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Some results on the block numerical range

Mostafa Zangiabadi[a,*], Hamid Reza Afshin[b]

[a]*Department of Mathematics, University of Hormozgan, Bandar Abbas, Islamic Republic of Iran.*
[b]*Department of Mathematics, Vali-e-Asr University, Rafsanjan, Islamic Republic of Iran.*

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Abstract

The main results of this paper are generalizations of classical results from the numerical range to the block numerical range. A different and simpler proof for the Perron-Frobenius theory on the block numerical range of an irreducible nonnegative matrix is given. In addition, the Wielandt's lemma and the Ky Fan's theorem on the block numerical range are extended.

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*Corresponding author
Email addresses: zangiabadi@hormozgan.ac.ir (Mostafa Zangiabadi), afshin@vru.ac.ir (Hamid Reza Afshin)

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1. Introduction

Let M_n be the set of $n \times n$ complex matrices and let $A \in M_n$. The numerical range of A is defined as follows (see, for instance, [2, 4]):

$$W(A) = \left\{x^*Ax \,:\, x \in \mathbb{C}^n,\ x^*x = 1\right\},$$

which is a useful concept in studying matrices and operators. However, it does not respect the block structure of block operator matrices, thus the new concept of the quadratic numerical range was introduced in [6]. The definition of the quadratic numerical range was generalized to the block numerical range in [11] as follows: Let $\mathfrak{D} := (\mathbb{C}^{k_1}, \ldots, \mathbb{C}^{k_n})$ be a decomposition of $\mathbb{C}^m$, i.e., $\mathbb{C}^m = \mathbb{C}^{k_1} \times \cdots \times \mathbb{C}^{k_n}$. With respect to this decomposition, the block matrix $\mathcal{A}$ has the following representation:

$$\mathcal{A} = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}, \tag{1.1}$$

where A_{ij} is a $k_i \times k_j$ matrix. Let us denote the spectrum of a matrix A by $\sigma(A)$. For $\mathfrak{D}$ define the block numerical range as follows:

$$W_{\mathfrak{D}}(\mathcal{A}) = \bigcup \{\sigma(\mathcal{A}_x) \ :\ x = (x_1, \ldots, x_n) \in \mathfrak{D},\ x_i^* x_i = 1,\ i = 1, \ldots, n\},$$

where

$$\mathcal{A}_x = \begin{pmatrix} \langle A_{11}x_1, x_1\rangle & \cdots & \langle A_{1n}x_n, x_1\rangle \\ \vdots & & \vdots \\ \langle A_{n1}x_1, x_n\rangle & \cdots & \langle A_{nn}x_n, x_n\rangle \end{pmatrix}, \quad x = (x_1, \ldots, x_n) \in \mathfrak{D}. \tag{1.2}$$

It has been investigated in [1, 11, 10].

The block numerical radius of $\mathcal{A}$ is defined as follows:

$$\omega_{\mathfrak{D}}(\mathcal{A}) = \max\{|\lambda| \ :\ \lambda \in W_{\mathfrak{D}}(\mathcal{A})\}.$$

Apparently, for $\mathfrak{D} = (\mathbb{C}^m)$, $\omega_{\mathfrak{D}}(\mathcal{A})$ yields the classical numerical radius of $\mathcal{A}$, i.e.,

$$\omega_{(\mathbb{C}^m)}(\mathcal{A}) = \omega(\mathcal{A}) = \max\{|z| : z \in W(\mathcal{A})\}.$$

Here is a summary of the results obtained by Tretter et al. in [11].

Lemma 1.1. *Let $\mathcal{A}$ be of the form (1.1) and $\mathfrak{D} = (\mathbb{C}^{k_1}, \ldots, \mathbb{C}^{k_n})$. Then*

(1) *For $n = 1$, the block numerical range coincides with the numerical range, i.e., $W_{(\mathbb{C}^m)}(\mathcal{A}) = W(\mathcal{A})$.*

(2) *For an $n \times n$ matrix A, the block numerical range of A coincides with the spectrum of A, i.e., $W_{(\mathbb{C},\ldots,\mathbb{C})}(A) = \sigma(A)$.*

(3) *If $\mathcal{A}$ is a lower or upper block triangular matrix, then $W_{\mathfrak{D}}(\mathcal{A}) = W(A_{11}) \cup \cdots \cup W(A_{nn})$.*

(4) $\sigma(\mathcal{A}) \subseteq W_{\mathfrak{D}}(\mathcal{A}) \subseteq W(\mathcal{A})$.

