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$\mathcal{A}$-$\mathcal{B}$-imprimitivity bimodule frames

Azadeh Alijani Zamani[a,*]

[a] *Department of Mathematics, Faculty of Sciences Vali-e-Asr University of Rafsanjan, P.O. Box 7719758457, Rafsanjan, Islamic Republic of Iran.*

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Abstract

Frames in Hilbert bimodules are a special case of frames in Hilbert C^*-modules. The paper considers $\mathcal{A}$-frames and $\mathcal{B}$-frames and their relationship in a Hilbert $\mathcal{A}$-$\mathcal{B}$-imprimitivity bimodule. Also, it is given that every frame in Hilbert spaces or Hilbert C^*-modules is a semi-tight frame. A relation between $\mathcal{A}$-frames and $\mathcal{K}(\mathcal{H}_{\mathcal{B}})$-frames is obtained in a Hilbert $\mathcal{A}$-$\mathcal{B}$-imprimitivity bimodule. Moreover, the last part of the paper investigates dual of an $\mathcal{A}$-frame and a $\mathcal{B}$-frame and presents a common property for all duals of a frame in a Hilbert $\mathcal{A}$-$\mathcal{B}$-imprimitivity bimodule.

1. Introduction

The theory of frames has been rapidly generalized in the elements of frame and framework space. For example, frames had been introduced in Banach spaces, Hilbert spaces, Hilbert C^*-modules and frames of subspaces (fusion frames), frames of operators (generalized frames) and etc., [2, 3, 4, 5, 9].

*Corresponding author
Email address: alijani@vru.ac.ir (Azadeh Alijani Zamani)

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It is well known that the theory of Hilbert C^*-modules and Hilbert bimodules have applications in the study of locally compact quantum groups, complete maps between C^*-algebras, non-commutative geometry, KK-theory, and dynamical systems. There are many differences between Hilbert C^*-modules (Hilbert bimodules) and Hilbert spaces. It is expected that problems about frames for Hilbert C^*-modules to be more complicated than those for Hilbert spaces. This makes the study of the frames for Hilbert C^*-modules and Hilbert bimodules important and interesting. Frames in Hilbert spaces have been extended to frames in Hilbert C^*-modules [5]. In this case, the study of frames was not easily done because there were need to compare and to use of elements of a C^*-algebra. Alijani-Dehghan [2], have studied frames in Hilbert C^*-modules with C^*-valued bounds. They have given interesting results about frames and C^*-valued bounds. On the other hand, inner products in Hilbert bimodules have some bilateral relationships, then frames with respect to them are found important properties. These properties can be used for frames in Hilbert C^*-modules and frames in Hilbert spaces.

The paper is organized as follows. The reminder of this section contains a brief account of definitions and cardinal properties of C^*-algebras, Hilbert C^*-modules, Hilbert bimodules and frames. The second section explains main results of the paper. First, frames and their frame operators in a Hilbert bimodule are introduced, $\mathcal{A}$-frames and $\mathcal{B}$-frames, and we see that there exists an $\mathcal{A}$-frame that is not a $\mathcal{B}$-frame and viceversa, by an example. So, one of the important results of the paper is given. Since Hilbert spaces and Hilbert C^*-modules are Hilbert bimodules, for every frame in these spaces is obtained a C^*-valued bound that this frame is semi-tight by this bound. It is important because it makes possible to compute inverse of the frame operator. Also, the paper presents that every $\mathcal{A}$-frame is a $\mathcal{K}(\mathcal{H}_{\mathcal{B}})$-frame in a Hilbert $\mathcal{A}$-$\mathcal{B}$-imprimitivity bimodule. At the last, a common property for all duals (general duals) of a frame in a Hilbert bimodule, a Hilbert module, and a Hilbert space is obtained.

In continue, some definitions of Hilbert bimodules and frames and their properties are recalled. For more detail, we refer the interested reader to [6, 7, 8, 10].

Suppose $\mathcal{A}$ is a C^*-algebra. A linear space $\mathcal{H}$ which is also an algebraic left $\mathcal{A}$-module together with an $\mathcal{A}$-inner product ${}_{\mathcal{A}}\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{A}$ and possesses the following properties is called a left pre-Hilbert C^*-module,

i. ${}_{\mathcal{A}}\langle f, f \rangle \geq 0$, for any $f \in \mathcal{H}$.

ii. ${}_{\mathcal{A}}\langle f, f \rangle = 0$ if and only if $f = 0$.

iii. ${}_{\mathcal{A}}\langle f, g \rangle = {}_{\mathcal{A}}\langle g, f \rangle^*$, for any $f, g \in \mathcal{H}$.

iv. ${}_{\mathcal{A}}\langle \lambda f, h \rangle = \lambda {}_{\mathcal{A}}\langle f, h \rangle$, for any $\lambda \in \mathbb{C}$ and $f, h \in \mathcal{H}$.

v. ${}_{\mathcal{A}}\langle af + bg, h \rangle = a {}_{\mathcal{A}}\langle f, h \rangle + b {}_{\mathcal{A}}\langle g, h \rangle$, for any $a, b \in \mathcal{A}$ and $f, g, h \in \mathcal{H}$.

