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Automatic continuity on continuous inverse algebras

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Abstract

In this paper, we first investigate the continuity of the spectral radius functions on continuous inverse algebras. Then we support our results by some examples. Finally, we continue our investigation by further determining the automatic continuity of linear mappings and homomorphisms on these algebras.

1. Introduction

Newburgh [13] introduced the concept of spectral continuity and proved that the spectrum function is upper semi-continuous on any Banach algebra. He gave a first sufficient condition for

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continuity of the spectrum function at a point of a Banach algebra. Since then, this topic has been studied widely by many researchers and mathematicians. The most outstanding results in this direction are due to Aupetit, and Burlando who have generalized the results of Newburgh in certain Banach algebras (see [1, 4]).
Continuity of the spectrum and spectral radius functions play a crucial role in automatic continuity. Automatic continuity of linear mappings and homomorphisms are very important in advanced studies on topological algebras and mathematical analysis. The starting point for automatic continuity theory is the easily proved fact that every homomorphism from a Banach algebra onto the complex field is automaticly continuous [1, 14]. It follows easily from the continuity of multiplicative linear functionals that every surjective homomorphism from a Banach algebra onto a commutative semisimple Banach algebra is continuous. A famous theorem due to Johnson [9] extends this result to arbitrary semisimple Banach algebras. Some results for automatic continuity in the area of Banach and Frechet algebras have also been obtained by Aupetit [1], and Husain [8]. For further information about the automatic continuity one can refer to [5, 6, 12].
An important class of topological algebras namely continuous inverse algebras have interesting properties regarding the automatic continuity problems. Hence we can extend and prove some automatic continuity results of Banach algebras to continuous inverse algebras.
Biller [3] showed that every multiplicative linear functional on a continuous inverse algebra is continuous, which leads easily to the continuity of all homomorphisms from a continuous inverse F-algebra into a semisimple commutative continuous inverse F-algebra, but it remains an intriguing open question, commonly known as Michael's problem, whether all multiplicative linear functionals on Frechet algebras are continuous.
In this paper, we first investigate the continuity of the spectral radius functions on continuous inverse algebras. Then several examples of spectral continuity are discussed as well. Finally, the automatic continuity of linear mappings and homomorphisms are studied on these algebras.
Throughout this paper, all algebras will be assumed unital and the units will be denoted by e.

2. Definitions and known results

In this section, we present a collection of definitions and known results, which are included in the list of our references.

Definition 2.1. Let A and B be two F-spaces and let $T : A \to B$ be a linear mapping. The separating space of T is defined by

$$G(T) = \{y \in B : \text{ there exists } (x_n)_n \text{ in } A \text{ s.t. } x_n \to 0 \text{ and } Tx_n \to y\}.$$

The separating space $G(T)$ is a closed linear subspace of B; moreover, by the Closed Graph Theorem, T is continuous if and only if $G(T) = \{0\}$ [5, 5.1.2].

Definition 2.2. Let A be an algebra. The set of all invertible elements of A is denoted by $Inv(A)$. A topological algebra A is called Q-algebra if $Inv(A)$ is open.

Definition 2.3. For an algebra A, the spectrum $sp_A(x)$ of an element $x \in A$ is the set of all $\lambda \in \mathbb{C}$ such that $\lambda e - x$ is not invertible in A. The spectral radius $r_A(x)$ of an element $x \in A$ is defined by $r_A(x) = \sup\{|\lambda| : \lambda \in sp_A(x)\}$.

Definition 2.4. [3, 2.1] Let A be an algebra.

(i) Define the Gelfand spectrum of A as $\Gamma_A := Hom(A, \mathbb{C})$ with the topology of pointwise convergence on A. Note that $0 \notin \Gamma_A$ because we require homomorphisms to preserve the unit elements.

(ii) Each element $a \in A$ gives rise to a continuous function $\hat{a} : \Gamma_A \to \mathbb{C}$ defined by $\hat{a}(\varphi) := \varphi(a)$. The function $\hat{a}$ is called the Gelfand transform of a. The map $a \to \hat{a} : A \to C(\Gamma_A)$, which is a homomorphism of unital algebras, is called the Gelfand homomorphism of the algebra A.

Definition 2.5. [8, p. 2] A complete metrizable algebra is called an F-algebra.

Definition 2.6. [3, p. 1034] A locally convex algebra is an algebra A with a locally convex Hausdorff vector space topology such that the algebra multiplication is separately continuous.