(5) $W_{\mathfrak{D}}(\mathcal{A})$ *is compact, but it is not necessarily convex.*

(6) *For all* $\alpha, \beta \in \mathbb{C}$ *we have* $W_{\mathfrak{D}}(\alpha\mathcal{A} + \beta I_m) = \alpha W_{\mathfrak{D}}(\mathcal{A}) + \beta$.

We call a matrix $A \in M_n$ irreducible if $n = 1$, or $n \geq 2$ and there does not exist a permutation matrix P such that

$$P^T AP = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix},$$

where B and D are nonempty square submatrices.

Given $A, B \in M_n$, A is said to be diagonally similar to B if there exists a nonsingular diagonal matrix D such that $A = D^{-1}BD$; if, in addition, D can be chosen to be unitary, then we say A is unitarily diagonally similar to B.

By the classical Perron-Frobenius theory, if A is a (square, entrywise) nonnegative matrix, then its spectral radius $\rho(A)$ is an eigenvalue of A and there is a corresponding nonnegative eigenvector. If, in addition, A is irreducible, then $\rho(A)$ is a simple eigenvalue and the corresponding eigenvector can be chosen to be positive. Moreover, for an irreducible nonnegative matrix with index of imprimitivity $h > 1$ (i.e., one having exactly h eigenvalues with modulus $\rho(A)$), Frobenius has also obtained a deeper structure theorem: The set of eigenvalues of A with modulus $\rho(A)$ consists precisely of $\rho(A)$ times all the $h-$th roots of unity, the spectrum of A is invariant under a rotation about the origin of the complex plane through an angle of $2\pi/h$ and A is an $h-$cyclic matrix, i.e., there is a permutation matrix P such that $P^T AP$ is a matrix of the form

$$\begin{pmatrix} 0 & A_{12} & 0 & \dots & 0 & 0 \\ 0 & 0 & A_{23} & \dots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & & \dots & 0 & A_{h-1,h} \\ A_{h,1} & 0 & & \dots & 0 & 0 \end{pmatrix}, \tag{1.3}$$

where the zero blocks along the diagonal are all square. This theory has been extended to the numerical range of a nonnegative matrix by Issos in his unpublished Ph.D. thesis [5] and then completed in [7, 8]. In [1], Forster et al. extended the main theorem of Issos on the block numerical range of a nonnegative irreducible matrix A. In Section 2, we redrive the main theorem of Issos on the block numerical range by simple method. The Wielandt's lemma [9, Chapter II, Theorem 2.1] and also the Ky Fan's theorem [3, Theorem 8.2.12] are the applications of the Perron-Frobenius theorem, which can be extended to the block numerical range. This we do in Section 3.

We first add a comment on the notation that is used. We always use $A = (a_{ij})_{i,j=1}^n$ to denote an $n \times n$ complex matrix and use $\mathcal{A} = (A_{ij})_{i,j=1}^n$ to denote an $m \times m$ $(m > n)$ matrix with complex entries, which is partitioned in the form (1.1), i.e., $A_{ij} \in M_{k_i \times k_j}$ is the (i, j) block of $\mathcal{A}$ and $k_1 + k_2 + \cdots + k_n = m$. Let ≥ 0 be the symbol for real arrays with nonnegative entries, and $\geq$ the symbol for the induced entrywise order. The symbol $|\,.\,|$ is reserved for the entrywise absolute value of an array.

For a vector $x \in \mathbb{C}^n$, we denote by $\|x\|$ the Euclidean norm of x, i.e., $\|x\| = (x^* x)^{1/2}$. For a matrix $A \in M_n$, we denote by $\|A\|$ the operator norm of A, i.e., $\|A\| = \max_{\|x\|=1} \|Ax\|$, where $\|\cdot\|$

is the Euclidean norm. Also we denote by $\lambda_{max}H(A)$ and $\lambda_{min}H(A)$ the largest eigenvalue and the smallest eigenvalue of $H(A)$, respectively, where $H(A) = (A + A^*)/2$ denotes the Hermitian part of the square matrix A.

2. Main result of Issos

In this section, we show that for a nonnegative irreducible matrix $\mathcal{A}$, the set of points of $W_{\mathfrak{D}}(\mathcal{A})$ with modulus $\omega_{\mathfrak{D}}(\mathcal{A})$, consists precisely of $\omega_{\mathfrak{D}}(\mathcal{A})$ times all the $h-$th roots of unity.