If $\mathcal{H}$ is Banach space with respect to the induced norm by the $\mathcal{A}$-valued inner product, $\| \cdot \| = \sqrt{\| {}_{\mathcal{A}}\langle \cdot, \cdot \rangle \|_{\mathcal{A}}}$, then $(\mathcal{H}, {}_{\mathcal{A}}\langle \cdot, \cdot \rangle)$ or ${}_{\mathcal{A}}\mathcal{H}$ is called a left Hilbert C^*-module over $\mathcal{A}$ or, simply, a left Hilbert $\mathcal{A}$-module. Similarly, a right Hilbert C^*-module has been defined. If the ideal $\{{}_{\mathcal{A}}\langle f, g \rangle; f, g \in \mathcal{H}\}$ is dense in $\mathcal{A}$, Hilbert $\mathcal{A}$-module is called full. Let $(\mathcal{H}, {}_{\mathcal{A}}\langle \cdot, \cdot \rangle_1)$ and $(\mathcal{K}, {}_{\mathcal{A}}\langle \cdot, \cdot \rangle_2)$ be two Hilbert $\mathcal{A}$-modules. The class of all adjointable maps from $\mathcal{H}$ into $\mathcal{K}$ is denoted by $B_*(\mathcal{H}, \mathcal{K})$, and if $\mathcal{K} = \mathcal{H}$, the notation $B_*(\mathcal{H})$ is used.

Now, Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. The vector space $\mathcal{H}$ is a Hilbert $\mathcal{A}$-$\mathcal{B}$-imprimitivity bimodule if it has the following conditions.

i. $\mathcal{H}$ is a full left Hilbert $\mathcal{A}$-module, and a full right Hilbert $\mathcal{B}$-module with respect to $\mathcal{A}$-inner product ${}_{\mathcal{A}}\langle\cdot,\cdot\rangle$ and $\mathcal{B}$-inner product $\langle\cdot,\cdot\rangle_{\mathcal{B}}$, respectively.

ii. For all $f, g \in \mathcal{H}$, $a \in \mathcal{A}$ and $b \in \mathcal{B}$,

$$\langle af, g\rangle_{\mathcal{B}} = \langle f, a^*g\rangle_{\mathcal{B}} \ \ and \ \ {}_{\mathcal{A}}\langle fb, g\rangle =_{\mathcal{A}} \langle f, gb^*\rangle.$$

iii. ${}_{\mathcal{A}}\langle f, g\rangle h = f\langle g, h\rangle_{\mathcal{B}}$, for $f, g, h \in \mathcal{H}$.

The closed linear subspace of $B_*(\mathcal{H})$ spanned by the rank-one operators $\{\theta_{f,g}; f, g \in \mathcal{H}\}$ is called C^*-algebra of compact operators and is shown by $\mathcal{K}({}_{\mathcal{A}}\mathcal{H})$. Note that for $f, g \in \mathcal{H}$,

$$\theta_{f,g} : \mathcal{H} \longrightarrow \mathcal{H}, \qquad \theta_{f,g}(h) = \ {}_{\mathcal{A}}\langle h, f\rangle g.$$

The compact operators $\theta_{f,g}$ are usually not compact operators in the generally sense.

Every Hilbert space $(H, \langle ., .\rangle)$ is a Hilbert $\mathcal{K}$-$\mathbb{C}$-imprimitivity bimodule when $\mathcal{K}$ is the C^*-algebra of compact operators with the following actions, for $f, g, h, k \in H$ and $T \in \mathcal{K}$,

$$\langle f, g\rangle_{\mathbb{C}} = \langle g, f\rangle, \quad {}_{\mathcal{K}}\langle f, g\rangle = f \otimes \overline{g}; \quad f \otimes \overline{g}(h) = f\langle g, h\rangle_{\mathbb{C}},$$

$$T.h = T(h), \quad f \otimes \overline{g}\ .\ h \otimes \overline{k} = \langle h, g\rangle f \otimes \overline{k}.$$

Some Hilbert C^*-modules have a Hilbert bimodule's structure. And we work with such Hilbert C^*-modules. The following proposition illustrates the connection.

