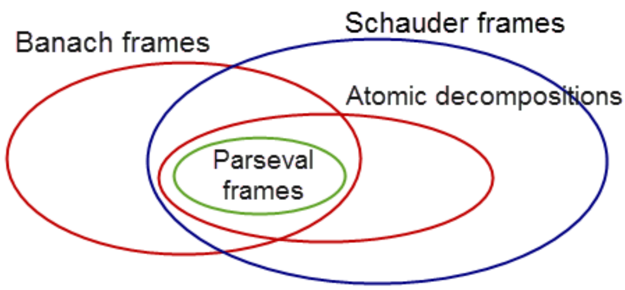

Figure 2: Inclusion relations associated to Parseval frames

We study some conditions under which two categories of pair frames and Banach pair frames coincide.

Proposition 3.14. *Suppose ℓ is a BK-space, $F = \{f_i\}_{i\in\mathbb{N}} \subset$ X and $G = \{g_i\}_{i\in\mathbb{N}} \subset$ X*. Also, assume that $T : \ell \to$ X is a bounded operator. Then*

1. *If ℓ is a Schauder sequence space with canonical basis $\{\delta_i\}_{i\in\mathbb{N}}$ and (G, T) is a Banach pair frame (Bessel) for* X *with respect to ℓ, then $(G, \{T(\delta_i)\}_{i\in\mathbb{N}})$ is a pair frame (Bessel) for* X *with respect to ℓ.*

2. *If (G, F) is a pair frame (Bessel) for* X *with respect to ℓ and U_G is onto or $F = \{f_i\}_{i\in\mathbb{N}} \subset$ X is a ℓ^*-Bessel for* X**, then (G, T_F) is a Banach pair frame (Bessel) for* X *with respect to ℓ.*

Proof. (1). For every $f \in$ X,

$$S_{TG}(f) = T\Big(\Big\{\Big\langle f, g_i\Big\rangle\Big\}_{i\in\mathbb{N}}\Big) = T\Big(\sum_{i\in\mathbb{N}}\langle f, g_i\rangle\delta_i\Big) = \sum_{i\in\mathbb{N}}\langle f, g_i\rangle T(\delta_i)$$
$$= S_{\{T(\delta_i)\}_i G}(f). \tag{3.9}$$

(2). Since U_G is onto, $\ell = \mathcal{R}(U_G) \subset \mathcal{D}(T_F)$. Hence, T_F is defined on the whole of ℓ. If $F = \{f_i\}_{i\in\mathbb{N}} \subset$ X is a ℓ^*-Bessel for X*, according to Theorem 3.7, there is an operator $T_F : \ell \to$ X such that $T_F(\{c_i\}_{i\in\mathbb{N}}) = \sum_{i\in\mathbb{N}} c_i f_i$ for all $\{c_i\}_{i\in\mathbb{N}} \in \ell$. So, $S_{FG} = S_{T_F G}$ and therefore (G, T_F) is a Banach pair frame for X with respect to ℓ. □

As a corollary, we can say that the notions of the atomic decomposition and Banach frame with respect to a BK-space ℓ can be regarded as the same concept.

Corollary 3.15. *Suppose ℓ is a BK-space. For $F = \{f_i\}_{i\in\mathbb{N}} \subset$ X, $G = \{g_i\}_{i\in\mathbb{N}} \subset$ X* and a bounded operator $T : \ell \to$ X,*

1. *If ℓ is a Schauder sequence space with canonical basis $\{\delta_i\}_{i\in\mathbb{N}}$ and (G, T) is a Banach frame for* X *with respect to ℓ, then $(G, \{T(\delta_i)\}_{i\in\mathbb{N}})$ is an atomic decomposition for* X *with respect to ℓ.*

2. *If (G, F) is an atomic decomposition for* X *with respect to ℓ and U_G is onto or $F = \{f_i\}_{i\in\mathbb{N}} \subset$ X is a ℓ^*-Bessel for* X**, then (G, T_F) is a Banach frame for* X *with respect to ℓ.*

Remark 3.16. An interpretation for Banach Pair Frames: From the practical point of view, when we construct the identity operator in the definition of Banach frames, atomic decompositions or Schauder frames, what we realy obtain is an approximation S of the identity operator I. With this observation, the frame operator S, in the definitions of pair frames and Banach pair frames can be regarded as a perturbation of the identity operator in the definitions of atomic decompositions and Banach frames (see also Example 3.9).

