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Jensen's inequality and m-convex functions

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Abstract

In this paper, we generalize the Jensen's inequality for m-convex functions and we present a correction of Jensen's inequality which is a better than the generalization of this inequality for m-convex functions. Finally we have found new lower and upper bounds for Jensen's discrete inequality.

1. Introduction

The theory of inequalities began its development from the time when C. F. Gacss, A. L. Cathy and P. L. Cebysey, to mention only the most important, laid the theoretical foundation for approximative methods (see[7]). In ([1],[2],[8],[9],[10]) the authors provided a brief survey of general

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properties of m-convex functions, uniformly functions and etc. In [5], the authors extended the Jensen inequality and apply it to derive some useful lower bounds for various entropy measures of discrete random variables. In [3], the authors obtained two converses of Jensen integral inequality for convex function. In this paper, we generalize the Jensen's inequality for m-convex functions and we have found new lower and upper bounds for Jensen's discrete inequality.

Definition 1.1. Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \longrightarrow \mathbb{R}$ is said to be convex if the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y),$$

holds for any $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 1.2. ([6], [4])Let $[0, c] \subset \mathbb{R}$ be a bounded closed interval with $c > 0$, and let $m \in [0, 1]$. A function $f : [0, c] \longrightarrow \mathbb{R}$ is said to be m-convex if the inequality

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y),$$

holds for any $x, y \in [0, c]$ and $t \in [0, 1]$.

Definition 1.3. Let $x_1, ..., x_n \in I$ and $p_1, ..., p_n \in [0, 1]$ be such that $\sum_{i=1}^{n} p_i = 1$. The sum

$$\sum_{i=1}^{n} p_i x_i$$

is called the convex combination of points x_i.

Theorem 1.4. *(**Jensen's Inequality**) Let $f : I \longrightarrow \mathbb{R}$ be a convex function. Then the inequality*

$$f(\sum_{i=1}^{n} p_i x_i) \leq \sum_{i=1}^{n} p_i f(x_i)$$

holds for every convex combination $\sum_{i=1}^{n} p_i x_i$ of points $x_i \in I$.

Theorem 1.5. *[11] If f is a convex function on I, then*

$$\begin{aligned} 0 &\leq \max_{0 \leq \mu \leq \nu \leq n} \{p_\mu f(x_\mu) + p_\nu f(x_\nu) - (p_\mu + p_\nu) f(\frac{p_\mu x_\mu + p_\nu x_\nu}{p_\mu + p_\nu})\} \\ &\leq \sum_{i=1}^{n} p_i f(x_i) - f(\sum_{i=1}^{n} p_i x_i). \end{aligned}$$

2. main results

In this section, we use numbers $c > 0$, $m \in (0, 1]$ and $n \in \mathbb{N}$.

Lemma 2.1. *[6] Let $f : [0, c] \longrightarrow \mathbb{R}$ be a m-convex function and $f(0) \leq 0$. Then $f(tx) \leq tf(x)$, for all $x \in [0, c]$ and $t \in [0, 1]$.*

In the sequel, we shall prove some properties of m-convex functions.

Lemma 2.2. *Let $f : [0,c] \longrightarrow \mathbb{R}$ be a m-convex function. If $r, s, n \in \mathbb{N}$ and $r < s < n$ and $y \in [0, m^n c]$ then*

$$m^r f(\frac{y}{m^r}) \leq m^s f(\frac{y}{m^s}) \leq m^n f(\frac{y}{m^n}).$$

Proof. By the use of Lemma 2.1 we have

$$f(my) \leq mf(y) \leq m^2 f(\frac{y}{m}) \leq m^3 f(\frac{y}{m^2}) \leq \ldots \leq m^{n+1} f(\frac{y}{m^n}),$$

thus $m^{r+1} f(\frac{y}{m^r}) \leq m^{s+1} f(\frac{y}{m^s})$, and $m^r f(\frac{y}{m^r}) \leq m^s f(\frac{y}{m^s}) \leq m^n f(\frac{y}{m^n})$. □

Lemma 2.3. *Let f be a m-convex function and $x \in [0,c]$ $0 \leq p, q$ and $p + q = 1$, then for every $x \in [0,c]$*