Proposition 1.1. *[8] Every full right Hilbert $\mathcal{B}$-module $\mathcal{H}_{\mathcal{B}}$ is a Hilbert $\mathcal{K}(H_{\mathcal{B}})$-$\mathcal{B}$-imprimitivity bimodule. Conversely, if $\mathcal{H}$ is a Hilbert $\mathcal{A}$-$\mathcal{B}$-impri- mitivity bimodule, then there is an isomorphism φ of $\mathcal{A}$ onto $\mathcal{K}(\mathcal{H}_{\mathcal{B}})$ such that*

$$\varphi({}_{\mathcal{A}}\langle f, g\rangle) =_{\mathcal{K}(\mathcal{H}_B)} \langle f, g\rangle, \quad \forall f, g \in \mathcal{H}.$$

Throughout the paper, let I be a countable index set and let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. Also, assume that Hilbert $\mathcal{A}$-$\mathcal{B}$-imprimitivity bimodules are finitely or countably generated with respect to both C^*-algebras $\mathcal{A}$ and $\mathcal{B}$.

The notion of frames for Hilbert spaces had been extended by Frank-Larson [5] to the notion of frames in a Hilbert $\mathcal{A}$-modules. In this case, frame bounds were real. Afterwards, Alijani-Dehghan [2] have been studied frames in a Hilbert $\mathcal{A}$-module with $\mathcal{A}$-valued bounds as a countable family $\{f_i\}_{i\in I}$ in a Hilbert $\mathcal{A}$-module $\mathcal{H}$ such that there exist two strictly nonzero elements A and B in $\mathcal{A}$ satisfying

$$A\ {}_{\mathcal{A}}\langle f, f\rangle A^* \leq \sum_{i\in I}\ {}_{\mathcal{A}}\langle f, f_i\rangle\ {}_{\mathcal{A}}\langle f_i, f\rangle \leq B\ {}_{\mathcal{A}}\langle f, f\rangle B^*, \quad \forall f \in \mathcal{H}.$$

The sequence $\{f_i\}_{i\in I}$ is called to be a $*$-frame and elements A and B are said lower and upper $*$-frame bounds, respectively. Moreover, they defined frame operator corresponding to a $*$-frame in a Hilbert $\mathcal{A}$-module.

Let $\{f_i\}_{i\in I}$ be a $*$-frame for $\mathcal{H}$ with lower and upper $*$-frame bounds A and B, respectively. The frame operator $S : \mathcal{H} \longrightarrow \mathcal{H}$ is defined by $Sf = \sum_{i\in I}\ {}_{\mathcal{A}}\langle f, f_i\rangle f_i$ that is positive, invertible and adjointable and the inequality $\|A^{-1}\|^{-2} \leq \|S\| \leq \|B\|^2$ holds, and the reconstruction formula $f = \sum_{i\in I}\ {}_{\mathcal{A}}\langle f, S^{-1}f_i\rangle f_i$ holds for all $f \in \mathcal{H}$, [2].

One of the important applications of frames is spanning of a framework. The coefficients in this composition are not unique. It means that for a frame $\{f_i\}_{i\in I}$,

$$f = \sum_{i\in I} {}_{\mathcal{A}}\langle S^{-1}f, f_i\rangle f_i = \sum_{i\in I} {}_{\mathcal{A}}\langle f, g_i\rangle f_i, \quad \forall f \in \mathcal{H},$$

where $\{g_i\}_{i\in I}$ is a dual frame for $\{f_i\}_{i\in I}$. Moreover, by general duals of frames, every $f \in \mathcal{H}$ is generated. If $\{g_i\}_{i\in I}$ is a sequence in $\mathcal{H}$ and Γ is an invertible element of $B_*({}_{\mathcal{A}}\mathcal{H})$, then a pair $(\{g_i\}_{i\in I}, \Gamma)$ is an operator dual (or general dual) of $\{f_i\}_{i\in I}$ when $f = \sum_{i\in I} {}_{\mathcal{A}}\langle \Gamma f, g_i\rangle f_i$, for all $f \in \mathcal{H}$.

Also, it is known that frames with real-valued bounds and frames with C^*-valued bounds have a bilateral relationship. So, for every frame in a Hilbert C^*-module, there are two collection of bounds (real-valued bounds and C^*-valued bounds). In the paper, we will study frames, regardless their bounds type.

2. Main Result

In this section contains definition of frame for a Hilbert $\mathcal{A}$-$\mathcal{B}$-imprimitivity bimodule with respect to two C^*-algebra $\mathcal{A}$ and $\mathcal{B}$. Here, frames are defined with real valued bounds but all of results are valid for frames with C^*-valued bounds by the relation between them in [2]. In continue, properties of frames in Hilbert $\mathcal{A}$-$\mathcal{B}$-imprimitivity bimodules are considered and we give important result for Hilbert C^*-module frames and Hilbert frames.