4. Frames and Pair Frames

In this section, we study some conditions under which a pair Bessel becomes a pair frame. These conditions are very similar to the inequalities in the frame definition. These results present close relations between the concept of (Banach) pair frames and frames. Some frame-like inequalities, (4.2)-(4.5), for (Banach) pair frames are presented. We need the next theorem.

Theorem 4.1. *[12, III.6.14] Let M be a linear subspace of X. M is dense in X if and only if for every bounded linear functional $g \in \mathrm{X}^*$ such that $g(M) = 0$ we have $g = 0$.*

The following lemma is proved in [12, III.12.Ex5].

Lemma 4.2. *An operator $V \in \mathcal{B}(\mathrm{X})$ is bounded below if and only if it is injective and has a closed range.*

Using above assertions we prove that:

Lemma 4.3. *Let $V \in \mathcal{B}(\mathrm{X})$. The following statements are equivalent:*

1. *V is invertible.*
2. *V and V^* are bounded below.*
3. *V and V^* are injective and have closed ranges.*

Proof. (1) $\Rightarrow$ (2). If V is invertible so is V^*. So, they are one to one and have closed ranges X and X^*. Hence, they are bounded below by Lemma 4.2.

(2) $\Leftrightarrow$ (3). Use Lemma 4.2.

(3) $\Rightarrow$ (1). Since V is one to one, it is enough to show that V is onto. First we show that $\mathcal{R}(V)$ is dense in X. If $\overline{\mathcal{R}(V)} \neq \mathrm{X}$, by putting $M = \mathcal{R}(V)$, Theorem 4.1 yields that there is a nonzero $g \in \mathrm{X}^*$ such that $\langle g, V(\mathrm{X})\rangle = \langle g, \mathcal{R}(V)\rangle \neq 0$ and so $\langle V^*(g), \mathrm{X}\rangle \neq 0$. If $V^*(g) \neq 0$, letting $M = \mathrm{X}$, Theorem 4.1 again implies that X is not dense in X; a contradiction. Therefore, $V^*(g) = 0$. Since V^* is one to one, then $g = 0$, which is a contradiction. So, $\overline{\mathcal{R}(V)} = \mathrm{X}$. Hence, from the fact that V has a closed range we get

$$\mathcal{R}(V) = \overline{\mathcal{R}(V)} = \mathrm{X}.$$

□

In the Hilbert space case, the proof of (3) $\Rightarrow$ (1) is strightforward, because

$$\mathcal{R}(V) = \overline{\mathcal{R}(V)} = \mathcal{N}(V^*)^{\perp} = \mathcal{H}.$$

Lemma 4.4. *Let $V \in \mathcal{B}(\mathrm{X})$. Suppose that there is a number $A > 0$ such that*

$$A \left|\langle f, g\rangle\right| \leqslant \left|\left\langle V(f), g\right\rangle\right|, \quad f \in \mathrm{X},\ g \in \mathrm{X}^*.$$

Then V is invertible.

Proof. For $f \in \mathrm{X}$,

$$\begin{aligned}\left\|V(f)\right\| &= \sup\left\{|\langle V(f), g\rangle| \;\middle|\; g \in \mathrm{X}^*, \|g\| = 1\right\} \\ &\geqslant A \sup\left\{|\langle f, g\rangle| \;\middle|\; g \in \mathrm{X}^*, \|g\| = 1\right\} = A \left\|f\right\|.\end{aligned}$$

Therefore, V is bounded below. Similarly, for $f \in \mathrm{X},\ g \in \mathrm{X}^*$,

$$A\left|\left\langle f, g\right\rangle\right| \leqslant \left|\left\langle f, V^*(g)\right\rangle\right|.$$

A conclusion like the above shows that V^* is bounded below. Then, Lemma 4.3 implies the invertibility of V. □

When the Banach space X is replaced by the Hilbert space $\mathcal{H}$, the result of Lemma 4.4 follows with a weaker condition.