Definition 2.7. [3, p. 1036] A locally convex algebra in which the topology can be described by a family of sub-multiplicative seminorms is called locally multiplicatively convex, or locally m-convex for short.

Definition 2.8. [8, p. 3] A complete metrizable locally m-convex algebra is called a Frechet algebra.

Definition 2.9. [3, 1.1] A continuous inverse algebra is a locally convex algebra in which the set of invertible elements is a neighbourhood of e and inversion is continuous at e.

Proposition 2.10. *[10, II. Proposition 4.1] Let A be a continuous inverse algebra. Then $Inv(A)$ is an open subset of A, and inversion is a continuous map from $Inv(A)$ into itself.*

Proposition 2.11. *[3, 1.5] Let A be a continuous inverse algebra. Then every element of A has non-empty compact spectrum.*

Proposition 2.12. *Let A be a commutative continuous inverse algebra. Then every maximal ideal of A is closed and it is the kernel of some continuous character (non-zero multiplicative linear functional) of A.*

Proof. Since $Inv(A)$ is open, every maximal ideal of A is closed. Hence the result follows from [10, II. Corollary 7.2]. □

In the sequel, Γ_A denotes the space of continuous characters of A.

3. Main Results

To prove some of the results, we need the following lemma.

Lemma 3.1. *Let A be a commutative continuous inverse algebra. Then the following assertions hold:*
(i) Every $a \in A$ satisfies

$$sp_A(a) = \{\varphi(a) : \varphi \in \Gamma_A\} = \hat{a}(\Gamma_A);$$

(ii) If $x, y \in A$, then

$$r_A(x + y) \leq r_A(x) + r_A(y), \quad r_A(xy) \leq r_A(x) r_A(y);$$

(iii) $\mathrm{rad}(A) = \bigcap_{\varphi \in \Gamma_A} \ker \varphi$, *where* $\mathrm{rad}(A)$ *is the Jacobson radical of A.*

The algebra A is called semisimple if $\mathrm{rad}(A) = \{0\}$.

Proof. (i) Let $\lambda \in sp_A(a)$, so that $a - \lambda e$ is not invertible. Since $(a - \lambda e)A$ is a proper ideal in A, by Zorn's lemma it is contained in a maximal ideal. So by Proposition 2.12, there exists $\varphi \in \Gamma_A$ with $\varphi((a - \lambda e)x) = 0$, for all $x \in A$. Hence $\hat{a}(\varphi) = \varphi(a) = \lambda$. Conversely, $\varphi(a)$ is an element of $sp_A(a)$ for each φ in Γ_A.
(ii) By the first part, we have

$$sp_A(x + y) = \mathrm{Im}(x + y)^\wedge = \mathrm{Im}(\hat{x} + \hat{y}) \subseteq \mathrm{Im}\,\hat{x} + \mathrm{Im}\,\hat{y} = sp_A(x) + sp_A(y),$$

$$sp_A(xy) = \mathrm{Im}(xy)^\wedge = \mathrm{Im}(\hat{x}\hat{y}) \subseteq \mathrm{Im}\,\hat{x}\,\mathrm{Im}\,\hat{y} = sp_A(x) sp_A(y),$$

where $\mathrm{Im}(\hat{x})$ is the range of $\hat{x}$. The analogous results hold for the spectral radius.
(iii) The kernel of the Gelfand transform consists of those $a \in A$ which satisfy $\hat{a}(\varphi) = \varphi(a) = 0$ for every $\varphi \in \Gamma_A$. So by Proposition 2.12, $\mathrm{rad}(A)$, which is the intersection of all maximal ideals of A, is equal to $\bigcap_{\varphi \in \Gamma_A} \ker \varphi$. □

Now, we first present the following theorem which plays a crucial role in this section. Then we support it by some examples.

Theorem 3.2. *Let A be a continuous inverse algebra. Then r_A is continuous at zero. Moreover, it is continuous on A if A is commutative.*

Proof. Since A is a Q-algebra by Proposition 2.10, the continuity of r_A at zero follows from [6, Theorem 9]. Let A be commutative. By Lemma 3.1, we have

$$r_A(x + y) \leq r_A(x) + r_A(y), \quad \text{for all } x, y \in A,$$

so,

$$|r_A(x) - r_A(y)| \leq r_A(x - y).$$

From this and the continuity of r_A at zero, we conclude that the spectral radius function is continuous on A. □

The following example shows that the converse of Theorem 3.2 may be false in general.

