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On n-weak biamenability of Banach algebras

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ABSTRACT

In this paper, the notion of n-weak biamenability of Banach algebras is introduced and for every $n \geq 3$, it is shown that n-weak biamenability of the second dual A^{**} of a Banach algebra A implies n-weak biamenability of A and this is true for $n = 1, 2$ under some mild conditions. As a concrete example, it is shown that for every abelian locally compact group G, $L^1(G)$ is 1-weakly biamenable and $\ell^1(G)$ is n-weakly biamenable for every odd integer n.

1. Introduction

Let X, Y and Z be normed spaces, and let $f : X \times Y \to Z$ be a bounded bilinear mapping. Following [1], the adjoint of f, denoted by $f^* : Z^* \times X \to Y^*$, is defined by

$$\langle f^*(z^*, x), y\rangle = \langle z^*, f(x, y)\rangle \quad (x \in X, y \in Y, z^* \in Z^*).$$

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Similarly $f^{**} : Y^{**} \times Z^{*} \to X^{*}$ and $f^{***} : X^{**} \times Y^{**} \to Z^{**}$ are defined by $f^{**} = (f^{*})^{*}$ and $f^{***} = (f^{**})^{*}$, respectively. Let $f^{t} : Y \times X \to Z$ be the flip map of f defined by $f^{t}(y, x) = f(x, y)$. Then both f^{***} and f^{t***t} are bounded bilinear mappings extending f. If $f^{***} = f^{t***t}$, then f is said to be (Arens) regular.

A derivation from a Banach algebra A into a Banach A-module X is a bounded linear mapping $d : A \to X$ such that

$$d(ab) = d(a)b + ad(b) \quad (a, b \in A).$$

For every $x \in X$, the mapping $\delta_x : a \to ax - xa$ $(a \in A)$ is called an inner derivation.
Let X be a Banach A-module. Then X^{*} is a dual Banach A-module, considering $a \cdot f$ and $f \cdot a$, for every $a \in A$ and $f \in X^{*}$, as

$$a \cdot f(x) = f(xa), \quad f \cdot a(x) = f(ax) \quad (x \in X).$$

Similarly, the higher duals $X^{(n)}$ can be made into Banach A-modules in a natural fashion. We denote these left and right A-module actions by π_{ℓ_n} and π_{r_n}, respectively. Obviously for every $n \in \mathbb{N}$, $\pi_{\ell_n} = \pi_{r_{n-1}}^{t*t}$ and $\pi_{r_n} = \pi_{\ell_{n-1}}^{*}$, where π_{ℓ_0} and π_{r_0} are the left and right A-module actions π_ℓ and π_r on X, respectively.

A Banach algebra A is called n-weakly amenable ($n \in \mathbb{N}$), if every derivation from A to $A^{(n)}$ is an inner derivation. The concept of n-weak amenability is initiated and developed by Dales, Ghahramani, and Gronbæk in [6].

Let A be a Banach algebra, and let X be an A-module. A bounded bilinear mapping $D : A \times A \to X$ is called a biderivation, if D is a derivation with respect to both arguments. That is, the mappings ${}_aD : A \to X$ and $D_b : A \to X$ with

$${}_aD(b) = D(a, b) = D_b(a) \quad (a, b \in A)$$

are derivations. We denote the space of such biderivations by $BZ^1(A, X)$.

Let $x \in Z(A, X)$, where

$$Z(A, X) = \{x \in X; ax = xa \quad (a \in A)\}.$$

The map $\Delta_x : A \times A \to X$ with

$$\Delta_x(a, b) = x[a, b] = x(ab - ba) \quad (a, b \in A)$$

is a basic example of a biderivation and is called an inner biderivation. We denote the space of such inner biderivations by $BN^1(A, X)$.
Biderivations are a subject of research in various areas. The algebraic aspects of biderivations on certain algebras are investigated by several authors; see, for example, [4, 8], where the structures of biderivations on triangular algebras and generalized matrix algebras are discussed, and particularly the question of whether biderivations on these algebras are inner, is investigated. For more applications of biderivations in some other fields, see the survey article [5, Section 3].

Definition 1.1. Let n be a natural number. A Banach algebra A is n-weakly biamenable, if every biderivation from A to $A^{(n)}$ is an inner biderivation. A is called permanently weakly biamenable if A is n-weakly biamenable, for every $n \in \mathbb{N}$.