Lemma 4.5. *Let $V \in \mathcal{B}(\mathcal{H})$. Suppose that there is a constant $A > 0$ such that*

$$A\left\|f\right\|^2 \leqslant \left|\left\langle V(f), f\right\rangle\right|, \quad f \in \mathcal{H},$$

Then V is invertible.

Proof. For $f \in \mathcal{H}$,

$$A\left\|f\right\|^2 \leqslant \left|\left\langle V(f), f\right\rangle\right| \leqslant \left\|V(f)\right\| \left\|f\right\|.$$

Hence

$$A\left\|f\right\| \leqslant \left\|V(f)\right\|.$$

So, V is bounded below. Similarly, we can show that V^* is bounded below. Lemma 4.3 implies the invertibility of V. □

Now, we apply the propositions above to obtain new conditions that turn Bessels into frames.

Proposition 4.6. *Let ℓ be a BK-space and $G = \{g_i\}_{i\in\mathbb{N}} \subset \mathrm{X}^*$ be a ℓ-Bessel for X. Assume that for a bounded operator $T : \ell \to \mathrm{X}$, (G, T) is a Banach pair Bessel for X with respect to ℓ and there is a constant $A > 0$ such that*

$$A\left|\left\langle f, g\right\rangle\right| \leqslant \left|\left\langle T\left(\{\langle f, g_i\rangle\}_{i\in\mathbb{N}}\right), g\right\rangle\right|, \quad f \in \mathrm{X},\ g \in \mathrm{X}^*.$$

Then (G, T) is a Banach pair frame for X with respect to ℓ and there is a positive number $B > 0$ such that

$$A\left|\left\langle f, g\right\rangle\right| \leqslant \left|\left\langle\left(\{\langle f, g_i\rangle\}_{i\in\mathbb{N}}\right), g\right\rangle\right| \leqslant B\left\|f\right\| \left\|g\right\|, \quad f \in \mathrm{X},\ g \in \mathrm{X}^*. \tag{4.1}$$

Proof. Lemma 4.4 implies the invertibility of S. Therefore, (G, T) is a Banach pair frame. Since

$$\left|\left\langle T\left(\{\langle f, g_i\rangle\}_{i\in\mathbb{N}}\right), g\right\rangle\right| \leqslant \left\|S\right\| \left\|f\right\| \left\|g\right\|,$$

the latter assertion follows by putting $B = \|S\|$. □

For the case of Hilbert spaces, Lemma 4.5 helps us obtain the result of Theorem 4.6 with a weaker condition.

Proposition 4.7. *Let ℓ be a BK-space and $G = \{g_i\}_{i\in\mathbb{N}} \subset \mathcal{H}$ be a ℓ-Bessel for $\mathcal{H}$. Assume that for a bounded operator $T : \ell \to \mathcal{H}$, (G, T) is a Banach pair Bessel for $\mathcal{H}$ with respect to ℓ and there is a positive constant A such that*

$$A\left\|f\right\|^2 \leqslant \left|\left\langle T\left(\{\langle f, g_i\rangle\}_{i\in\mathbb{N}}\right), f\right\rangle\right|, \quad f \in \mathcal{H}.$$

Then (G, T) is a Banach pair frame for $\mathcal{H}$ with respect to ℓ and there is a positive constant B such that

$$A\left\|f\right\|^2 \leqslant \left|\left\langle T\left(\{\langle f, g_i\rangle\}_{i\in\mathbb{N}}\right), f\right\rangle\right| \leqslant B\left\|f\right\|^2, \quad f \in \mathcal{H}. \tag{4.2}$$

Proof. Lemma 4.5 implies the invertibility of S. The remainder is similar to the proof of Proposition 4.6. □