The following lemma shows that if $\mathcal{A}$ is $h-$cyclic, then the block numerical range of $\mathcal{A}$ is invariant under rotations about the origin of the complex plane through the angles $2\pi t/h$ for all $t = 0, 1, \ldots, h-1$.

Lemma 2.1. *Let $\mathcal{A} = (A_{ij})_{i,j=1}^n \in M_m$ be permutationally similar to the matrix of the form (1.3). Then we have:*

(1) $W_{\mathfrak{D}}(\mathcal{A}) = W_{\mathfrak{D}}(e^{2\pi ti/h}\mathcal{A})$, *for all* $t = 0, 1, \ldots, h-1$.

(2) $\alpha \in W_{\mathfrak{D}}(\mathcal{A})$ *if and only if* $\alpha e^{2\pi ti/h} \in W_{\mathfrak{D}}(\mathcal{A})$, *for all* $t = 0, 1, \ldots, h-1$.

(3) *If $h = 2t$, then $W_{\mathfrak{D}}(\mathcal{A})$ is symmetric with respect to the origin.*

Proof. (1). Let $\mathcal{B} = P^T\mathcal{A}P$ be an $h-$cyclic matrix for a permutation matrix P. By the proof of Theorem 6 in [5], $D^{-1}\mathcal{B}D = e^{i\theta}\mathcal{B}$, where $D = I_{n_1} \oplus e^{i\theta}I_{n_2} \oplus \cdots \oplus e^{i(h-1)\theta}I_{n_h}$ is unitary diagonal matrix with $\theta = 2\pi t/h$ and $n_1 + \cdots + n_h = m$, where $n_i(i = 1, \ldots, h)$ is the dimension of the i-th diagonal block of $\mathcal{B}$. This shows that

$$e^{2\pi ti/h}\mathcal{A} = (PDP^T)^{-1}\mathcal{A}(PDP^T). \tag{2.1}$$

As the block numerical range is invariant under diagonal unitary transformation [1, Equation (2.13)], so we have $W_{\mathfrak{D}}(\mathcal{A}) = W_{\mathfrak{D}}(e^{2\pi ti/h}\mathcal{A})$, for all $t = 0, \ldots, h-1$.

(2). It is clear from part (1) and Lemma 1.1.

(3). Let $h = 2t$. Then by part (2), $\alpha \in W_{\mathfrak{D}}(\mathcal{A})$ if and only if $\alpha e^{2\pi ti/2t} \in W_{\mathfrak{D}}(\mathcal{A})$, i.e., $-\alpha \in W_{\mathfrak{D}}(\mathcal{A})$. □

The main theorem of Issos in [5] for the block numerical range can now be stated as follows:

Theorem 2.2. *Let $\mathcal{A} = (A_{ij})_{i,j=1}^n \in M_m$ be an irreducible nonnegative matrix with index of imprimitivity h. Then*

$$\{\lambda \in W_{\mathfrak{D}}(\mathcal{A}) : |\lambda| = \omega_{\mathfrak{D}}(\mathcal{A})\} = \{\omega_{\mathfrak{D}}(\mathcal{A})e^{2\pi ti/h} : t = 0, 1, \ldots, h-1\}.$$

Proof. By the Perron-Frobenius theorem, there is a permutation matrix P such that $P^T\mathcal{A}P$ is a matrix of the form (1.3). By [1, Proposition 3.1] $\omega_{\mathfrak{D}}(\mathcal{A}) \in W_{\mathfrak{D}}(\mathcal{A})$ and then by (2.1) and Lemma 2.1 (2), we have $\omega_{\mathfrak{D}}(\mathcal{A})e^{2\pi ti/h} \in W_{\mathfrak{D}}(\mathcal{A})$ for all $t = 0, \ldots, h-1$, i.e.,

$$\{\omega_{\mathfrak{D}}(\mathcal{A})e^{2\pi ti/h} : t = 0, 1, \ldots, h-1\} \subseteq \{\lambda \in W_{\mathfrak{D}}(\mathcal{A}) : |\lambda| = \omega_{\mathfrak{D}}(\mathcal{A})\}.$$

Again by (2.1) and the Perron-Frobenius theorem, h is equal to the largest positive integer such that the matrix $\mathcal{A}$ is unitarily diagonally similar to the matrix $e^{2\pi ti/h}\mathcal{A}$ for $t = 0, 1, \ldots, h-1$.