Note that despite the apparent similarity between derivations and biderivations and also inner derivations and inner biderivations, there are fundamental differences between them. Especially when a biderivation wants to be an inner bidetivation these differences become more apparent. A part of these differences comes from the nature of biderivations, which depends on two components. Another essential part of these differences goes back to the definition of inner biderivations, which the implemented elements should be in $Z(A, X)$. Accordingly, the concepts of amenability and also weak amenability are different from biamenability and weak biamenability, respectively [2].

Let A be a non-unital Banach algebra, and let $A^{\sharp} = A \oplus \mathbb{C}$ be its unitization with the product

$$(a, \alpha)(b, \beta) = (ab + \alpha b + \beta a, \alpha\beta) \quad (a, b \in A, \alpha, \beta \in \mathbb{C}).$$

Then $A^{\sharp}$ is a Banach algebra, and we can consider for every natural number n, $(A^{\sharp})^{(2n)} = A^{(2n)} \oplus \mathbb{C}$ and for every non-negative integer n, $(A^{\sharp})^{(2n+1)} = A^{(2n+1)} \oplus \mathbb{C}$. The module actions of $A^{\sharp}$ on $(A^{\sharp})^{(2n+1)}$ are given by

$$(a, \alpha)(\lambda, \gamma) = (a\lambda + \alpha\lambda, \alpha\gamma + \langle \lambda, a\rangle), \text{ and}$$

$$(\lambda, \gamma)(a, \alpha) = (\lambda a + \alpha\lambda, \alpha\gamma + \langle \lambda, a\rangle) \quad (a \in A, \lambda \in A^{(2n+1)}, \alpha, \gamma \in \mathbb{C}).$$

In this paper, we study the n-weak biamenability of a Banach algebra and its unitization and introduce some permanently weakly biamenable Banach algebras. In [3] it is shown that for every $n \geq 2$, n-weak amenability of A^{**} implies n-weak amenability of A and the same hold for $n = 1$ under some mild conditions. We obtain similar results for n-weak biamenability for $n \geq 3$, in general, and for $n = 1$ or $n = 2$, in certain cases. Also, we show that for every abelian locally compact group G, $L^1(G)$ is 1-weakly biamenable and, for every discrete abelian locally compact group G, $\ell^1(G)$ is n-weakly biamenable, for every odd integer n. For this, consider $L^{\infty}(G)$ as an $M(G)$-module with module actions

$$\langle f.\mu, g\rangle = \langle f, \mu * g\rangle, \quad \langle \mu.f, g\rangle = \langle f, g * \mu\rangle \quad (f \in L^{\infty}(G), g \in L^1(G), \mu \in M(G)).$$

We consider $L^{\infty}(G)$ with the w^*-topology and $M(G)$ with the strict topology. In strict topology, a net (μ_{α}) in $M(G)$ converges to $\mu \in M(G)$, if in the norm topology of $L^1(G)$ two convergences $\mu_{\alpha} * f \to \mu * f$ and $f * \mu_{\alpha} \to f * \mu$ hold for every $f \in L^1(G)$.

2. *n*-weak biamenability of a Banach algebra and its unitization

In [2] it is shown that, despite the apparent similarities between the concepts of amenability and biamenability of Banach algebras, they have very different and somewhat opposite properties in some cases. In this regard, it is shown that commutative Banach algebras and also unitization of Banach algebras are not biamenable, although they may be amenable. On the other hand, it is shown that there are some noncommutative Banach algebras which are biamenable while they are not amenable.

In this section, we study the n-weak biamenability of a Banach algebra and its unitization. Some known results on n-weak amenability (see [6] and [10]) are also studied for the concept of n-weak biamenability. Before these, Let us give some examples of permanently weakly biamenable Banach algebras.

If A is a commutative Banach algebra and $n \in \mathbb{N}$, then A is n-weakly biamenable if and only if the only biderivation $D : A \times A \to A^{(n)}$ is zero. For example, since $\mathbb{C}$ is amenable, there exists an $\alpha(a) \in \mathbb{C}$ such that

$$D(a, b) = \delta_{\alpha(a)}(b) = 0$$

for every biderivation $D : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and every $a, b \in \mathbb{C}$. This says that $\mathbb{C}$ is permanently weakly biamenable. Also [2, Example 2.7] states that the module extension Banach algebras $B(H) \oplus K(H)$ and $B(H) \oplus B(H)^{(2n)}$, and also $B(H)^{(2n)}$ are permanently weakly biamenable for every infinite-dimensional Hilbert space H and every integer $n \geq 0$.

Let A be a Banach algebra, and let $A^2 = span\{ab; a, b \in A\}$. We commence with the following lemmas.