Definition 2.1. Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-$\mathcal{B}$-imprimitivity bimodule. A sequence $\{f_i\}_{i\in I}$ is an $\mathcal{A}$-frame if it is a frame for Hilbert $\mathcal{A}$-module $\mathcal{H}$,

$$A_1 \, {}_{\mathcal{A}}\langle f, f\rangle \leq \sum_{i\in I} {}_{\mathcal{A}}\langle f, f_i\rangle \, {}_{\mathcal{A}}\langle f_i, f\rangle \leq A_2 \, {}_{\mathcal{A}}\langle f, f\rangle, \quad \forall f \in \mathcal{H},$$

when $0 < A_1 \leq A_2 < \infty$. And, $\{f_i\}_{i\in I}$ is a $\mathcal{B}$-frame if it is a frame for Hilbert $\mathcal{B}$-module $\mathcal{H}$,

$$B_1\langle f, f\rangle_{\mathcal{B}} \leq \sum_{i\in I}\langle f, f_i\rangle_{\mathcal{B}}\langle f_i, f\rangle_{\mathcal{B}} \leq B_2\langle f, f\rangle_{\mathcal{B}}, \quad \forall f \in \mathcal{H},$$

when $0 < B_1 \leq B_2 < \infty$.

In every case, the frame operators are

$$S_{\mathcal{A}}f = \sum_{i\in I} {}_{\mathcal{A}}\langle f, f_i\rangle f_i, \quad S_{\mathcal{B}}f = \sum_{i\in I} f_i\langle f_i, f\rangle_{\mathcal{B}}, \quad \forall f \in \mathcal{H}.$$

It is well known that if $\{f_i\}_{i\in I}$ is an $\mathcal{A}$-frame (or B-frame), then the frame operator $S_{\mathcal{A}}$ (or $S_{\mathcal{B}}$) are well define, adjointable and positive [5].

The following example says that there exists an $\mathcal{A}$-frame that is not a $\mathcal{B}$-frame. So, it is not necessarily bilateral relationship between $\mathcal{A}$-frames and $\mathcal{B}$-frames in a Hilbert $\mathcal{A}$-$\mathcal{B}$-imprimitivity bimodule. Also, a sequence is given such that is tight frame with respect to a C^*-algebra and is not tight with respect to another C^*-algebra.

Example 2.2. Let H be a Hilbert space. Then H is a Hilbert $\mathcal{K}$-$\mathbb{C}$-imprimitivity bimodule. Every orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ for Hilbert space H is a $\mathbb{C}$-frame for the Hilbert $\mathcal{K}$-$\mathbb{C}$-imprimitivity bimodule H, but it is not a $\mathcal{K}$-frame for the Hilbert $\mathcal{K}$-$\mathbb{C}$-imprimitivity bimodule H because

$$\begin{aligned}\sum_{n\in\mathbb{N}} {}_{\mathcal{K}}\langle f, e_n\rangle\, {}_{\mathcal{K}}\langle e_n, f\rangle &= \sum_{n\in\mathbb{N}} f\otimes\overline{e_n}\ .\ e_n\otimes\overline{f} \\ &= \sum_{n\in\mathbb{N}}\langle e_n, e_n\rangle f\otimes\overline{f} = \Big(\sum_{n\in\mathbb{N}}\langle e_n, e_n\rangle\Big) f\otimes\overline{f}.\end{aligned}$$

Since $\sum_{n\in\mathbb{N}}\langle e_n, e_n\rangle$ is not finite, $\{e_n\}_{n\in\mathbb{N}}$ is not a $\mathcal{K}$-Bessel sequence for $\mathcal{H}$.

Now, we consider the sequence $\{\frac{e_n}{n}\}_{n\in\mathbb{N}}$. It is a $\mathbb{C}$-frame but it is not a tight $\mathbb{C}$-frame. On the other hand, it is a tight $\mathcal{K}$-frame. Since

$$\sum_{n\in\mathbb{N}} {}_{\mathcal{K}}\langle f, e_n\rangle\, {}_{\mathcal{K}}\langle e_n, f\rangle = \Big(\sum_{n\in\mathbb{N}}\frac{1}{n^2}\langle e_n, e_n\rangle\Big) f\otimes\overline{f} = \Big(\sum_{n\in\mathbb{N}}\frac{1}{n^2}\Big) f\otimes\overline{f}, \quad \forall f\in H.$$

The third property of Hilbert bimodules gives an interesting result for Hilbert module frames and Hilbert space frames. In some applications of frames, we need computing frame operator and it's inverse operator but they are difficult in very cases. By C^*-valued bounds and Hilbert bimodules, they are possible. For given this aim, we present the semi-tight frames and some results about them.