Therefore, the set

$$\left\{0, \frac{2\pi}{h}, \dots, \frac{2\pi(h-1)}{h}\right\},$$

is the cyclic group modulo 2π of the largest order, concluding that there does not exist $\upsilon = \frac{2\pi}{p} < \frac{2\pi}{h}$ such that $\omega_{\mathfrak{D}}(\mathcal{A})e^{i\upsilon} \in W_{\mathfrak{D}}(\mathcal{A})$. Hence we establish the equality

$$\left\{\lambda \in W_{\mathfrak{D}}(\mathcal{A}) \ : \ |\lambda| = \omega_{\mathfrak{D}}(\mathcal{A})\right\} = \left\{\omega_{\mathfrak{D}}(\mathcal{A})e^{2\pi t i/h} \ : \ t = 0, 1, \dots, h-1\right\}.$$

So, the proof is complete. □

We illustrate Theorem 2.2 in the following example.

Example 2.3. Consider the nonnegative irreducible matrix

$$\mathcal{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

with index of imprimitivity $h = 4$. Here the spectrum of $\mathcal{A}$ is given by $\sigma(\mathcal{A}) = \{-1, -i, i, 1\}$. If $\mathfrak{D} = \{\mathbb{C}^2, \mathbb{C}^2\}$, then $\omega_{\mathfrak{D}}(\mathcal{A}) = 1$. Figure 1 shows that the numbers $-1, -i, i, 1$ belong to $W_{\mathfrak{D}}(\mathcal{A})$, with modulus $\omega_{\mathfrak{D}}(\mathcal{A}) = 1$. Note that $W(\mathcal{A})$ is the convex hull of the eigenvalues of $\mathcal{A}$.

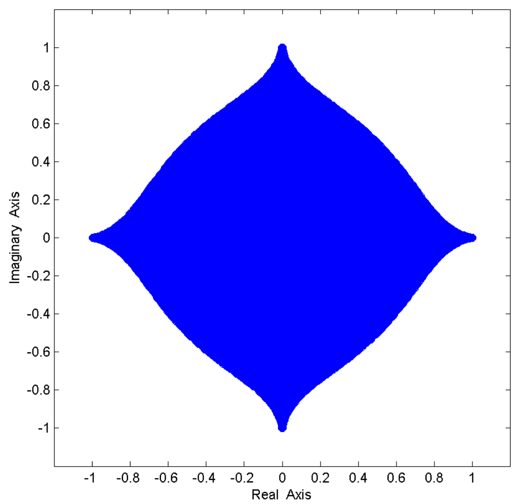

Figure 1: The quadratic numerical range of the matrix $\mathcal{A}$ with respect to the decomposition $\mathbb{C}^4 = \mathbb{C}^2 \times \mathbb{C}^2$.

3. The block numerical radius

In this section, we generalize the Wielandt's lemma and the Ky Fan's theorem on the block numerical range.

For proving the main result of this section, we need the following lemma due to Wielandt [9, Chapter II, Theorem 2.1].

Lemma 3.1 (Wielandt's lemma)**.** *Let $A, C \in M_n$ and assume that A is nonnegative. If $|C| \leq A$, then $\rho(C) \leq \rho(A)$. Suppose, in addition, that A is irreducible. If $\rho(C) = \rho(A)$ and ε is a unit complex number such that $\varepsilon\rho(C) \in \sigma(C)$, then $C = \varepsilon DAD^{-1}$ for some unitary diagonal matrix D.*

Now, we can generalize the above lemma on the block numerical range.

Theorem 3.2. *Assume that $C = (C_{ij})_{i,j=1}^n \in M_m$ is partitioned as in (1.1). Let $A = (a_{ij})_{i,j=1}^n$ be such that*

$$\|C_{ij}\| \leq a_{ij}, \quad i, j = 1, \ldots, n. \tag{3.1}$$

Then

$$\omega_{\mathfrak{D}}(C) \leq \omega(A). \tag{3.2}$$

Suppose, in addition, that A is irreducible. If $\omega_{\mathfrak{D}}(C) = \omega(A)$ and ξ is a unit complex number such that $\xi\omega_{\mathfrak{D}}(C) \in W_{\mathfrak{D}}(C)$, then