Lemma 2.1. *If A is 1-weakly biamenable, then A^2 is dense in A.*

Proof. If A^2 is not dense in A, then there exists a nonzero linear functional $f \in A^*$ such that it is zero on A^2. Now the bilinear map

$$\begin{aligned} D : A \times A &\to A^* \\ (a, b) &\mapsto f(a)f(b)f \end{aligned}$$

is a biderivation, which is not inner. Indeed, since f is zero on A^2 we conclude that $D(ab, c), aD(b, c)+ D(a, c)b, D(a, bc)$ and $bD(a, c) + D(a, b)c$ are zero, for every $a, b, c \in A$. On the other hand since f is nonzero, there is $\xi \in A$ such that $f(\xi)$ and so $D(\xi, \xi)$ are nonzero. While for every inner biderivation Δ, we have $\Delta(\xi, \xi) = 0$.

So A is not 1-weakly biamenable, which is a contradiction. So A^2 is dense in A. □

Lemma 2.2. *If A is $(n + 2)$-weakly biamenable, then it is n-weakly biamenable.*

Proof. Let $D \in BZ^1(A, A^n)$. Then by embedding A^n in $A^{(n+2)}$ we can consider $D \in BZ^1(A, A^{(n+2)})$. Hence there is $\phi \in Z(A, A^{(n+2)})$ such that $D = \Delta_\phi$ in $A^{(n+2)}$. Now for the projection map $P : A^{(n+2)} \to A^n$, $a \in A$, $\psi \in A^{(n+2)}$ and $\Omega \in A^{(n-1)}$ we have

$$\begin{aligned} \langle P(a\psi), \Omega \rangle &= \langle a\psi, \Omega \rangle \\ &= \langle \psi, \Omega a \rangle \\ &= \langle P(\psi), \Omega a \rangle \\ &= \langle aP(\psi), \Omega \rangle; \end{aligned}$$

and similarly $P(\psi a) = P(\psi)a$. Therefore

$$\begin{aligned} P(\phi)a &= P(\phi a) \\ &= P(a\phi) \\ &= aP(\phi); \end{aligned}$$

and

$$\begin{aligned} P \circ \Delta_\phi(a,b) &= P(\phi[a,b]) \\ &= P(\phi)[a,b] \\ &= \Delta_{P(\phi)}(a,b). \end{aligned}$$

That is $P(\phi) \in Z(A, A^n)$ and $D = \Delta_{P(\phi)}$ in A^n. □

Theorem 2.3. *Let A be a non-unital Banach algebra, and let $n \in \mathbb{N}$.*

(i) Suppose that A is $(2n-1)$-weakly biamenable. Then $A^\sharp$ is $(2n-1)$-weakly biamenable.

(ii) If A is commutative or it has an approximate identity, then $(2n)$-weak biamenability of $A^\sharp$ implies $(2n)$-weak biamenability of A.

Proof. (i) Let $D : A \times \mathbb{C} \to A^{(n)}$ be a biderivation, for some n. Then

$$D(a, \beta) = \beta D(a, 1) = 0, \quad (a \in A, \beta \in \mathbb{C}).$$

Similarly the only biderivations $D : \mathbb{C} \times A \to A^{(n)}$ and $D : \mathbb{C} \times \mathbb{C} \to A^{(n)}$ are zero.
Now let $D : A^\sharp \times A^\sharp \to (A^\sharp)^{(2n-1)}$ be a biderivation, and let $P : (A^\sharp)^{(2n-1)} \to A^{(2n-1)}$ be the projection map. Then $\mathcal{D} : A \times A \to A^{(2n-1)}$ defined by $\mathcal{D}(a, b) = P \circ D((a, 0), (b, 0))$ is a biderivation, and therefore $\mathcal{D} = \Delta_F$ for some $F \in Z(A, A^{(2n-1)})$.
Now the first part of the proof implies that

$$\begin{aligned} D((a,\alpha),(b,\beta)) &= D((a,0),(b,0)) + D((a,0),(0,\beta)) \\ &\quad + D((0,\alpha),(b,0)) + D((0,\alpha),(0,\beta)) \\ &= D((a,0),(b,0)) \qquad (a,b \in A \text{ and } \alpha,\beta \in \mathbb{C}). \end{aligned} \tag{2.1}$$

Also for every $a, b, c \in A$ we have $D((ab, 0), (c, 0)) = (F[ab, c], \gamma)$ for some $\gamma \in \mathbb{C}$. On the other hand, there are $\zeta, \eta \in \mathbb{C}$ such that

$$\begin{aligned} D((ab,0),(c,0)) &= (a,0)D((b,0),(c,0)) + D((a,0),(c,0))(b,0) \\ &= (a,0)(F[b,c],\zeta) + (F[a,c],\eta)(b,0) \\ &= (aF[b,c],\langle F[b,c],a\rangle) + (F[a,c]b,\langle F[a,c],b\rangle) \\ &= (F[ab,c],\langle F,[ab,c]\rangle). \end{aligned}$$