Proposition 4.8. *Let $F = \{f_i\}_{i\in\mathbb{N}} \subset \mathrm{X}$ and $G = \{g_i\}_{i\in\mathbb{N}} \subset \mathrm{X}^*$. Suppose that (G, F) is a pair Bessel for X. and there exists a constant $A > 0$ such that*

$$A\Big|\langle f, g\rangle\Big| \leqslant \Big|\sum_{i\in\mathbb{N}}\langle f, g_i\rangle\langle f_i, g\rangle\Big|, \quad f \in \mathrm{X},\ g \in \mathrm{X}^*,$$

then (G, F) is a pair frame for X and also there is some positive constant $B > 0$ such that

$$A\Big|\langle f, g\rangle\Big| \leqslant \Big|\sum_{i\in\mathbb{N}}\langle f, g_i\rangle\langle f_i, g\rangle\Big| \leqslant B\,\Big\|f\Big\|\,\Big\|g\Big\|, \quad f \in \mathrm{X},\ g \in \mathrm{X}^*. \tag{4.3}$$

Proof. For $f \in \mathrm{X}$ and $g \in \mathrm{X}^*$,

$$A\Big|\langle f, g\rangle\Big| \leqslant \Big|\Big\langle S_{FG}(f), g\Big\rangle\Big|.$$

Now, use Lemma 4.4. □

The following proposition is another version of Proposition 4.8 for Hilbert spaces.

Proposition 4.9. *Let $F = \{f_i\}_{i\in\mathbb{N}}, G = \{g_i\}_{i\in\mathbb{N}} \subset \mathcal{H}$. Suppose that (G, F) is a pair Bessel for $\mathcal{H}$. If there exists a constant $A > 0$ such that*

$$A\,\Big\|f\Big\|^2 \leqslant \Big|\sum_{i\in\mathbb{N}}\langle f, g_i\rangle\langle f_i, f\rangle\Big|, \quad f \in \mathcal{H},$$

then (G, F) is a pair frame for $\mathcal{H}$. In this case we obtain a frame-like inequality, i.e., there is a constant $B > 0$ such that

$$A\,\Big\|f\Big\|^2 \leqslant \Big|\sum_{i\in\mathbb{N}}\langle f, g_i\rangle\langle f_i, f\rangle\Big| \leqslant B\,\Big\|f\Big\|^2, \quad f \in \mathcal{H}. \tag{4.4}$$

Remark 4.10. The frame inequality (2.1) can be rewritten in the form

$$A\,\Big\|f\Big\|^2 \leqslant \Big|\sum_{i\in\mathbb{N}}\langle f, f_i\rangle\langle f_i, f\rangle\Big| \leqslant B\,\Big\|f\Big\|^2, \quad f \in \mathcal{H}. \tag{4.5}$$

By this, it can be seen that the inequalities (4.3) and (4.4) are similar to (4.5). Furthermore, let $\mathrm{X} = \mathcal{H}$ and instead of using the *pairs* F,G put $F = G$ and $f = g$, the relations (4.3) and (4.4) coincide with the frame inequalities.

5. Some Characterizations of Pair Frames in Banach Spaces

In this section, we extend the results of [6] to Banach pair frames. Given $G = \{g_i\}_{i\in\mathbb{N}} \subset \mathrm{X}^*$, by saying "*G participates in the construction of a pair frame for* X" we mean that there exists a set $F = \{f_i\}_{i\in\mathbb{N}} \subset \mathrm{X}$ such that (G, F) is a pair frame X. In a similar way, we say that "*G participates in the construction of a Banach pair frame for* X" if there is a bounded operator $T : \ell \to \mathrm{X}$ such that (G, T) is a Banach pair frame for X.

Theorem 5.1. *Let $G = \{g_i\}_{i\in\mathbb{N}} \subset \mathrm{X}^*$. If $G = \{g_i\}_{i\in\mathbb{N}}$ participates in the construction of a (Banach) pair frame for* X*, then G is total on* X*.*

Proof. Suppose that there is $F = \{f_i\}_{i\in\mathbb{N}} \subset \mathrm{X}$ (bounded operator $T : \ell \to \mathrm{X}$) so that (G, F) is a pair frame ((G, T) is a Banach pair frame) for X. Therefore, the operator $S = T_F U_G$ ($S = TU_G$) is invertible on X. If there is some nonzero $f \in \mathrm{X}$ such that $\{\langle f, g_i\rangle\}_{i\in\mathbb{N}} = \{0\}$, then

$$S(f) = \sum_{i\in\mathbb{N}} \langle f, g_i\rangle f_i = 0 \quad \Big(S(f) = T\big(\{\langle f, g_i\rangle\}_{i\in\mathbb{N}}\big) = 0\Big).$$

Thus $\mathcal{N}(S) \neq \{0\}$, which contradicts the invertibility of S. Hence, $f = 0$ and therefore G is total on X. □

The necessity part of the next proposition is stated in [6] for Banach frames. We give necessary and sufficient conditions for Banach pair frames.