Definition 2.3. A sequence $\{f_i\}_{i\in I}$ is a semi-tight frame in a Hilbert $\mathcal{A}$-$\mathcal{B}$-bimodule $\mathcal{H}$ if there exists an element $A\in\mathcal{A}$ such that

$$\sum_{i\in I}\langle f, f_i\rangle_{\mathcal{B}} f_i = Af, \quad \forall f\in\mathcal{H},$$

or there exists an element $B\in\mathcal{B}$ such that

$$\sum_{i\in I} {}_{\mathcal{A}}\langle f, f_i\rangle f_i = Bf, \quad \forall f\in\mathcal{H}.$$

In the first case, the semi-tight frame is called $\mathcal{A}$-semi-tight $\mathcal{B}$-frame and in the second case, it is said $\mathcal{B}$-semi-tight $\mathcal{A}$-frame.

Frames in Hilbert $\mathcal{A}$-$\mathcal{B}$-imprimitivity bimodule have some interesting properties. Every $\mathcal{A}$-frame or $\mathcal{B}$-frame is a semi-tight frame, also frame operator $S_{\mathcal{A}}$ commutes with $S_{\mathcal{B}}$. The following remark illustrates them.

Here, we say that $\{f_i\}_{i\in I}$ is a semi-tight frame in a Hilbert $\mathcal{A}$-$\mathcal{B}$-bimodule if is a tight frame such that the inner product in definition of frame is with respect to a C^*-algebra and C^*-valued bound is in another C^*-algebra.

Remark 2.4. Let $\{f_i\}_{i\in I}$ be an $\mathcal{A}$-frame for Hilbert $\mathcal{A}$-$\mathcal{B}$-imprimitivity bimodule $\mathcal{H}$ with the frame operator $S_{\mathcal{A}}$. The following properties for $\{f_i\}_{i\in I}$ and its frame operator are valid.

i. If $\{f_i\}_{i\in I}$ is an $\mathcal{A}$-frame, then the frame operator $S_{\mathcal{A}}$ is well define [5], and $S_{\mathcal{A}}f \in \mathcal{H}$, for all $f \in \mathcal{H}$. By the third property of definition of Hilbert $\mathcal{A}$-$\mathcal{B}$-imprimitivity bimodule, for $f \in \mathcal{H}$

$$S_{\mathcal{A}}f = \sum_{i\in I} {}_{\mathcal{A}}\langle f, f_i\rangle f_i = \sum_{i\in I} f\langle f_i, f_i\rangle_{\mathcal{B}} = f\Big(\sum_{i\in I}\langle f_i, f_i\rangle_{\mathcal{B}}\Big), \quad \forall f \in \mathcal{H},$$

therefore, the series $\sum_{i\in I}\langle f_i, f_i\rangle_{\mathcal{B}}$ is converges in $\mathcal{B}$ and $\beta = \sum_{i\in I}\langle f_i, f_i\rangle_{\mathcal{B}} \in \mathcal{B}$. Similarly, if $\{f_i\}_{i\in I}$ is a $\mathcal{B}$-frame for $\mathcal{H}$ with the frame operator $S_{\mathcal{B}}$, then

$$S_{\mathcal{B}}f = \alpha f; \quad \alpha = \sum_{i\in I} {}_{\mathcal{A}}\langle f_i, f_i\rangle \in \mathcal{A}.$$

ii. $S_{\mathcal{A}}$ is an $\mathcal{A}$-module linear map and also

$$S_{\mathcal{A}}^{-1}(f\beta) = S_{\mathcal{A}}^{-1}(S_{\mathcal{A}}f) = f = S_{\mathcal{A}}(S_{\mathcal{A}}^{-1}f) = (S_{\mathcal{A}}^{-1}f)\beta \Longrightarrow S_{\mathcal{A}}^{-1}(f\beta) = (S_{\mathcal{A}}^{-1}f)\beta.$$

Similarly, if $\{f_i\}_{i\in I}$ is a $\mathcal{B}$-frame for $\mathcal{H}$ with frame operator $S_{\mathcal{B}}$, then

$$S_{\mathcal{B}}^{-1}(\alpha f) = \alpha S_{\mathcal{B}}^{-1}f, \quad \forall f \in \mathcal{H}.$$

iii. If $\{f_i\}_{i\in I}$ is both $\mathcal{A}$-frame and $\mathcal{B}$-frame for Hilbert $\mathcal{A}$-$\mathcal{B}$-imprimitivity bimodule $\mathcal{H}$ with the frame operators $S_{\mathcal{A}}$ and $S_{\mathcal{A}}$, then

$$S_{\mathcal{A}}S_{\mathcal{B}}f = S_{\mathcal{A}}(\alpha f) = \alpha S_{\mathcal{A}}f = \alpha f\beta = (S_{\mathcal{B}}f)\beta = S_{\mathcal{B}}(f\beta) = S_{\mathcal{B}}S_{\mathcal{A}}f, \ \ \forall f \in \mathcal{H}.$$

It shows that $S_{\mathcal{A}}$ and $S_{\mathcal{B}}$ commute with each other.

