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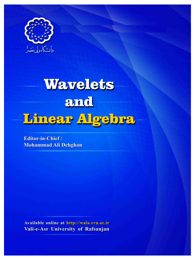

A note on zero Lie product determined nest algebras as Banach algebras

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Abstract

A Banach algebra $\mathcal{A}$ is said to be zero Lie product determined Banach algebra if for every continuous bilinear functional $\phi : \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ the following holds: if $\phi(a, b) = 0$ whenever $ab = ba$, then there exists some $\tau \in \mathcal{A}^*$ such that $\phi(a, b) = \tau(ab - ba)$ for all $a, b \in \mathcal{A}$. We show that any finite nest algebra over a complex Hilbert space is a zero Lie product determined Banach algebra.

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1. Introduction

Let $\mathcal{A}$ be an algebra over the complex field $\mathbb{C}$. For $a, b \in \mathcal{A}$, we will write $[a, b] = ab - ba$. We say that $\mathcal{A}$ is a *zero Lie product determined algebra* if for every $\mathcal{A}$-bimodule $\mathcal{X}$ and every bilinear map $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{X}$ satisfying

$$a, b \in \mathcal{A}, \ [a, b] = 0 \Rightarrow \phi(a, b) = 0, \tag{1.1}$$

there exists a linear map $T : [\mathcal{A}, \mathcal{A}] \to \mathcal{X}$, ($[\mathcal{A}, \mathcal{A}]$ is the linear span of all commutators of the algebra $\mathcal{A}$) such that $\phi(a, b) = T([a, b])$ for all $a, b \in \mathcal{A}$. An analytic analogue of this definition is as follows. Let $\mathcal{A}$ be a Banach algebra. We say that $\mathcal{A}$ is a *zero Lie product determined Banach algebra* if, for every continuous bilinear functional $\phi : \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ satisfying (1.1) there exists $\tau \in \mathcal{A}^*$ such that $\phi(a, b) = \tau([a, b])$ for all $a, b \in \mathcal{A}$. The concept of zero Lie product determined algebras was introduced in [8] and further studied in [16, 17, 19]. The original motivation for introducing this concept is arising from the paper [7]. Recently, the notion of zero Lie product determined Banach algebras has been defined and studied in [4, 3, 7]. In [4] it was shown that if $\mathcal{A}$ is a weakly amenable Banach algebra with property $\mathbb{B}$ and having a bounded approximate identity, then $\mathcal{A}$ is a zero Lie product determined Banach algebra. According to this result each C^*-algebra and the group algebra $L^1(G)$ of each locally compact group G are zero Lie product determined Banach algebras. In [15, Theorem 4.2] it was proved that a class of non-self adjoint operator algebras, called finite-dimensional nest algebras on a Hilbert space are zero Lie product determined. Hence they are zero Lie product determined Banach algebras. In this paper, we want to generalize this result to finite nest algebras on a Hilbert space. In particular, we show that every finite nest algebras on a Hilbert space is a zero Lie product determined Banach algebra. For this purpose we show that every finite nest algebra is weakly amenable and generated by its idempotents, that these can be interesting in their own right. Note that in [15] it was proved that any finite nest algebra on a Hilbert space is a zero product determined algebra (so it has property $\mathbb{B}$), and in [9] it was shown that every algebra which is generated by its idempotents is a zero product determined algebra. So our result that any finite nest algebra is generated by its idempotents is stronger than [15, Theorem 2.5 and Corollary 2.6].

In the second section of this article, the preliminaries and primary tools are given, and the third section is devoted to the results and proofs.

2. Preliminaries and tools

Let $\mathcal{A}$ be an algebra and $\mathcal{X}$ be an $\mathcal{A}$-bimodule. A *derivation* $\delta : \mathcal{A} \to \mathcal{X}$ is a linear map which satisfies $\delta(ab) = a\delta(b) + \delta(a)b$ for all $a, b \in \mathcal{A}$. The derivation δ is said to be *inner* if there exists $x \in \mathcal{X}$ such that $\delta(a) = ax - xa$ for all $a \in \mathcal{A}$. Recall that a Banach algebra $\mathcal{A}$ is said to be *weakly amenable* if every continuous derivation from $\mathcal{A}$ into $\mathcal{A}^*$ is inner. See [11] for a comprehensive survey of results and many interesting examples of weakly amenable Banach algebras. It should be noted that each C^*-algebra and the group algebra $L^1(G)$ of each locally compact group G are weakly amenable [11, Theorems 5.6.48 and 5.6.77].