Theorem 5.2. *The sequence $G = \{g_i\}_{i\in\mathbb{N}} \subset \mathrm{X}^*$ participates in the construction of a Banach pair frame for* X *if and only if G is total on* X*.*

Proof. The sufficient condition is established in the previous proposition. For the converse, suppose that G is total on X. Define

$$\ell_G = \Big\{\big\{\langle f, g_i\rangle\big\}_{i\in\mathbb{N}} \;\Big|\; f \in \mathrm{X}\Big\},$$

and put the following norm on ℓ_G,

$$\Big\|\big\{\langle f, g_i\rangle\big\}_{i\in\mathbb{N}}\Big\| = \big\|f\big\|, \; f \in \mathrm{X}.$$

So X is isomorphic to ℓ_G via the operator

$$U_G : \mathrm{X} \to \ell_G, \quad U_G(f) = \big\{\langle f, g_i\rangle\big\}_{i\in\mathbb{N}},$$

Hence, ℓ_G is a Banach space and $\ell_G = \mathcal{R}(U_G)$. Therefore, $(G, {U_G}^{-1})$ is a Banach frame for X with respect to ℓ_G. □

In what follows, we need the next lemma.

Lemma 5.3. *Let ℓ be a Banach scalar sequence space and ℓ' be a closed subspace of ℓ. Then the following statments are equivalent:*

1. ℓ' is complemented in ℓ.

2. There exists a projection P from ℓ onto ℓ'.

3. Every bounded operator V' on ℓ' can be extended to a bounded operator V on the whole of ℓ.

If the statements above hold and P is as in (2), then $\mathcal{R}(P) = \ell'$ and

$$\ell = \ell' \oplus \mathcal{N}(P).$$

Proof. (1) $\Rightarrow$ (2). Assume that ℓ' is complemented in ℓ by a closed subspace ℓ'', i.e., $\ell = \ell' \oplus \ell''$. Define the operator $P : \ell \to \ell'$ such that $P(k) = k'$, where $k = k' + k''$ with $k' \in \ell', k'' \in \ell''$. Hence, P is a projection from ℓ onto ℓ'.

(2) $\Rightarrow$ (1). Suppose that there is a projection P from ℓ onto ℓ'. Since $I = P \oplus (I - P)$, then $\ell = \mathcal{R}(P) \oplus \mathcal{R}(I - P)$. On the other hand, $\mathcal{R}(I - P) = \mathcal{N}(P)$ and $\mathcal{R}(P) = \ell'$. Therefore,

$$\ell = \ell' \oplus \mathcal{N}(P).$$

(2) $\Rightarrow$ (3). Suppose that Y is a Banach space and $V' : \ell' \to \mathrm{Y}$ is a bounded operator. Put

$$V : \ell \to \mathrm{Y}, \quad V := V'P.$$

Then V is a bounded extension of V', $V|_{\ell'} = V'$.

(3) $\Rightarrow$ (2). Consider the identity operator $I' : \ell' \to \ell'$. By (3), since I' is a bounded operator on ℓ', I' can be extended to a bounded operator $P : \ell \to \ell'$ such that $P|_{\ell'} = I'$. Since $P(\{c_i\}_{i\in\mathbb{N}}) \in \ell'$ for $\{c_i\}_{i\in\mathbb{N}} \in \ell$, then

$$P^2\Big(\{c_i\}_{i\in\mathbb{N}}\Big) = P\Big(P(\{c_i\}_{i\in\mathbb{N}})\Big) = P|_{\ell'}\Big(P(\{c_i\}_{i\in\mathbb{N}})\Big) = I'\Big(P(\{c_i\}_{i\in\mathbb{N}})\Big) = P\Big(\{c_i\}_{i\in\mathbb{N}}\Big).$$

Hence, $P^2 = P$ and $\mathcal{R}(P) = \ell'$. □

We consider some conditions under which $G = \{g_i\}_{i\in\mathbb{N}} \subset \mathrm{X}^*$ participates in the construction of a Banach pair frame with respect to a given BK-space ℓ.