In [2], we saw that there exist some non-tight frames in Hilbert spaces that are tight Hilbert module frames. Here, we can obtain a C^*-valued bound for every frame in Hilbert bimodules that it is semi-tight frame. Then for every frame in Hilbert spaces will be given a C^*-valued bound.

Proposition 2.5. *Let $\mathcal{H}_{\mathcal{B}}$ be a full right Hilbert $\mathcal{B}$-module. Then for every $\mathcal{B}$-frame $\{f_i\}_{i\in I}$, there exist an element $\xi \in \mathcal{K}(\mathcal{H}_{\mathcal{B}})$ such that $S_{\mathcal{B}}f = \xi f$, for all $f \in \mathcal{H}_{\mathcal{B}}$. We say that $\mathcal{B}$-frame $\{f_i\}_{i\in I}$ is a $\mathcal{K}(\mathcal{H}_{\mathcal{B}})$-semi-tight $\mathcal{B}$-frame.*

Proof. By Proposition 1.1, $\mathcal{H}$ is a Hilbert $\mathcal{K}(\mathcal{H}_{\mathcal{B}})$-$\mathcal{B}$-imprimitivity bimodule. Then for $f \in \mathcal{H}_{\mathcal{B}}$,

$$S_{\mathcal{B}}f = \sum_{i\in I} f_i\langle f_i, f\rangle_{\mathcal{B}} = \sum_{i\in I} {}_{\mathcal{K}(\mathcal{H}_{\mathcal{B}})}\langle f_i, f_i\rangle \,.\, f = \sum_{i\in I}\theta_{f_i,f_i} \,.\, f = (\sum_{i\in I}\theta_{f_i,f_i}).f.$$

Since $S_{\mathcal{B}}$ is well define, $\xi = \sum_{i\in I}\theta_{f_i,f_i}$ converges in $\mathcal{K}(\mathcal{H}_{\mathcal{B}})$. □

Corollary 2.6. *Every frame in a Hilbert space H is a $\mathcal{K}$-semi-tight $\mathbb{C}$-frame in Hilbert $\mathcal{K}$-$\mathbb{C}$-imprimitivity bimodule H. By the assumption in Proposition 2.5, if $\sum_{i\in I}\theta_{f_i,f_i}$ is invertible, then S^{-1} is easily calculated.*

In [2], the relation between frames for Hilbert $\mathcal{A}$-module $\mathcal{H}$ and Hilbert $\mathcal{B}$-module $\mathcal{H}$ has been considered. By this theorem, we get an interesting result for frames in a Hilbert $\mathcal{A}$-$\mathcal{B}$-imprimitivity bimodule. In the following, we first recall Theorem 3.2 [2], and in the second step, this subject is investigated for Hilbert $\mathcal{A}$-$\mathcal{B}$-imprimitivity bimodules.

Theorem 2.7. *[2] Suppose $\mathcal{H}$ is a Hilbert $\mathcal{A}$-module and a Hilbert $\mathcal{B}$-module. The operator $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a $*$-homomorphism and let θ be a map on $\mathcal{H}$ such that $\langle \theta f, \theta g\rangle_{\mathcal{B}} = \varphi({}_{\mathcal{A}}\langle f, g\rangle)$ for all $f, g \in \mathcal{H}$. Also, assume that $\{f_i\}_{i\in I}$ is a $*$-frame for Hilbert $\mathcal{A}$-module $\mathcal{H}$ with $*$-frame operator $S_{\mathcal{A}}$ and lower and upper $*$-frame bounds α_1, α_2, respectively. If θ is surjective, then $\{\theta f_i\}_{i\in I}$ is a $*$-frame for Hilbert $\mathcal{B}$-module $\mathcal{H}$ with $*$-frame operator $S_{\mathcal{B}}$ and lower and upper $*$-frame bounds $\varphi(\alpha_1), \varphi(\alpha_2)$, respectively, and*

$$\langle S_{\mathcal{B}}\theta f, \theta g\rangle_{\mathcal{B}} = \varphi({}_{\mathcal{A}}\langle S_{\mathcal{A}} f, g\rangle), \quad \forall f \in \mathcal{H}. \tag{2.1}$$