1. *$f(px) \leq pf(x) + qmf(0)$ and $f(qx) \leq qf(x) + pmf(0)$,*
2. *$f(px) + f(qx) \leq f(x) + mf(0)$.*

Proof.
1. $f(px) = f(px + q \times 0) \leq pf(x) + mqf(\frac{0}{m}) = pf(x) + mqf(0)$, and similarly $f(qx) \leq qf(x) + pmf(0)$.
2. Suppose that $p + q = 1$. By using (1), we have

$$f(px) + f(qx) \leq pf(x) + qmf(0) + qf(x) + pmf(0) = f(x) + mf(0).$$

□

Lemma 2.4. *If f is a m-convex function such that $f(0) \leq 0$ and n is a natural number then f is a m^n-convex function.*

Proof. Suppose that $x, y \in [0,c]$

$$f(tx + m^n(1-t)y) \leq tf(x) + m(1-t)f(m^{n-1}y) \leq tf(x) + m^n(1-t)f(y).$$

Therefore, f is a m^n-convex function. □

In the following we give the Jensen inequality for m-convex function.

Theorem 2.5. *Let $f : [0,c] \longrightarrow \mathbb{R}$ be a m-convex function. Then the inequality*

$$f(\sum_{i=0}^{n} p_i x_i) \leq \sum_{i=0}^{n} m^i p_i f(\frac{x_i}{m^i}) \tag{2.1}$$

holds for any convex combination $\sum_{i=0}^{n} p_i x_i$ of points $x_i \in [0, m^n c]$.

Proof. Note that points $\frac{x_i}{m^i}$ belong to $[0,c]$. Namely, since $x_i \leq m^n c$, it follows that

$$\frac{x_i}{m^i} \leq m^{n-i} c \leq c.$$

The proof can be done by applying mathematical induction. For $n = 1$ is obvious. To prove for $n \geq 2$, we assume that the inequality in (2.1) applies to all convex combinations having less than or equal to n members. So,

$$\begin{aligned}
f(\sum_{i=0}^{n+1} p_i x_i) &= f(p_0 x_0 + (1-p_0)\sum_{i=1}^{n+1} \frac{p_i x_i}{1-p_0}) \\
&\leq p_0 f(x_0) + m(1-p_0) f(\sum_{i=1}^{n+1} \frac{p_i x_i}{m(1-p_0)}) \\
&= p_0 f(x_0) + m(1-p_0) f(\sum_{i=0}^{n} \frac{p_{i+1} x_{i+1}}{m(1-p_0)}) \\
&\leq p_0 f(x_0) + \sum_{i=0}^{n} m^i \frac{p_{i+1}}{1-p_0} f(\frac{x_{i+1}}{m \times m^i}) \\
&= p_0 f(x_0) + \sum_{i=0}^{n} m^i \frac{p_{i+1}}{1-p_0} f(\frac{x_{i+1}}{m^{i+1}}) \\
&= \sum_{i=0}^{n+1} m^i p_i f(\frac{x_i}{m^i}).
\end{aligned}$$

□

Theorem 2.6. *Let f be a m-convex function and r, s, n be natural numbers such that $0 \leq r < s \leq n$, then*

$$\begin{aligned}
0 &\leq \max_{0 \leq r < s \leq n} \{m^r p_r f(\frac{x_r}{m^r}) + m^s p_s f(\frac{x_s}{m^s}) - m^r (p_r + p_s) f(\frac{p_r x_r + p_s x_s}{m^r (p_r + p_s)})\} \\
&\leq \sum_{i=0}^{n} m^i p_i f(\frac{x_i}{m^i}) - f(\sum_{i=0}^{n} p_i x_i).
\end{aligned}$$