Theorem 5.4. *Let ℓ be a BK-space and $G = \{g_i\}_{i\in\mathbb{N}} \subset \mathrm{X}$ be a ℓ-Bessel for X. There exists a bounded operator $T : \ell \to \mathrm{X}$ so that (G, T) is a Banach pair frame for X with respect to ℓ if and only if $G = \{g_i\}_{i\in\mathbb{N}}$ is an ℓ-frame for X and $\mathcal{R}(U_G)$ is closed and complemented in ℓ. In this situation, U_G is bounded below.*

Proof. Assume that (G, T) is a Banach pair frame for X with respect to ℓ. Then $S = TU_G$ is invertible and for every $f \in \mathrm{X}$ we have

$$\Big\|f\Big\| = \Big\|S^{-1}S(f)\Big\| \leqslant \Big\|S^{-1}\Big\|\,\Big\|T\Big\|\,\Big\|U_G(f)\Big\|.$$

Namely, $U_G : \mathrm{X} \to \ell$ is bounded below with the lower bound $\frac{1}{\|S^{-1}\|\|T\|}$ and hence $\mathcal{R}(U_G)$ is closed in ℓ. In other words, G is an ℓ-frame for X. Define

$$P = U_G S^{-1} T : \ell \to \mathcal{R}(U_G).$$

Then

$$P^2 = U_G S^{-1} T U_G S^{-1} T = U_G S^{-1} T = P.$$

This shows that P is a projection from ℓ onto $\mathcal{R}(U_G)$. Lemma 5.3 implies that for the projection P, $\ell = \mathcal{R}(P) \oplus \mathcal{N}(P)$. Hence,

$$\ell = \mathcal{R}(U_G) \oplus \mathcal{N}(P).$$

For the converse, suppose that $G = \{g_i\}_{i\in\mathbb{N}}$ is an ℓ-frame for X. Then U_G is bounded below and therefore $\mathcal{R}(U_G)$ is closed. By the operator U_G, X is isomorphic to $\mathcal{R}(U_G)$. Consequently, $U_G^{-1} : \mathcal{R}(U_G) \to \mathrm{X}$ is bounded. By the assumption, $\mathcal{R}(U_G)$ is complemented in ℓ. Lemma 5.3 implies that U_G^{-1} can be extended to a bounded operator $T : \ell \to \mathrm{X}$. Now, for every $f \in \mathrm{X}$,

$$S(f) = TU_G(f) = U_G^{-1} U_G(f) = f = I(f).$$

So (G, T) is a Banach frame. □

Corollary 5.5. *The sequence $G = \{g_i\}_{i\in\mathbb{N}} \subset \mathrm{X}^*$ participates in the construction of a Banach pair frame for* X *with respect to ℓ^2 if and only if G is a 2-frame for* X.

Proof. Every closed subspace of ℓ^2 is complemented in ℓ^2. Combining this fact with Theorem 5.4, gives the result. □

By ignoring the direct presence of $G = \{g_i\}_{i\in\mathbb{N}}$ in Theorem 5.4, we obtain a slightly more generalized version of that theorem.

Theorem 5.6. *Let ℓ be a BK-space. The Banach space* X *has a Banach pair frame with respect to ℓ if and only if* X *is isomorphic to a closed and and complemented subspace of ℓ.*

Proof. Suppose that there is a ℓ-Bessel $G = \{g_i\}_{i\in\mathbb{N}} \subset \mathrm{X}^*$ and a bounded operator $T : \ell \to \mathrm{X}$ such that (G, T) is a Banach pair frame. Theorem 5.4 yields that X is isomorphic to some closed complemented subspace of ℓ.