Moreover, the map θ is surjective if the following conditions are valid:

1. *φ is surjective;*
2. *$\{\theta f_i\}_{i\in I}$ is a $*$-frame for $\mathcal{H}$; and*
3. *$\theta(af) = \varphi(a)\theta f$, for all $a \in \mathcal{A}$, $f \in \mathcal{H}$.*

Theorem 2.8. *Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-$\mathcal{B}$-imprimitivity bimodule. If $\{f_i\}_{i\in I}$ is an $\mathcal{A}$-frame for $\mathcal{H}$, then $\{f_i\}_{i\in I}$ is a $\mathcal{K}(\mathcal{H}_{\mathcal{B}})$-frame for $\mathcal{H}$ with the same frame operators. Similar result is valid for $\mathcal{B}$-frames and $\mathcal{K}({}_{\mathcal{A}}\mathcal{H})$-frames in $\mathcal{H}$.*

Proof. By Proposition 1.1, there is an isomorphism φ from $\mathcal{A}$ onto $\mathcal{K}(\mathcal{H}_{\mathcal{B}})$ such that $\varphi({}_{\mathcal{A}}\langle f, g\rangle) = \langle f, g\rangle_{\mathcal{K}(\mathcal{H}_{\mathcal{B}})}$, for all $f, g \in \mathcal{H}$. On the other hand, Theorem 2.7 concludes that $\{f_i\}_{i\in I}$ is a $\mathcal{K}(\mathcal{H}_{\mathcal{B}})$-frame for $\mathcal{H}$ when θ is identity operator on $\mathcal{H}$. Now, suppose that $S_{\mathcal{A}}$ and $S_{\mathcal{K}(\mathcal{H}_{\mathcal{B}})}$ are the frame operators for $\mathcal{A}$-frame and $\mathcal{K}(\mathcal{H}_{\mathcal{B}})$-frame $\{f_i\}_{i\in I}$, respectively. For $f \in \mathcal{H}$,

$$\begin{aligned} S_{\mathcal{K}(\mathcal{H}_{\mathcal{B}})} f = \sum_{i\in I} {}_{\mathcal{K}(\mathcal{H}_{\mathcal{B}})}\langle f, f_i\rangle . f_i = \sum_{i\in I} \theta_{f,f_i} . f_i = \sum_{i\in I} \theta_{f,f_i}(f_i) \\ = \sum_{i\in I} f\langle f_i, f_i\rangle_{\mathcal{B}} = f \sum_{i\in I} \langle f_i, f_i\rangle_{\mathcal{B}} = f\beta, \end{aligned}$$

where $\beta = \sum_{i\in I}\langle f_i, f_i\rangle_{\mathcal{B}}$. By the equality of Remark 2.4, i, frame operator $S_{\mathcal{A}} f = S_{\mathcal{K}(\mathcal{H}_{\mathcal{B}})} f$ on $\mathcal{H}$. □

The subject attractive in frame theory is studying of dual frames. For this in Hilbert bimodules, we need the following lemma.

Lemma 2.9. *Let $\mathcal{H}$ be a full Hilbert $\mathcal{A}$-module and $\gamma \in \mathcal{A}$. If $\gamma f = 0$, for all $f \in \mathcal{H}$, then $\gamma = 0$.*

Proof. Since $\gamma f = 0$, for $f \in \mathcal{H}$,

$$\|\gamma f\| = 0 \implies {}_{\mathcal{A}}\langle \gamma f, \gamma f\rangle = 0 \implies \gamma\, {}_{\mathcal{A}}\langle f, f\rangle \gamma^* = 0.$$

By Polarization identity, for $f, g \in \mathcal{H}$,

$$\gamma\, {}_{\mathcal{A}}\langle f, g\rangle\gamma^* = \gamma[\frac{1}{4}\sum_{k=0}^{3} {}_{\mathcal{A}}\langle f + i^k g, f + i^k g\rangle]\gamma^* = \frac{1}{4}\sum_{k=0}^{3} \gamma\, {}_{\mathcal{A}}\langle f + i^k g, f + i^k g\rangle\gamma^* = 0.$$

Then $\gamma\, {}_{\mathcal{A}}\langle f, g\rangle\gamma^* = 0$, for all $f, g \in \mathcal{H}$. On the other hand, $\mathcal{H}$ is full and $\overline{span}\{ {}_{\mathcal{A}}\langle f, g\rangle, f, g \in \mathcal{H}\} = \mathcal{A}$. Then $\gamma a \gamma^* = 0$, for $a \in \mathcal{A}$. Set $a = 1_{\mathcal{A}}$, and

$$\gamma\gamma^* = 0 \implies 0 = \|\gamma\gamma^*\| = \|\gamma\|^2 \implies \gamma = 0.$$

□

Now, by the last theorem, a common property for all of dual of a given frame in a Hilbert bimodule is obtained.