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $\mathcal{M}$ be a Banach $(\mathcal{A}, \mathcal{B})$-bimodule. The *triangular Banach algebra* introduced in [13] is $Tri(\mathcal{A}, \mathcal{M}, \mathcal{B}) := \begin{pmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{pmatrix}$, with the usual matrix operations and l^1-norm. We will need the following result for weak amenability of triangular Banach algebras.

Lemma 2.1. (*[14, Corollary 3.5]*) *Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras and $\mathcal{M}$ be a unital Banach $(\mathcal{A}, \mathcal{B})$-bimodule. Then the triangular Banach algebra $Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is weakly amenable if and only if both $\mathcal{A}$ and $\mathcal{B}$ are weakly amenable.*

The algebra $\mathcal{A}$ is a *zero product determined algebra* if for every linear space $\mathcal{X}$ and every bilinear map $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{X}$ satisfying $\phi(a, b) = 0$ whenever $ab = 0$, there exists a linear map $T : \mathcal{A}^2 \to \mathcal{X}$ ($\mathcal{A}^2$ is the linear span of all elements of the form ab where $a, b \in \mathcal{A}$) such that $\phi(a, b) = T(ab)$ for all $a, b \in \mathcal{A}$. The notion of zero product determined algebras was introduced in [8], and has been studied by several authors (for instance, see [6] and the references therein). Brešar in [9] showed that every algebra which is generated by its idempotents is zero product determined. An analytic analogue of the concept of a zero product determined algebra is as follows. A Banach algebra $\mathcal{A}$ is said to have *property* $\mathbb{B}$ if for every Banach space $\mathcal{X}$ and every continuous bilinear map $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{X}$ with the property that $\phi(a, b) = 0$ whenever $ab = 0$, we have $\phi(ab, c) = \phi(a, bc)$ for all $a, b, c \in \mathcal{A}$. This concept was introduced in [2] and subsequently studied in several papers (see [1] and references therein). The class of Banach algebras with property $\mathbb{B}$ turns out to be quite large, in particular it includes C^*-algebras and group algebras of arbitrary locally compact groups. It is clear that every zero product determined Banach algebra has property $\mathbb{B}$. The concepts of a zero Lie product determined algebra and a zero Lie product determined Banach algebra can be seen as the Lie version of the notions of a zero product determined algebra and a Banach algebra having property $\mathbb{B}$, respectively. The next result in this article is essential.

Lemma 2.2. *[4, Corollary 2.8] Let $\mathcal{A}$ be a weakly amenable Banach algebra with property $\mathbb{B}$ and having a bounded approximate identity. Then $\mathcal{A}$ is a zero Lie product determined Banach algebra.*

Let $\mathcal{H}$ be a complex Hilbert space. We denote by $B(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$, and the identity element of $B(\mathcal{H})$ will be denoted by I, and an element P in an algebra is called an *idempotent* if $P^2 = P$. A nest $\mathcal{N}$ on $\mathcal{H}$ is a chain of closed (under norm topology) subspaces of $\mathcal{H}$ which is closed under the formation of arbitrary intersection (denoted by $\bigwedge$) and closed linear span (denoted by $\bigvee$), and which includes $\{0\}$ and $\mathcal{H}$. The nest algebra associated to the nest $\mathcal{N}$, denoted by $Alg\mathcal{N}$, is the weak closed operator algebra of the form

$$Alg\mathcal{N} = \{T \in B(\mathcal{H}) \mid T(N) \subseteq N \text{ for all } N \in \mathcal{N}\}.$$

When $\mathcal{N} \neq \{\{0\}, \mathcal{H}\}$, we say that $\mathcal{N}$ is non-trivial. It is clear that if $\mathcal{N}$ is trivial, then $Alg\mathcal{N} = B(\mathcal{H})$. If $\mathcal{N}$ is a finite set, then $Alg\mathcal{N}$ is called a *finite nest algebra* and if $\mathcal{H}$ is finite-dimensional, then $Alg\mathcal{N}$ is called a *finite-dimensional nest algebra*.