Example 3.3. Let l_p, $0 < p < 1$ be the algebra of numerical sequences $x = (\eta_i)_{i=1}^{\infty}$ for which

$$\|x\|_p = \sum_{i=1}^{\infty} |\eta_i|^p < \infty.$$

For sequences $(\eta_i)_{i=1}^{\infty}$ and $(\xi_i)_{i=1}^{\infty}$ in l_p, we define their convolution as

$$(\eta_i)_{i=1}^{\infty} * (\xi_i)_{i=1}^{\infty} = (\sum_{k=1}^{i} \eta_k \xi_{i-k})_{i=1}^{\infty}.$$

Hence, l_p with the multiplication defined as convolution is a complete p-normed algebra [2, 3.4.7] . By [2, 3.6.23(b)], it is also a Q-algebra. Since $\|x\|_p$ is a complete p-norm topology, l_p is complete with the topology given by the metric
$d(x, y) = \|x - y\|_p$, and so it is an F-algebra. By [6, Theorem 9], the spectral radius function is continuous at zero. Also the inversion is continuous for l_p [5, 2.2.39]. Now, we show that l_p with $0 < p < 1$ is not locally convex. If it would be locally convex, then the unit ball $B_1(0)$ would contain a convex neighbourhood U of 0. Then there must be $\delta > 0$ with $B_{2\delta}(0) \subset U$, hence also $\mathrm{conv}(B_{2\delta}(0)) \subset U \subset B_1(0)$, where $\mathrm{conv}(B_{2\delta}(0))$ is the convex hull of the ball $B_{2\delta}(0)$. Let $e_i := (\underbrace{0, ..., 0, 1}_{i}, 0, 0, ...)$. Then $\delta^{\frac{1}{p}} e_i \in B_{2\delta}(0)$. We get

$$\sum_{i=1}^{n} \frac{1}{n} \delta^{\frac{1}{p}} e_i \in \mathrm{conv}(B_{2\delta}(0)) \subset B_1(0), \quad \text{for all } n \in \mathbb{N}.$$

This implies that

$$1 > \|\sum_{i=1}^{n} \frac{1}{n} \delta^{\frac{1}{p}} e_i\|_p^p = \delta n^{1-p}, \quad \text{for all } n \in \mathbb{N},$$

which is a contradiction because $1 - p > 0$. Thus, l_p is not locally convex and hence not a continuous inverse algebra.

Example 3.4. Let A and B be two continuous inverse algebras. Then $A \times B$ is a continuous inverse algebra. This is due to the fact that basic neighbourhoods are given by $U_A \times U_B$, where U_A and U_B are open. For given two points $x \in A$ and $y \in B$, we may choose convex neighbourhoods $V_A \subseteq U_A$ and $V_B \subseteq U_B$ containing x and y respectively for the nontrivial components, and then $V_A \times V_B$ containing (x, y) is a convex neighbourhood. Since A and B are locally convex algebras, so $A \times B$ with product topology and pointwise algebraic operations is a locally convex algebra.
Let i_A and i_B be inversions in A and B respectively. Then

$$i_{A\times B} : Inv(A \times B) \to Inv(A \times B) \ by$$

$$(x, y) \mapsto (x, y)^{-1} = (x^{-1}, y^{-1}),$$

is also an inversion. Since i_A and i_B are continuous at e_A and e_B, so $i_{A\times B}$ is continuous at (e_A, e_B). Since $Inv(A)$ and $Inv(B)$ are open, from general topology we get

$$Inv(A \times B) = Inv(A) \times Inv(B).$$

By the definition of continuous inverse algebras, $Inv(A)$ and $Inv(B)$ are neighbourhoods of e_A and e_B, and so $Inv(A \times B)$ is a neighbourhood of (e_A, e_B) as well. Thus, $A \times B$ is a continuous inverse algebra. By Theorem 3.2, $r_{A\times B}$ is continuous at zero. Moreover, if A and B are commutative, then $r_{A\times B}$ is continuous on $A \times B$.

Example 3.5. The algebra $C(\mathbb{R})$ of all continuous complex-valued functions on the real line $\mathbb{R}$ with the sequence $\{p_n\}$ of seminorms defined by $p_n(f) = \sup_{|x|\leq n} |f(x)|$ is a Frechet algebra which is not a Q-algebra, (see Example 2.2.46 (iii) of [5]). So the spectral radius function is not continuous at zero. In general, this example shows that the spectral radius function may be discontinuous at zero. Hence, $C(\mathbb{R})$ is not a continuous inverse algebra.