Hence $\gamma = \langle F, [ab, c]\rangle$ and $D((ab, 0), (c, 0)) = (F, 0)([ab, c], 0)$. Also since A is $(2n-1)$-weakly biamenable, Lemmas 2.2 and 2.1 imply that A^2 is dense in A, and by applying (2.1), we have

$$\begin{aligned} D((a,\alpha),(b,\beta)) &= D((a,0),(b,0)) \\ &= (F,0)([a,b],0) \\ &= (F,0)[(a,\alpha),(b,\beta)] \quad (a,b \in A, \alpha,\beta \in \mathbb{C}). \end{aligned}$$

Now since $(F, 0) \in Z(A^\sharp, (A^\sharp)^{(2n-1)})$, D is inner and hence $A^\sharp$ is $(2n-1)$-weakly biamenable.

(ii) Let $D : A \times A \to A^{(2n)}$ be a biderivation. Then $\mathcal{D} : (A^\sharp) \times (A^\sharp) \to (A^\sharp)^{(2n)}$ defined by $\mathcal{D}((a,\alpha),(b,\beta)) = (D(a,b),0)$ is a biderivation. So there is $(\phi,\gamma) \in Z(A^\sharp,(A^\sharp)^{(2n)})$ such that

$$\begin{aligned}(D(a,b),0) &= \mathcal{D}((a,\alpha),(b,\beta))\\ &= (\phi,\gamma)[(a,\alpha),(b,\beta)]\\ &= (\phi,\gamma)([a,b],0)\\ &= (\phi[a,b]+\gamma[a,b],0).\end{aligned}$$

That is, $D(a,b) = \phi[a,b] + \gamma[a,b]$.
Now if A is commutative, then $D = 0$, and so A is $(2n)$-weakly biamenable.
Also if A has an approximate identity (e_α) and E is a w^*-cluster point of (e_α), then $\psi = \phi + \gamma E \in Z(A, A^{(2n)})$ and $D = \Delta_\psi$. □

3. *n*-weak biamenability of the second dual of a Banach algebra

We know that for every $n \geq 2$, n-weak amenability of A^{**} implies n-weak amenability of A and the same hold for $n = 1$ under some mild conditions [3]. In this section, we obtain similar results for n-weak biamenability for $n \geq 3$, in general, and for $n = 1$ or $n = 2$, under some conditions.

Lemma 3.1. *Let $a \in A$, and let $\varphi \in X^{(n)}$ for $n > 1$. Then*

(i) $\pi_{\ell_n}(a,\varphi) = \pi^{***}_{\ell_{n-2}}(a,\varphi)$.

(ii) $\pi_{r_n}(\varphi,a) = \pi^{***}_{r_{n-2}}(\varphi,a)$.

Proof. (i) Let $\lambda \in X^{(n-1)}$, and let $\{\omega_\alpha\}$ be a net in $X^{(n-2)}$ w^*-converging to φ. We have

$$\begin{aligned}\langle \pi_{\ell_n}(a,\varphi),\lambda\rangle &= \langle \varphi, \pi_{r_{n-1}}(\lambda,a)\rangle\\ &= \lim_\alpha \langle \pi_{r_{n-1}}(\lambda,a),\omega_\alpha\rangle\\ &= \lim_\alpha \langle \lambda, \pi_{\ell_{n-2}}(a,\omega_\alpha)\rangle\\ &= \langle \pi^{***}_{\ell_{n-2}}(a,\varphi),\lambda\rangle\end{aligned}$$

(ii) can be proved similarly. □

Note that the above lemma implies, by an inductive argument, that

$$\pi_{\ell_{2k}}(a,\varphi) = \pi_\ell^{(3k)}(a,\varphi), \quad \pi_{r_{2k}}(\varphi,a) = \pi_r^{(3k)}(\varphi,a)$$

and

$$\pi_{\ell_{2k+1}}(a,\psi) = \pi_{\ell_1}^{(3k)}(a,\psi) = \pi_r^{t*t3k}(a,\psi), \quad \pi_{r_{2k+1}}(\psi,a) = \pi_{r_1}^{(3k)}(\psi,a) = \pi_\ell^{3k+1}(\psi,a),$$

for every non-negative integer k and for every $a \in A, \varphi \in X^{(2k)}$ and $\psi \in X^{(2k+1)}$. So we arrive at the next lemma, where a Banach algebra A with a product π is considered as an A-module and also A^{**} with the first Arens product π^{***} is considered as an A^{**}-module. Although we have similar results with the second Arens product.