Conversely, assume that there is a closed subspace ℓ' of ℓ which is complemented in ℓ and an operator $V : \mathrm{X} \to \ell'$ is the associated isomorphism. By Lemma 5.3, there exists a projection P from ℓ onto ℓ'. If $\{\eta_i\}_{i\in\mathbb{N}}$ is the set of coordinate functionals of ℓ, it also is the coordinate functionals of ℓ'. For every $i \in \mathbb{N}$ put $g_i = V^*\eta_i$. Then for $f \in \mathrm{X}$,

$$\left\{\langle f, g_i\rangle\right\}_{i\in\mathbb{N}} = \left\{\langle f, V^*\eta_i\rangle\right\}_{i\in\mathbb{N}} = \left\{\langle V(f), \eta_i\rangle\right\}_{i\in\mathbb{N}} = V(f) \in \ell,$$

and

$$\left\|\{\langle f, g_i\rangle\}_{i\in\mathbb{N}}\right\| = \left\|V(f)\right\| \leqslant \left\|V\right\| \left\|f\right\|.$$

Hence, G is a ℓ-Bessel for X with bound $\|V\|$. By letting

$$T := V^{-1}P : \ell \to \mathrm{X},$$

we get

$$S(f) := TV(f) = V^{-1}PV(f) = V^{-1}V(f) = f = I(f).$$

for $f \in \mathrm{X}$. So, (G, T) is a Banach pair frame. □

We study some conditions under which these two concepts coincide. Before, we need the following lemma.

Lemma 5.7. *[6] For $F = \{f_i\}_{i\in\mathbb{N}} \subset \mathrm{X}\backslash\{0\}$, the space*

$$\ell_F := \mathcal{D}(T_F) = \Big\{\{c_i\}_{i\in\mathbb{N}} \subset \mathbb{C} \,\Big|\, \sum_{i\in\mathbb{N}} c_i f_i \quad converges\Big\},$$

with the norm

$$\Big\|\{c_i\}_{i\in\mathbb{N}}\Big\|_{\ell_F} = \sup_{n\in\mathbb{N}} \Big\|\sum_{i=1}^{n} c_i f_i\Big\|, \tag{5.1}$$

is a Schauder sequence space. With this norm $T_F : \ell_F \to \mathrm{X}$ is a bounded operator.

For $G = \{g_i\}_{i\in\mathbb{N}} \subset \mathrm{X}^*$, if we were allowed to choose the scalar sequence space ℓ by ourselves, the concepts of pair frames, pair frames with respect to ℓ and Banach pair frame with respect to ℓ coincide.

Proposition 5.8. *Let $G = \{g_i\}_{i\in\mathbb{N}} \subset \mathrm{X}^*$. The following statements are equivalent:*

1. *There is a sequence $F = \{f_i\}_{i\in\mathbb{N}} \subset \mathrm{X}$ such that (G, F) is a pair frame (Bessel) for X.*
2. *There is a sequence $F = \{f_i\}_{i\in\mathbb{N}} \subset \mathrm{X}$ and a Schauder sequence space ℓ, such that (G, F) is a pair frame (Bessel) for X with respect to ℓ.*
3. *There is a Schauder sequence space ℓ and a bounded operator $T : \ell \to \mathrm{X}$ such that (G, T) is a Banach pair frame (Bessel) for X with respect to ℓ.*

If the above statements hold, then $f_i = T(\delta_i)$ for all $i \in \mathbb{N}$, so $T = T_F$. Moreover, G is a ℓ-Bessel for X with the bound $\sup_{n\in\mathbb{N}} \|S_n\|$, where $S_n(f) = \sum_{i=1}^{n}\langle f, g_i\rangle f_i$ for every $n \in \mathbb{N}$ and $f \in \mathrm{X}$.

In the frame cases, G is a ℓ-frame for X with the frame bounds $\left\|S^{-1}\right\|^{-1}$ and $\sup_{n\in\mathbb{N}} \|S_n\|$, where S is the frame operator.