$$\|C_{ij}\| = a_{ij}, \quad i, j = 1, \ldots, n \tag{3.3}$$

and there exists a unitary matrix U such that

$$U^*CU = \xi \begin{pmatrix} A & 0 \\ 0 & * \end{pmatrix}. \tag{3.4}$$

Proof. Let $\mathfrak{D} = (\mathbb{C}^{k_1}, \ldots, \mathbb{C}^{k_n})$ be a decomposition of $\mathbb{C}^m$. In view of the Cauchy-Schwarz inequality, the entries of the matrix C_x, defined by (1.2), satisfy the relations

$$|(C_x)_{ij}| \leq \|C_{ij}x_j\|\|x_i\| \leq \|C_{ij}\|, \quad i, j = 1, 2, \ldots, n, \tag{3.5}$$

for all $x = (x_1, \ldots, x_n) \in \mathfrak{D}$ with $x_i^*x_i = 1,\ i = 1, 2, \ldots, n$. Taking into account (3.1) and (3.5), we can apply Wielandt's lemma to C_x and A, which yields the inequality

$$\rho(C_x) \leq \rho(A). \tag{3.6}$$

On the other hand, by definition of the block numerical range, there exists some $y = (y_1, \ldots, y_n) \in \mathfrak{D}$ with $y_i^*y_i = 1,\ i = 1, 2, \ldots, n$ such that $\omega_{\mathfrak{D}}(C) = \rho(C_y)$. So by (3.6) and [4, Property 1.2.9], we have

$$\omega_{\mathfrak{D}}(C) = \rho(C_y) \leq \rho(A) \leq \omega(A), \tag{3.7}$$

which shows (3.2). Now assume that the matrix A is irreducible and that $\omega_{\mathfrak{D}}(C) = \omega(A)$. So the inequalities in (3.7) all become equalities, i.e.,

$$\rho(C_y) = \rho(A). \tag{3.8}$$

In view of $|C_y| \leq A$ and equality (3.8), by the second half of Wielandt's lemma, it follows that there is a unitary diagonal matrix D, say $D = diag(d_1, \dots, d_n)$, such that

$$(C_y)_{ij} = y_i^* C_{ij} y_j = \xi d_i a_{ij} d_j^{-1}, \quad i, j = 1, \dots, n. \tag{3.9}$$

Using (3.9), (3.1) and (3.5), we derive $|(C_y)_{ij}| = a_{ij} \geq \|C_{ij}\| \geq |(C_y)_{ij}|$, for all $i, j = 1, \dots, n$ and conclude that (3.3) holds true. By setting $z_i = d_i y_i, \;\; i = 1, \dots, n$ in (3.9), we have

$$z_i^* C_{ij} z_j = \xi \, a_{ij}, \quad i, j = 1, \dots, n. \tag{3.10}$$

In order to obtain (3.4), we form the matrix

$$U_1 = \begin{pmatrix} z_1 & 0 & \cdots & \cdots & 0 \\ 0 & z_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & z_n \end{pmatrix} \in M_{m \times n}.$$

Obviously, the columns of U_1 are orthonormal, i.e., $U_1^* U_1 = I_n$. Therefore, one can complete U_1 to a unitary matrix $U = [U_1 \; U_2] \in M_m$. Using relations (3.10), we derive the equality $U_1^* C U_1 = \xi A$. By (3.3) and (3.10), we have

$$\|C_{ij}\| = a_{ij} = \left| \left\langle C_{ij} z_j, z_i \right\rangle \right| \leq \|C_{ij} z_j\| \, \|z_i\| \leq \|C_{ij}\|,$$

which implies that there exists $\alpha \in \mathbb{C}$ such that $C_{ij} z_j = \alpha \, z_i$ for all $i, j = 1, \dots, n$. In view of the unitarity of the matrix U and the collinearity of the column vectors $C_{ij} z_j$ and z_i for all $i, j = 1, \dots, n$, we then obviously have $U_2^* C U_1 = 0$. The remaining equality $U_1^* C U_2 = 0$ is established in a similar way, based on the collinearity of the row vectors $z_i^* C_{ij}$ and z_j^* for all $i, j = 1, \dots, n$. □

Remark 3.3. In Theorem 3.2, let $\mathfrak{D} = \mathbb{C}^m$, i.e., the block numerical range coincides with the numerical range (see Lemma 1.1). We consider the following two cases:

(1) Assume that $C = (C_{ij})_{i,j=1}^n \in M_m$ is partitioned as in (1.1). In this case Theorem 3.2 can be reformulated for the numerical range.