Lemma 3.2. *Let $F \in A^{**}$, $\phi \in A^{(n+2)}$ and $n \geq 0$. Then*

(i) $\pi^{***}_{\ell_n}(F, \phi) = (\pi^{***})_{\ell_n}(F, \phi)$.

(ii) $\pi^{***}_{r_n}(\phi, F) = (\pi^{***})_{r_n}(\phi, F)$.

Proof. (i) It is easy to see that

$$\pi^{***t*t}(a, \lambda) = \pi^{t*t***}(a, \lambda)$$

for every $a \in A$ and $\lambda \in A^{***}$. Consider the nets $\{a_\alpha\}$ in A and $\{\varphi_\beta\}$ in $A^{(n)}$, which are w^*-convergent to F and ϕ, respectively. In the case when $n = 2k + 1$, the latter lemma yields

$$\begin{aligned}
\pi^{***}_{\ell_n}(F, \phi) &= w^* - \lim_\alpha w^* \lim_\beta \pi_{\ell_n}(a_\alpha, \varphi_\beta) \\
&= w^* - \lim_\alpha w^* \lim_\beta \pi^{t*t(3k)}(a_\alpha, \varphi_\beta) \\
&= w^* - \lim_\alpha w^* \lim_\beta (\pi^{t*t***})^{(3k-3)}(a_\alpha, \varphi_\beta) \\
&= w^* - \lim_\alpha w^* \lim_\beta (\pi^{***t*t})^{(3k-3)}(a_\alpha, \varphi_\beta) \\
&= w^* - \lim_\alpha w^* \lim_\beta (\pi^{***})^{t*t(3k-3)}(a_\alpha, \varphi_\beta) \\
&= (\pi^{***})^{t*t(3k)}(F, \phi) \\
&= (\pi^{***})_{\ell_n}(F, \phi).
\end{aligned}$$

Also if n is even, then one can prove (i) with the same argument.
The proof of (ii) is similar. □

For every $n \geq 0$, let $J_n : A^{(n)} \to A^{(n+2)}$ be the inclusion map, and consider $A^{(n)}$ as a subspace of $A^{(n+2)}$. In the following lemma, we consider A^{**} with the first Arens product. Also, we can prove similar results with the second Arens product.

Lemma 3.3.

(i) For every $n \geq 3$, if $D : A \times A \to A^{(n)}$ is a biderivation, then $[J^*_{n-1} \circ D^{***}] : A^{**} \times A^{**} \to A^{(n+2)}$ is a biderivation.

(ii) Fix $n \in \mathbb{N}$. If $D : A \times A \to A^{(n)}$ is a biderivation and π_{r_n} and π_{ℓ_n} are Arens regular, then $D^{***} : A^{**} \times A^{**} \to A^{(n+2)}$ is a biderivation.

Proof. (i) Let F, G and H be elements of A^{**}, and let $\omega \in A^{(n-1)}$. Also suppose that $\{a_\alpha\}, \{b_\beta\}$, and $\{c_\gamma\}$ are three nets in A w^*-converging to F, G and H, respectively. Then since $n \geq 3$, we have

$$\pi^{***}_{\ell_{n-1}}(A^{**}, A^{(n-1)}) = \pi^{******}_{\ell_{n-3}}(A^{**}, A^{(n-1)}) = \pi^{***}_{\ell_{n-3}}(A^{**}, A^{(n-1)}) \subseteq A^{(n-1)}.$$