Theorem 2.10. *Let $\{f_i\}_{i\in I}$ be an $\mathcal{A}$-frame for Hilbert $\mathcal{A}$-$\mathcal{B}$-imprimitivity bimodule $\mathcal{H}$ with frame operator $S_{\mathcal{A}}$ and let $\{g_i\}_{i\in I}$ be a dual $\mathcal{A}$-frame for $\{f_i\}_{i\in I}$. Then $\sum_{i\in I}\langle S_{\mathcal{A}}^{-1}f_i, f_i\rangle_{\mathcal{B}} = \sum_{i\in I}\langle g_i, f_i\rangle_{\mathcal{B}}$.*

Moreover, if a pair $(\{g_i\}_{i\in I}, \Gamma)$ is an operator daul of $\{f_i\}_{i\in I}$, then

$$\sum_{i\in I}\langle S_{\mathcal{A}}^{-1}f_i, f_i\rangle_{\mathcal{B}} = \sum_{i\in I}\langle \Gamma^* g_i, f_i\rangle_{\mathcal{B}}.$$

Proof. Set

$$\gamma_F = \sum_{i\in I}\langle S_{\mathcal{A}}^{-1}f_i, f_i\rangle_{\mathcal{B}}, \qquad \gamma_G = \sum_{i\in I}\langle g_i, f_i\rangle_{\mathcal{B}}.$$

By the reconstruction formula and the definition of dual frame, for $f \in \mathcal{H}$, obtain

$$f = \sum_{i\in I} {}_{\mathcal{A}}\langle S_{\mathcal{A}}^{-1}f, f_i\rangle f_i = f\sum_{i\in I}\langle S_{\mathcal{A}}^{-1}f_i, f_i\rangle_{\mathcal{B}} = f\gamma_F,$$

and

$$f = \sum_{i\in I} {}_{\mathcal{A}}\langle f, g_i\rangle f_i = f\sum_{i\in I}\langle g_i, f_i\rangle_{\mathcal{B}} = f\gamma_G.$$

Then

$$f\gamma_F = f\gamma_G, \quad \forall f \in \mathcal{H} \Longrightarrow f(\gamma_F - \gamma_G) = 0, \quad \forall f \in \mathcal{H},$$

by Lemma 2.9, get $\gamma_F = \gamma_G$. □

Corollary 2.11. *In the composition of elements of a full left Hilbert $\mathcal{A}$-module (or a Hilbert space) with respect to a frame and their duals, the summand of coefficients are unique and is $\sum_{i\in I}\langle S_{\mathcal{A}}^{-1}f_i, f_i\rangle_{\mathcal{K}(_{\mathcal{A}}\mathcal{H})}$ (or $\sum_{i\in I} S_{\mathcal{A}}^{-1}f_i \otimes f_i$), where $\{f_i\}_{i\in I}$ is a frame with frame operator $S_{\mathcal{A}}$.*

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References

[1] A. Alijani, *Dilations of ∗-Frames and their operator Duals*, Preprint.
[2] A. Alijani and M.A. Dehghan, ∗-Frames in Hilbert C^*-modules, *U.P.B. Sci. Bull., Ser. A,* **73**(4) (2011), 89–106.
[3] P. Casazza, D. Han and D. Larson, Frames for Banach spaces, *Contemp. Math.,* **247** (1999), 149–181.
[4] P.G. Casazza and G. Kutyniok, Frames of subspaces, *Contemp. Math.,* **345** (2004), 87–113.
[5] M. Frank and D.R. Larson, Frames in Hilbert C^*-modules and C^*-algebra, *J. Oper. Theory,* **48** (2002), 273–314.
[6] E.C. Lance, *Hilbert C^*-modules, A Toolkit for Operator Algebraists*, University of Leeds, Cambridge University Press, 1995.
[7] G.J. Morphy, *C^*-Algebras and Operator Theory*, San Diego, California, Academic Press, 1990.

[8] L. Raeburn and D.P. Williams, *Morita Equivalence and Continuous-Trace C^*-Algebras*, Matemathical Surveys and Monographs, 1998.
[9] W. Sun, G-frames and g-Riesz bases, *J. Math. Anal. Appl.*, **322**(1) (2006), 437–452.
[10] N.E. Wegge Olsen, *K-Theory and C^*-Algebras, A Friendly Approch*, Oxford University Press, Oxford, England, 1993.