Proof. Suppose that,

$$\begin{cases} q_i = p_i, & i \neq r, s \\ q_r = p_r + p_s \\ q_s = 0 \end{cases}$$

and

$$\begin{cases} y_i = x_i, & i \neq r \\ y_r = \frac{p_r x_r + p_s x_s}{p_r + p_s} \end{cases}$$

then $\sum_{i=0}^{n} q_i = 1$ and Theorem 2.5 follows that

$$\begin{aligned} f(\sum_{i=0}^{n} p_i x_i) &= f(\sum_{i=0}^{n} q_i y_i) \\ &\leq \sum_{i=0}^{n} m^i q_i f(\frac{y_i}{m^i}) \\ &= \sum_{i\neq r,s}^{n} m^i q_i f(\frac{y_i}{m^i}) + m^r q_r f(\frac{y_r}{m^r}) + m^s q_s f(\frac{y_s}{m^s}) \\ &= \sum_{i\neq r,s}^{n} m^i p_i f(\frac{x_i}{m^i}) + m^r (p_r + p_s) f(\frac{p_r x_r + p_s x_s}{m^r (p_r + x_r)}). \end{aligned}$$

Hence,

$$\begin{aligned} m^r p_r f(\frac{x_r}{m^r}) + m^s p_s f(\frac{x_s}{m^s}) - m^r (p_r + p_s) f(\frac{p_r x_r + p_s x_s}{m^r (p_r + p_s)}) \\ \leq \sum_{i=0}^{n} m^i p_i f(\frac{x_i}{m^i}) - f(\sum_{i=0}^{n} p_i x_i). \end{aligned}$$

Thus the theorem is proved. □

Corollary 2.7. *Let f be a m-convex function and $0 \leq r < s \leq n$ are arbitrary, then*

$$\begin{aligned} f(\sum_{i=0}^{n} p_i x_i) &\leq \sum_{i\neq r,s} m^i p_i f(\frac{x_i}{m^i}) + m^r (p_r + p_s) f(\frac{p_r x_r + p_s x_s}{m^r (p_r + p_s)}) \\ &\leq \sum_{i\neq r,s} m^i p_i f(\frac{x_i}{m^i}) + m^s (p_r + p_s) f(\frac{p_r x_r + p_s x_s}{m^s (p_r + p_s)}) \\ &\leq \sum_{i\neq r,s} m^i p_i f(\frac{x_i}{m^i}) + m^n (p_r + p_s) f(\frac{p_r x_r + p_s x_s}{m^n (p_r + p_s)}) \\ &\leq m^n \{\sum_{i\neq r,s} p_i f(\frac{x_i}{m^n}) + (p_r + p_s) f(\frac{p_r x_r + p_s x_s}{m^n (p_r + p_s)})\}. \end{aligned}$$

Corollary 2.8. *Let f be a m-convex function and $0 \leq r < s \leq n$ are arbitrary, then*

$$\begin{aligned} &\max_{0\leq r<s\leq n} \{m^r p_r f(\frac{x_r}{m^r}) + m^s p_s f(\frac{x_s}{m^s}) - m^n (p_r + p_s) f(\frac{p_r x_r + p_s x_s}{m^n (p_r + p_s)})\} \\ &\leq \max_{0\leq r<s\leq n} \{m^r p_r f(\frac{x_r}{m^r}) + m^s p_s f(\frac{x_s}{m^s}) - m^s (p_r + p_s) f(\frac{p_r x_r + p_s x_s}{m^s (p_r + p_s)})\} \\ &\leq \max_{0\leq r<s\leq n} \{m^r p_r f(\frac{x_r}{m^r}) + m^s p_s f(\frac{x_s}{m^s}) - m^r (p_r + p_s) f(\frac{p_r x_r + p_s x_s}{m^r (p_r + p_s)})\} \\ &\leq \sum_{i=0}^{n} m^i p_i f(\frac{x_i}{m^i}) - f(\sum_{i=0}^{n} p_i x_i). \end{aligned}$$

Another results with respect to m-convex function as follows:

Theorem 2.9. *Let* $f : [0, c] \longrightarrow [0, +\infty)$ *be a m-convex function and* $0 \le x_i \le m^n c$ *for every* $0 \le i \le n$. *If* f *is continuous on* $[0, c]$ *and* $f'(x)$ *exists on* $(0, c)$ *and* $f(0) = f'(0^+) = 0$, *then*

$$0 \le \sum_{i=0}^{n} m^i p_i f(\frac{x_i}{m^i}) - f(\sum_{i=0}^{n} p_i x_i) \le f(c).$$