Hence

$$\begin{aligned}
\langle [J^*_{n-1} \circ D^{***}](F\Box G, H), \omega\rangle &= \lim_\alpha \lim_\beta \lim_\gamma \langle D(a_\alpha b_\beta, c_\gamma), \omega\rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \langle \pi_{r_n}(D(a_\alpha, c_\gamma), b_\beta), \omega\rangle \\
&+ \lim_\alpha \lim_\beta \lim_\gamma \langle \pi_{\ell_n}(a_\alpha, D(b_\beta, c_\gamma)), \omega\rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \langle D(a_\alpha, c_\gamma), \pi_{\ell_{n-1}}(b_\beta, \omega\rangle \\
&+ \lim_\alpha \lim_\beta \lim_\gamma \langle D(b_\beta, c_\gamma), \pi_{r_{n-1}}(\omega, a_\alpha)\rangle \\
&= \lim_\alpha \langle [J^*_{n-1} \circ D^{***}](a_\alpha, H), \pi^{***}_{\ell_{n-1}}(G, \omega)\rangle \\
&+ \langle [J^*_{n-1} \circ D^{***}](G, H), \pi^{***}_{r_{n-1}}(\omega, F)\rangle \\
&= \langle [J^*_{n-1} \circ D^{***}](F, H), \pi^{***}_{\ell_{n-1}}(G, \omega)\rangle \\
&+ \langle [J^*_{n-1} \circ D^{***}](G, H), (\pi^{***})_{r_{n-1}}(\omega, F)\rangle \\
&= \langle \pi^{***}_{r_n}([J^*_{n-1} \circ D^{***}](F, H), G), \omega)\rangle \\
&+ \langle (\pi^{***})_{\ell_n}(F, [J^*_{n-1} \circ D^{***}](G, H)), \omega\rangle \\
&= \langle (\pi^{***})_{r_n}([J^*_{n-1} \circ D^{***}](F, H), G), \omega)\rangle \\
&+ \langle (\pi^{***})_{\ell_n}(F, [J^*_{n-1} \circ D^{***}](G, H)), \omega\rangle.
\end{aligned}$$

Similarly we have

$$[J^*_{n-1} \circ D^{***}](F, G\Box H) = (\pi^{***})_{r_n}([J^*_{n-1} \circ D^{***}](F, G), H) + (\pi^{***})_{\ell_n}(G, [J^*_{n-1} \circ D^{***}](F, H)).$$

Therefore $[J^*_{n-1} \circ D^{***}]$ is a biderivation.
(ii) We have

$$\begin{aligned}
D^{***}(F\Box G, H) &= w^* - \lim_\alpha w^* - \lim_\beta w^* - \lim_\gamma D(a_\alpha b_\beta, c_\gamma) \\
&= w^* - \lim_\alpha w^* - \lim_\beta w^* - \lim_\gamma \pi_{r_n}(D(a_\alpha, c_\gamma), b_\beta) \\
&+ w^* - \lim_\alpha w^* - \lim_\beta w^* - \lim_\gamma \pi_{\ell_n}(a_\alpha, D(b_\beta, c_\gamma)) \\
&= \pi^{***}_{r_n}(D^{***}(F, H), G) + \pi^{***}_{\ell_n}(F, D^{***}(G, H)) \\
&= (\pi^{***})_{r_n}(D^{***}(F, H), G) + (\pi^{***})_{\ell_n}(F, D^{***}(G, H)).
\end{aligned}$$

Also, a similar argument shows that

$$D^{***}(F, G\Box H) = (\pi^{***})_{r_n}(D^{***}(F, G), H) + (\pi^{***})_{\ell_n}(G, D^{***}(F, H)).$$

□

Theorem 3.4.

(i) For every $n \geq 3$, n-weak biamenability of A^{**} implies n-weak biamenability of A.

(ii) If π_{ℓ_2} and π_{r_2} are Arens regular, then 2-weak biamenability of A^{**} implies 2-weak biamenability of A.

(iii) If π_{ℓ_1} and π_{r_1} are Arens regular, then 1-weak biamenability of A^{**} implies 1-weak biamenability of A.

Proof. (i) Let $D : A \times A \to A^{(n)}$ be a biderivation. Then by the latter lemma, $[J^*_{n-1} \circ D^{***}]$ is a biderivation and so there exists $\phi \in Z(A^{**}, A^{(n+2)})$ such that, for each $F, G \in A^{**}$,

$$[J^*_{n-1} \circ D^{***}](F, G) = (\pi^{***})_{r_n}(\phi, [F, G]).$$

In particular, for every $a, b \in A$, since $D(a, b) \in A^{(n)}$, we have

$$D(a, b) = [J^*_{n-1} \circ D^{***}](a, b) = (\pi^{***})_{r_n}(\phi, [a, b]) = \pi_{r_n}(J^*_{n-1}(\phi), [a, b]).$$

Now since $J^*_{n-1}(\phi) \in Z(A, A^{(n)})$, D is an inner biderivation and so A is n-weakly biamenable.
(ii) Consider $D : A \times A \to A^{**}$. Then the part (ii) of Lemma 3.3 implies that D^{***} is a biderivation. Therefore there exists $\rho \in A^{****}$ such that for every $F, G \in A^{**}$,

$$D^{***}(F, G) = (\pi^{***})_{r_2}(\rho, [F, G]).$$

In particular, for every $a, b \in A$, since $D(a, b) \in A^{**}$, we have

$$D(a, b) = D^{***}(a, b) = (\pi^{***})_{r_2}(\rho, [a, b]) = \pi_{r_2}(J^*_1(\rho), [a, b]).$$