(2) Assume that $C = (c_{ij})_{i,j=1}^m$ is an $m \times m$ matrix. In this case Theorem 3.2 can be reformulated in the following way, which is a version of the Wielandt's lemma on the numerical range [7, Lemma 3.8].

Lemma 3.4. *Let $A, C \in M_n$ and assume that A is nonnegative. If $|C| \leq A$, then $\omega(C) \leq \omega(A)$. Suppose, in addition, that A is irreducible. If $\omega(C) = \omega(A)$ and ε is a unit complex number such that $\varepsilon\omega(C) \in W(C)$, then $C = \varepsilon D A D^{-1}$ for some unitary diagonal matrix D.*

In the following result we present a version of the Ky Fan's theorem on the block numerical range.

Theorem 3.5. *Assume that* $C = (C_{ij})_{i,j=1}^n \in M_m$ *is partitioned as in (1.1). Let* $A = (a_{ij})_{i,j=1}^n$ *be such that*

$$\|C_{ij}\| \le a_{ij}, \quad i, j = 1, \dots, n.$$

Then

$$W_{\mathfrak{D}}(C) \subseteq \bigcap_{\theta \in [0,2\pi]} \left\{ z \in \mathbb{C} \,:\, m_\theta \le \cos\theta \operatorname{Re} z - \sin\theta \operatorname{Im} z \le M_\theta \right\},$$

where

$$m_\theta = \min_j \left(\lambda_{min} H\left(e^{i\theta} C_{jj}\right) - \omega(A) + a_{jj} \right), \; M_\theta = \max_j \left(\lambda_{max} H\left(e^{i\theta} C_{jj}\right) + \omega(A) - a_{jj} \right).$$

Proof. Let $\mathfrak{D} = (\mathbb{C}^{k_1}, \dots, \mathbb{C}^{k_n})$ be a decomposition of $\mathbb{C}^m$. By (3.5), it is easy to see that for any $\theta \in [0, 2\pi]$ and for all $x \in \mathfrak{D}$ with $x_i^* x_i = 1, \ i = 1, 2, \dots, n$, we have $|H(e^{i\theta} C_x)| \le H(|C_x|) \le H(A)$. Then by the Ky Fan's theorem

$$\sigma\left(H(e^{i\theta} C_x)\right) \subseteq \bigcup_{j=1}^{n} \left\{ z \in \mathbb{C} \,:\, \left| z - \operatorname{Re}(e^{i\theta} x_j^* C_{jj} x_j) \right| \le \rho(H(A)) - a_{jj} \right\}.$$

Since $\rho(H(A)) = \omega(A)$ (see [7, Proposition 3.3]), it follows that

$$\begin{aligned} &\operatorname{Re} W(e^{i\theta} C_x) \\ &\subseteq \left[\min_j \left(\operatorname{Re}(e^{i\theta} x_j^* C_{jj} x_j) - \omega(A) + a_{jj} \right), \; \max_j \left(\operatorname{Re}(e^{i\theta} x_j^* C_{jj} x_j) + \omega(A) - a_{jj} \right) \right]. \end{aligned}$$

By [4, Properties 1.2.5 and 1.2.9] for all $j = 1, \dots, n$, we have

$$\operatorname{Re}(e^{i\theta} x_j^* C_{jj} x_j) \in W\left(H\left(e^{i\theta} C_{jj}\right)\right) = \left[\lambda_{min} H\left(e^{i\theta} C_{jj}\right), \; \lambda_{max} H\left(e^{i\theta} C_{jj}\right) \right]$$

which implies that

$$\begin{aligned} &\min_j \left(\operatorname{Re}(e^{i\theta} x_j^* C_{jj} x_j) - \omega(A) + a_{jj} \right) \ge \min_j \left(\lambda_{min} H\left(e^{i\theta} C_{jj}\right) - \omega(A) + a_{jj} \right) \\ &\max_j \left(\operatorname{Re}(e^{i\theta} x_j^* C_{jj} x_j) + \omega(A) - a_{jj} \right) \le \max_j \left(\lambda_{max} H\left(e^{i\theta} C_{jj}\right) + \omega(A) - a_{jj} \right). \end{aligned}$$

Hence $W(C_x)$ lies in the zone $\{z \in \mathbb{C} \,:\, m_\theta \le \operatorname{Re}(e^{i\theta} z) \le M_\theta\}$. Since $\sigma(C_x) \subseteq W(C_x)$ and so, by definition of the block numerical range, the proof is completed by taking the intersection of these zones for all $\theta \in [0, 2\pi]$. □

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