Note that every Banach algebra is a continuous inverse algebra but the converse does not hold in general, as the following example shows:

Example 3.6. [11, 3.6] Let A be the algebra of all formal power series $x = \sum_{k=1}^{\infty} \xi_k(x)t^k$ with the topology of pointwise convergence of the coefficients $\xi_k(x)$ and with the Cauchy multiplication of power series. It is a commutative locally m- convex Frechet algebra with the seminorms

$$||x||_k = \sum_{i=0}^{k-1} |\xi_i(x)|.$$

Since the inversion is continuous and $Inv(A)$ is open, A is a continuous inverse algebra. It has no topological zero divisor and hence it is not equal to the field of complex numbers. This example shows that A is not a Banach algebra because the Gelfand-Mazur theorem [11, 3.5] does not hold for A.

In the sequel, we continue our investigation by further determining the automatic continuity of linear mappings and homomorphisms on continuous inverse algebras.

Theorem 3.7. *Let A and B be continuous inverse F-algebras such that B is semisimple and commutative. If $T : A \to B$ is a linear mapping satisfying*

$$r_B(Tx) \leq r_A(x), \quad \text{for all } x \in A,$$

then T is continuous.

Proof. Let $x_n \to 0$ in A and $Tx_n \to b$ in B. Since r_A is continuous at zero and r_B is continuous on B by Thereom 3.2, we have

$$r_A(x_n) \to r_A(0) = 0, \qquad r_B(Tx_n) \to r_B(b).$$

on the other hend,

$$r_B(Tx_n) \leq r_A(x_n) \to 0.$$

Thus, $r_B(b) = 0$. Let y be an arbitrary element in B. By Lemma 3.1, we have

$$r_B(by) \leq r_B(b)r_B(y) = 0,$$

hence, $r_B(by) = 0$ for each y in B, so $b \in \mathrm{rad}(B)$. Since B is semisimple, $b = 0$. Therefore, by the Closed Graph Theorem T is continuous. □

Corollary 3.8. *Let A and B be continuous inverse F-algebras such that B is semisimple and commutative. Then every homomorphism $T : A \to B$ is continuous.*

Proof. Let $T : A \to B$ be a homomorphism, then

$$r_B(Tx) \leq r_A(x), \quad \text{for all } x \in A.$$

The result follows from Theorem 3.7. □

Corollary 3.9. *If A is a continuous inverse algebra, then every multiplicative linear functional $T : A \to \mathbb{C}$ is continuous.*

Proof. It is interesting to note that the above corollary is still valid if A is a Q-algebra, not necessarily a continuous inverse algebra [5, Theorem 2.2.28]. For a more general case, see [6, Corollary 12]. □

Theorem 3.10. *Let A be a continuous inverse F-algebra and B be an F-algebra. If there exists a continuous surjective homomorphism $T : A \to B$, then every multiplicative linear functional on B is continuous, i.e. B is functionally continuous.*

Proof. If $T : A \to B$ is continuous and surjective, then by the Open Mapping Theorem, it is open. Let φ be a multiplicative linear functional on B, then $\varphi \circ T$ is a multiplicative linear functional on A, and hence it is continuous by Corollary 3.9. Now suppose that U is an open subset of $\mathbb{C}$, then $(\varphi \circ T)^{-1}(U)$ is an open subset of A. Since T is an open mapping and

$$\varphi^{-1}(U) = ((T \circ T^{-1}) \circ \varphi^{-1})(U) = T((\varphi \circ T)^{-1}(U)),$$

it follows that $\varphi^{-1}(U)$ is open and hence φ is continuous. □

Lemma 3.11. *[7, 10.1.6] Let A be a Frechet algebra with rationally finitely many generators. Then every multiplicative linear functional on A is continuous.*

We now extend Lemma 3.11 as follows.