That is D is an inner biderivation, and so A is 2-weakly biamenable.
(iii) The proof is similar to the proof of (ii). □

4. *n*-Weak biamenability of the group algebra $\ell^1(G)$

[9, Theorem 3], [6, Theorem 4.1] and the results of [7] show that for a locally compact group G, $L^1(G)$ is n-weakly amenable for every $n \in \mathbb{N}$. In what follows, we show that if G is abelian, then $\ell^1(G)$ is $(2n + 1)$-weakly biamenable. First of all note that if A is a unital Banach algebra and X is a unital Banach A-module, then for every biderivation $D : A \times A \to X$ and every invertible element a in A we have for every $b \in A$, $a^{-1}D(a, b) = D(a^{-1}a, b) - D(a^{-1}, b)a = -D(a^{-1}, b)a$, and similarly $a^{-1}D(b, a) = -D(b, a^{-1})a$.

Lemma 4.1. *Let G be a locally compact group, and let $D : L^1(G) \times L^1(G) \to L^\infty(G)$ be a biderivation. Then D has an extension to a biderivation $\mathcal{D} : M(G) \times M(G) \to L^\infty(G)$.*

Proof. Let (e_α) be the approximate identity of $L^1(G)$, and let $f \in L^1(G)$ and $\mu, \nu \in M(G)$. Cohen's factorization theorem implies that there are $g, h, s, t, u, v \in L^1(G)$ such that $f = g * h$, $h = s * t$, and

$h * \mu = u * v$. So

$$\begin{aligned}
\lim_\beta \lim_\alpha \langle D(\mu * e_\alpha, v * e_\beta), f\rangle &= \lim_\beta \lim_\alpha \langle D(\mu * e_\alpha, v * e_\beta), g * h\rangle \\
&= \lim_\beta \lim_\alpha \langle D(\mu * e_\alpha, v * e_\beta).g, h\rangle \\
&= \lim_\beta \lim_\alpha \langle D(\mu * e_\alpha * g, v * e_\beta) \\
&- \mu * e_\alpha.D(g, v * e_\beta), h\rangle \\
&= \lim_\beta (\langle D(\mu * g, v * e_\beta), h\rangle - \langle D(g, v * e_\beta), h * \mu\rangle) \\
&= \lim_\beta (\langle D(\mu * g, v * e_\beta), s * t\rangle - \langle D(g, v * e_\beta), u * v\rangle) \\
&= \lim_\beta (\langle D(\mu * g, v * e_\beta).s, t\rangle - \langle D(g, v * e_\beta).u, v\rangle) \\
&= \lim_\beta (\langle D(\mu * g, v * e_\beta * s) - v * e_\beta.D(\mu * g, s), t\rangle) \\
&- \lim_\beta (\langle D(g, v * e_\beta * u) - v * e_\beta.D(g, u), v\rangle) \\
&= \langle D(\mu * g, v * s), t\rangle - \langle D(\mu * g, s), t * v\rangle \\
&- \langle D(g, v * u), v\rangle - \langle D(g, u), v * v\rangle.
\end{aligned}$$

Therefore the existence of this limit shows that we can define

$$\mathcal{D}(\mu, v) = w^* - \lim_\beta w^* - \lim_\alpha D(\mu * e_\alpha, v * e_\beta).$$

The continuity of $\mathcal{D}$ on each argument follows from the latter equation. Also a similar argument as above, which shows that $\mathcal{D}(\mu * g, v) = \mathcal{D}(\mu, v).g + \mu.\mathcal{D}(g, v)$, can be applied to prove that $\mathcal{D}(\mu * \lambda, v) = \mathcal{D}(\mu, v).\lambda + \mu.\mathcal{D}(\lambda, v)$ and $\mathcal{D}(\mu, v * \lambda) = \mathcal{D}(\mu, v).\lambda + v.\mathcal{D}(\mu, \lambda)$ for every $\mu, v, \lambda \in M(G)$. □

Now we show that $L^1(G)$ is 1-weakly biamenable, if G is abelian.