Proof. The theorem is proved in the frame case; the Bessel case is an implicit result.
(1) ⇒ (2). Assume that (G, F) is a pair frame for X. By letting $S_n(f) = \sum_{i=1}^{n}\langle f, g_i\rangle f_i$ for every $n \in \mathbb{N}$ and $f \in \mathrm{X}$, the principle of uniform boundedness implies that $S(\cdot) = \sum_{i\in\mathbb{N}}\langle \cdot, g_i\rangle f_i = \lim_{n\to\infty} S_n(\cdot)$ is bounded and $\|S\| \leqslant \sup_{n\in\mathbb{N}} \|S_n\| < \infty$. Moreover,

$$\Big\|\langle f, g_i\rangle\Big\|_{\ell_F} = \sup_{n\in\mathbb{N}} \Big\|\sum_{i=1}^{n}\langle f, g_i\rangle f_i\Big\| = \sup_{n} \Big\|S_n(f)\Big\| \leqslant \Big(\sup_{n\in\mathbb{N}} \|S_n\|\Big)\Big\|f\Big\| \tag{5.2}$$

for $f \in \mathrm{X}$. This shows that G is a ℓ_F-Bessel for X with the bound $\sup_{n\in\mathbb{N}} \|S_n\|$. Since ℓ_F is a BK-space, (G, F) is a pair frame for X with respect to ℓ_F.
(2) $\Rightarrow$ (1). It is obvious.
(2) $\Rightarrow$ (3). Let $T = T_F$. Then $S_{T_F G} = S_{FG}$.
(3) $\Rightarrow$ (2). Proposition 3.14 (1).
After proving the equivalences of assertions (1)-(3), we prove the other statements. Since S is invertible,

for $f \in \mathrm{X}$,

$$\begin{aligned}\left\|f\right\| &= \left\|S^{-1}S(f)\right\| = \left\|S^{-1}\right\| \left\|S(f)\right\| = \left\|S^{-1}\right\| \left\|\sum_{i=1}^{\infty}\langle f, g_i\rangle f_i\right\| \\ &= \left\|S^{-1}\right\| \left\|\lim_{n\to\infty}\sum_{i=1}^{n}\langle f, g_i\rangle f_i\right\| \leqslant \left\|S^{-1}\right\| \sup_{n\in\mathbb{N}}\left\|\sum_{i=1}^{n}\langle f, g_i\rangle f_i\right\| \\ &= \left\|S^{-1}\right\| \left\|\{\langle f, g_i\rangle\}_{i\in\mathbb{N}}\right\|_{\ell_F}.\end{aligned} \tag{5.3}$$

Relations (5.2) and (5.3) imply that $\{f_i\}_{i\in\mathbb{N}}$ is a ℓ_F-frame for X with bounds $\|S^{-1}\|^{-1}$ and $\sup_{n\in\mathbb{N}} \|S_n\|$. Other assertions are proved implicitly. □

Similarly, for $G = \{g_i\}_{i\in\mathbb{N}} \subset \mathrm{X}^*$, if we can choose a scalar sequence space ℓ, then the concepts of Schauder frame, atomic decomposition and Banach frame with respect to ℓ are the same.

Corollary 5.9. *[6] Let $G = \{g_i\}_{i\in\mathbb{N}} \subset \mathrm{X}^*$. The following statments are equivalent:*

1. *There is a sequence $F = \{f_i\}_{i\in\mathbb{N}} \subset \mathrm{X}$ such that (G, F) is a Schauder frame for X.*

2. *There is a sequence $F = \{f_i\}_{i\in\mathbb{N}} \subset \mathrm{X}$ and a Schauder sequence space ℓ_F, such that (G, F) is an atomic decomposition for X with respect to ℓ.*

3. *There is a Schauder sequence space ℓ and a bounded operator $T : \ell \to \mathrm{X}$ such that (G, T) is a Banach frame for X with respect to ℓ.*

In this case, G is a ℓ-frame for X with the frame bounds 1 and $\sup_{n\in\mathbb{N}} \|S_n\|$.

Corollary 5.10. *Let $G = \{g_i\}_{i\in\mathbb{N}} \subset \mathrm{X}^*$ be given. If G participates in the construction of a pair frame for X, then there is a Schauder sequence space ℓ such that G is an ℓ-frame for X.*

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