Theorem 3.12. *Let A be a Frechet algebra with rationally finitely many generators and B be a commutative semisimple continuous inverse F-algebra. Then every homomorphism $T : A \to B$ is continuous.*

Proof. Let $\varphi \in \Gamma_B$. Then $\varphi \circ T$ is a continuous multiplicative linear functional on A by Lemma 3.11. Suppose that $x_n \to 0$ in A and $Tx_n \to y$ in B. By the continuity of $\varphi \circ T$, we have $(\varphi \circ T)(x_n) \to 0$. On the other hand, $(\varphi \circ T)(x_n) = \varphi(Tx_n) \to \varphi(y)$. Consequently, $\varphi(y) = 0$. So we obtain

$$y \in \bigcap_{\varphi \in \Gamma_B} \ker \varphi = \operatorname{rad}(B) = \{0\}.$$

Hence, by the Closed Graph Theorem, T is continuous. □

Lemma 3.13. *Let $T : A \to B$ be a homomorphism between continuous inverse F-algebras A and B. If T is surjective, then $G(T)$ is a proper ideal of B.*

Proof. Let $b \in B$ and $y \in G(T)$. Then there exists a sequence $(x_n)_n \subseteq A$ such that $x_n \to 0$ and $Tx_n \to y$. Since T is surjective, there exists $x \in A$ such that $Tx = b$. So $x_n x \to 0$ and

$$T(x_n x) = Tx_n Tx = Tx_n b \to yb.$$

Thus, $G(T)$ is an ideal of B. Now we claim that $G(T)$ is a proper ideal. To see this, Let e_A and e_B be the unit elements of A and B respectively. Since T is a homomorphism, $r_B(Tx) \leq r_A(x)$, for all $x \in A$. Now suppose on the contrary that $e_B \in G(T)$. Then there exists a sequence $(x_n)_n \subseteq A$ such that $x_n \to 0$ and $Tx_n \to e_B$. By applying Lemma 3.1, we have

$$r_B(Te_A) = r_B(e_B) \leq r_B(e_B - Tx_n) + r_A(x_n).$$

By Thereom 3.2, $r_A(x_n)$ and $r_B(e_B - Tx_n)$ converge to zero. Thus, $r_B(e_B) = 0$. This contradicts the fact that $r_B(e_B) = 1$. Therefore, $G(T)$ is a proper ideal of B. □

Theorem 3.14. *Let $T : A \to B$ be a surjective homomorphism between continuous inverse F-algebras A and B. If B is simple, then T is continuous.*

Proof. Since $G(T)$ is an ideal of B and B is simple, we have $G(T) = \{0\}$ or $G(T) = B$. By Lemma 3.13, $G(T)$ is a proper ideal. So, $G(T) = \{0\}$ and hence T is continuous. □

Theorem 3.15. *Let A be a continuous inverse F-algebra and B be a topological algebra. If B satisfies the following property (C), then every homomorphism*
$T : A \to B$ is continuous.
(C) for every sequence $(y_n)_n \subseteq B$, $y_n \neq 0$ and $y_n \nrightarrow 0$, there is a sequence $(\varphi_m)_m$ of multiplicative linear functionals on B such that $\inf_{m,n} |\varphi_m(y_n)| = \epsilon > 0$.

Proof. Suppose that T is not continuous. Let $(x_n)_n \subseteq A$ be a sequence such that $x_n \to 0$, but $T(x_n) \nrightarrow 0$. Put $y_n = T(x_n)$. We may assume that $y_n \neq 0$ for all $n \geq 1$ (otherwise choose a subsequence). By hypothesis, there exists a sequence of multiplicative linear functionals $(\varphi_m)_m$ on B such that $\inf_{m,n} |\varphi_m(y_n)| = \epsilon > 0$. Thus, we have $|\varphi_m(T(\epsilon^{-1}x_n))| = |\epsilon^{-1}\varphi_m(T(x_n))| \geq 1$ for all $m, n \geq 1$. Set $z_n = \epsilon^{-1}x_n$. Then $z_n \to 0$. Since $\varphi_m \circ T$ is a multiplicative linear functional on A,it is continuous by Corollary 3.9, and so $\varphi_m \circ T(z_n) \to 0$. On the other hand,

$$|\varphi_m \circ T(\epsilon^{-1}x_n)| = |\varphi_m T(z_n)| \geq 1.$$

This contradiction implies that T is continuous. □

Remark 3.16. T. Husain [8, p. 45] introduced property (C) for a class of topological algebras (in particular, Frechet algebras). He has also indicated that if a Frechet algebra A satisfies property (C), then

$$r_A(x) = \sup\{|\varphi(x)| : \varphi \in \Gamma_A\} = \infty,$$

[8, p. 77]. This implies that a continuous inverse algebra B (in particular, a Banach algebra) cannot satisfy property (C) because the spectrum $sp_B(x)$ of every $x \in B$ is compact and so $r_B(x) < \infty$.

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