Theorem 4.2. *Let G be a locally compact abelian group. Then $L^1(G)$ is 1-weakly biamenable.*

Proof. By Lemma 4.1, it is sufficient to show that every biderivation $D : M(G) \times M(G) \to L^\infty(G)$ is inner. For this, since G is abelian, we have $f\delta_x = \delta_x f$ for every $x \in G$ and $f \in L^\infty(G)$. So we have for every x, y, s, and t in G,

$$\begin{aligned}
D(\delta_x, \delta_y) &= D(\delta_{t^{-1}tx}, \delta_{s^{-1}sy}) \\
&= \delta_{t^{-1}} D(\delta_{tx}, \delta_{s^{-1}})\delta_{sy} + \delta_{(st)^{-1}} D(\delta_{tx}, \delta_{sy}) \\
&+ \delta_{s^{-1}} D(\delta_{t^{-1}}, \delta_{sy})\delta_{tx} + D(\delta_{t^{-1}}, \delta_{s^{-1}})\delta_{sytx} \\
&= \delta_x(\delta_{(tx)^{-1}} D(\delta_{tx}, \delta_{s^{-1}})\delta_s)\delta_y + \delta_{x(tx)^{-1}}\delta_{y(sy)^{-1}} D(\delta_{tx}, \delta_{sy}) \\
&+ \delta_y(\delta_{(sy)^{-1}} D(\delta_{t^{-1}}, \delta_{sy})\delta_t)\delta_x + D(\delta_{t^{-1}}, \delta_{s^{-1}})\delta_{styx} \\
&= (\delta_{(tx)^{-1}} D(\delta_{tx}, \delta_{s^{-1}})\delta_s + \delta_{(tx)^{-1}} D(\delta_{tx}, \delta_{sy})\delta_{y(sy)^{-1}})\delta_{xy} \\
&- (\delta_{(t)^{-1}} D(\delta_t, \delta_{s^{-1}})\delta_s + \delta_{t^{-1}} D(\delta_t, \delta_{sy})\delta_{(sy)^{-1}})\delta_{yx}.
\end{aligned}$$

Now if we define $\phi_y(s)$ as an element of $L^\infty(G)_{\mathbb{R}}$ by

$$\phi_y(s) = \sup\{\delta_{(t)^{-1}} ReD(\delta_t, \delta_{s^{-1}})\delta_s + \delta_{t^{-1}} ReD(\delta_t, \delta_{sy})\delta_{(sy)^{-1}}; t \in G\},$$

then $ReD(\delta_x, \delta_y) + \phi_y(s)\delta_{yx} = \phi_y(s)\delta_{xy}$. Therefore $ReD(\delta_x, \delta_y) = \phi_y(s)[\delta_x, \delta_y] = 0$. Similarly, we have $ImD(\delta_x, \delta_y) = 0$. Hence $D = 0$ and by Lemma 4.1 the only biderivation $D : L^1(G) \times L^1(G) \to L^\infty(G)$ is zero. That is $L^1(G)$ is 1-weakly biamenable. □

Theorem 4.3. *For every abelian locally compact group G and every $n \in \mathbb{N} \cup \{0\}$, $\ell^1(G)$ is $(2n+1)$-weakly biamenable.*

Proof. As we see in Theorem 4.2, $\ell^1(G)$ is 1-weakly amenable. Now let $n \in \mathbb{N}$, $F \in \ell^\infty(G)^{(2n)}$ and $\delta_x \in \ell^1(G)$, then since $F\delta_x = \delta_x F$, similar to proof of Theorem 4.2 we conclude for every biderivation $D : \ell^1(G) \times \ell^1(G) \to \ell^\infty(G)^{(2n)}$,

$$\begin{aligned} D(\delta_x, \delta_y) = &(\delta_{(tx)^{-1}} D(\delta_{tx}, \delta_{s^{-1}})\delta_s + \delta_{(tx)^{-1}} D(\delta_{tx}, \delta_{sy})\delta_{y(sy)^{-1}})\delta_{xy} \\ &- (\delta_{(t)^{-1}} D(\delta_t, \delta_{s^{-1}})\delta_s + \delta_{t^{-1}} D(\delta_t, \delta_{sy})\delta_{(sy)^{-1}})\delta_{yx}. \end{aligned}$$

and since $\ell^\infty(G)^{(2n)}$ is a commutative Von Neumann algebra and so an L^∞-space, similar the proof of Theorem 4.2, since the set of real parts of an L^∞-space is a complete vector latice and the set

$$\Gamma = \{\delta_{(t)^{-1}} ReD(\delta_t, \delta_{s^{-1}})\delta_s + \delta_{t^{-1}} ReD(\delta_t, \delta_{sy})\delta_{(sy)^{-1}}; t \in G\}$$

is bounded above, we can define $\phi_y(s)$ as the suprimum of Γ. Then a similar argument as Theorem 4.2 shows that every biderivation from $\ell^1(G) \times \ell^1(G)$ to $\ell^\infty(G)^{(2n)}$ is zero and so $\ell^1(G)$ is $(2n + 1)$-weakly biamenable. □

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