Proof. Since $0 \le x_i \le m^n c$ ($0 \le \frac{x_i}{m^i} \le \frac{x_i}{m^n} \le c$), there is a sequence $\{\lambda_i\}_i$, $0 \le \lambda_i \le 1$, such that $x_i = \lambda_i m^i c$. Hence,

$$\begin{aligned}
I &= \sum_{i=0}^{n} m^i p_i f(\frac{x_i}{m^i}) - f(\sum_{i=0}^{n} p_i x_i) \\
&= \sum_{i=0}^{n} m^i p_i f(\frac{\lambda_i m^i c}{m^i}) - f(\sum_{i=0}^{n} p_i \lambda_i m^i c) \\
&= \sum_{i=0}^{n} m^i p_i f(\lambda_i c) - f(pc), \text{where } \sum_{i=0}^{n} p_i \lambda_i m^i = p \\
&\le \sum_{i=0}^{n} m^i p_i (\lambda_i f(c) + m(1 - \lambda_i) f(0)) - f(pc), \text{ Lemma 2.3} \\
&= pf(c) - f(pc), \text{ since } f(0) = 0.
\end{aligned}$$

Thus

$$\sum_{i=0}^{n} m^i p_i f(\frac{x_i}{m^i}) - f(\sum_{i=0}^{n} p_i x_i) \le pf(c) - f(pc). \tag{2.2}$$

On the other hand for any $x \in [0, c]$ have

$$f(\frac{1}{2}px) = f(\frac{1}{2}px + \frac{1}{2} \times 0) \le \frac{1}{2} f(px) + \frac{m}{2} f(\frac{0}{m}) = \frac{1}{2} f(px) + \frac{m}{2} f(0).$$

So $-f(px) \le -2f(\frac{1}{2}px)$. Hence,

$$-f(px) \le -2f(\frac{1}{2}px) \le -4f(\frac{1}{4}px) \le \dots.$$

Therefore by using (2.2),

$$\begin{aligned}
I &= \sum_{i=0}^{n} m^i p_i f(\frac{x_i}{m^i}) - f(\sum_{i=0}^{n} p_i x_i) \\
&\le pf(c) - f(pc) \le pf(c) - 2f(\frac{1}{2}pc) \\
&\le pf(c) - 4f(\frac{1}{4}pc) \le \dots \le pf(c) - 2^k f(\frac{1}{2^k}pc)
\end{aligned} \tag{2.3}$$

for any $k \in \mathbb{N}$. But $f(0) = 0$, so

$$\lim_{k\to\infty} 2^k f(\frac{1}{2^k}pc) = pc \lim_{x\to 0^+} \frac{f(x)}{x} = pcf'(0^+) = 0. \tag{2.4}$$

Two relations 2.3 and 2.4 mean that

$$I = \sum_{i=0}^{n} m^i p_i f(\frac{x_i}{m^i}) - f(\sum_{i=0}^{n} p_i x_i) \le pf(c) \le f(c).$$

By the use of Theorem 2.5 we have $0 \le I$. So $0 \le I \le f(c)$. □

Theorem 2.10. *Let* $f : [0, c] \longrightarrow \mathbb{R}$ *be a m-convex function and* $\{x_i\}_{i=0}^n \subseteq [0, m^n c]$. *If* f *is continuous on* $[0, c]$ *and* $f'(x)$ *exists on* $(0, c)$ *also* $f(0) = f'(0^+) = 0 = f(c)$, *then*

$$\sum_{i=0}^{n} m^i p_i f(\frac{x_i}{m^i}) = f(\sum_{i=0}^{n} p_i x_i).$$

Proof. Let $I = \sum_{i=0}^{n} m^i p_i f(\frac{x_i}{m^i}) - f(\sum_{i=0}^{n} p_i x_i)$. By the use of Theorem 2.9 we have $0 \le I \le pf(c)$. Hence, if $f(c) = 0$ then $I = 0$. □

Theorem 2.11. *Let* $f : [0, c] \longrightarrow \mathbb{R}$ *be a m-convex function and* $\{x_i\}_{i=0}^n \subseteq [0, m^n c]$, *then*

$$f(\sum_{i=0}^{n} p_i x_i) \le m^n \sum_{i=0}^{n} p_i f(\frac{x_i}{m^n}).$$

Proof. By the Lemma 2.2 we have $m^i f(\frac{x_i}{m^i}) \le m^n f(\frac{x_i}{m^n})$ and by the Theorem 2.5 have $f(\sum_{i=0}^{n} p_i x_i) \le \sum_{i=0}^{n} m^i p_i f(\frac{x_i}{m^i})$, thus

$$\begin{aligned} f(\sum_{i=0}^{n} p_i x_i) &\le \sum_{i=0}^{n} m^i p_i f(\frac{x_i}{m^i}) \\ &\le \sum_{i=0}^{n} m^n p_i f(\frac{x_i}{m^n}) = m^n \sum_{i=0}^{n} p_i f(\frac{x_i}{m^n}). \end{aligned}$$

□

Theorem 2.12. *Let* $f : [0, c] \longrightarrow \mathbb{R}$ *be a m-convex function and* $\{x_i\}_{i=0}^n \subseteq [0, m^n c]$, *then*

$$\begin{aligned} 0 &\le m^n \max_{0\le r<s\le n} \{p_r f(\frac{x_r}{m^n}) + p_s f(\frac{x_s}{m^n}) - (p_r + p_s) f(\frac{p_r x_r + p_s x_s}{m^n(p_r + p_s)})\} \\ &\le m^n \sum_{i=0}^{n} p_i f(\frac{x_i}{m^n}) - mf(\sum_{i=0}^{n} p_i x_i). \end{aligned}$$

Proof. By the use of Theorem 2.11 have

$$f(\sum_{i=0}^{n} p_i x_i) = f((p_r + p_s)\frac{p_r x_r + p_s x_s}{p_r + p_s} + \sum_{i\neq r,s} p_i x_i)$$
$$\leq m^{n-1}((p_r + p_s) f(\frac{p_r x_r + p_s x_s}{p_r + p_s}) + \sum_{i\neq r,s} p_i f(\frac{x_i}{m^n})).$$

Hence,

$$m^n \sum_{i\neq r,s} p_i f(\frac{x_i}{m^n}) - m f(\sum_{i=0}^{n} p_i x_i) \geq -m^n((p_r + p_s) f(\frac{p_r x_r + p_s x_s}{p_r + p_s})).$$

Therefore,

$$m^n \sum_{i=0}^{n} p_i f(\frac{x_i}{m^n}) - m f(\sum_{i=0}^{n} p_i x_i) \geq m^n\{p_r f(\frac{x_r}{m^n}) + p_s f(\frac{x_s}{m^n})$$
$$- (p_r + p_s) f(\frac{p_r x_r + p_s x_s}{m^n(p_r + p_s)})\}.$$

So,

$$m^n \sum_{i=0}^{n} p_i f(\frac{x_i}{m^n}) - m f(\sum_{i=0}^{n} p_i x_i)$$
$$\geq m^n \max_{0\leq r<s\leq n} \{p_r f(\frac{x_r}{m^n}) + p_s f(\frac{x_s}{m^n}) - (p_r + p_s) f(\frac{p_r x_r + p_s x_s}{m^n(p_r + p_s)})\} \geq 0.$$

□

In view of the above theorem we have the following theorem.

Theorem 2.13. *Let* $f : [0, c] \longrightarrow \mathbb{R}$ *be a m-convex function and* $\mu = \min_{0\leq i\leq n}\{x_i\}$ *and* $\nu = \max_{0\leq i\leq n}\{x_i\}$*, then*

$$\frac{m^n}{n+1} \sum_{i=0}^{n} f(\frac{x_i}{m^n}) - m f(\frac{\sum_{i=0}^{n} x_i}{n+1})$$
$$\geq \frac{m^n}{n+1}(f(\frac{\mu}{m^n}) + f(\frac{\nu}{m^n}) - 2f(\frac{\mu+\nu}{2m^n})).$$

Proof. Applying Theorem 2.12 and putting $\mu = \min_{0\leq i\leq n}\{x_i\}$, $\nu = \max_{0\leq i\leq n}\{x_i\}$ and $p_i = \frac{1}{n+1}$, for every $0 \leq i \leq n$. □

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