Remark 2.3. Let $\mathcal{N}$ be a non-trivial nest on a Hilbert space $\mathcal{H}$. If $N \in \mathcal{N} \backslash \{\{0\}, \mathcal{H}\}$ and P_N is the orthogonal projection onto N, then we have $(I - P_N)(Alg\mathcal{N})P_N = \{0\}$ and hence

$$Alg\mathcal{N} = P_N(Alg\mathcal{N})P_N \dotplus P_N(Alg\mathcal{N})(I - P_N) \dotplus (I - P_N)(Alg\mathcal{N})(I - P_N)$$

as sum of linear spaces. The sets $P_N(Alg\mathcal{N})P_N$ and $(I - P_N)(Alg\mathcal{N})(I - P_N)$ are unital Banach subalgebras of $Alg\mathcal{N}$ whose unit elements are P_N and $I-P_N$, respectively and $P_N(Alg\mathcal{N})(I-P_N)$ is a unital Banach $(P_N(Alg\mathcal{N})P_N, (I-P_N)(Alg\mathcal{N})(I-P_N))$-bimodule. The operator norm and l^1-norm are equivalent norms on $Alg\mathcal{N}$. On the other hand $\mathcal{M}_1 = P_N(\mathcal{N})$ and $\mathcal{M}_2 = (I-P_N)(\mathcal{N})$ are nests of N and $N^\perp = (I - P_N)(\mathcal{H})$, respectively. Moreover, we have the isometric isomorphisms $Alg\mathcal{M}_1 \cong P_N(Alg\mathcal{N})P_N$ and $Alg\mathcal{M}_2 \cong (I - P_N)(Alg\mathcal{N})(I - P_N)$, and so we can convert $P_N(Alg\mathcal{N})(I - P_N)$ to a unital Banach $(Alg\mathcal{M}_1, Alg\mathcal{M}_2)$-bimodule. Hence we have

$$Alg\mathcal{N} \cong Tri(Alg\mathcal{M}_1, P_N(Alg\mathcal{N})(I - P_N), Alg\mathcal{M}_2)$$

as isomorphism of Banach algebras.

For more information on nest algebras, we refer to [12].

3. Main result and proofs

The following is our main result.

Theorem 3.1. *Any finite nest algebra on a complex Hilbert space is a zero Lie product determined Banach algebra.*

We prove this theorem through the following lemmas.

Lemma 3.2. *Any finite nest algebra on a complex Hilbert space is weakly amenable.*

Proof. Let $Alg\mathcal{N}$ be a finite nest algebra on the complex Hilbert space $\mathcal{H}$. The proof is by induction on k, the number of elements in the nest. If $k = 2$, then the nest $\mathcal{N}$ with two elements is the trivial nest and hence $Alg\mathcal{N} = B(\mathcal{H})$. So by [11, Theorem 5.6.77] the result is obvious in this case.

Assume $k \geq 2$ and for each nest $\mathcal{N}$ with k elements on any complex Hilbert space $\mathcal{H}$ the nest algebra $Alg\mathcal{N}$ is weakly amenable.

Let $\mathcal{N} = \{\{0\} = N_0, N_1, \cdots, N_k = \mathcal{H}\}$ be a nest with $k + 1$ elements, then it is clear that $\mathcal{M}_1 = P_{N_1}(\mathcal{N})$ is a trivial nest on N_1 and $\mathcal{M}_2 = (I - P_{N_1})(\mathcal{N})$ is a finite nest with k elements on $N_1^\perp$. It follows from Remark 2.3 that

$$Alg\mathcal{N} \cong Tri(Alg\mathcal{M}_1, P_{N_1}(Alg\mathcal{N})(I - P_{N_1}), Alg\mathcal{M}_2)$$

as isomorphism of Banach algebras, where $Alg\mathcal{M}_1$ and $Alg\mathcal{M}_2$ are unital Banach algebras and $P_{N_1}(Alg\mathcal{N})(I - P_{N_1})$ is a unital Banach $(Alg\mathcal{M}_1, Alg\mathcal{M}_2)$-bimodule. The Banach algebra $Alg\mathcal{M}_1 = B(N_1)$ is weakly amenable and by induction hypothesis the nest algebra $Alg\mathcal{M}_2$ is weakly amenable. Therefore, by Lemma 2.1 the Banach algebra $Alg\mathcal{N}$ is weakly amenable. This completes the proof of the lemma. □

Lemma 3.3. *Any finite nest algebra on a complex Hilbert space is generated by its idempotents.*

Proof. Let $Alg\mathcal{N}$ be a finite nest algebra on the complex Hilbert space $\mathcal{H}$. The proof is by induction on k, the number of elements in the nest. If $k = 2$, then the nest $\mathcal{N}$ with two elements is the trivial nest and hence $Alg\mathcal{N} = B(\mathcal{H})$. If $\mathcal{H}$ is finite-dimensional, then $B(\mathcal{H}) \cong M_n(\mathbb{C})$ (the full matrix algebra over $\mathbb{C}$) is a von Neumann algebra so it is equal to the linear span of the orthogonal projections which it contains. If $\mathcal{H}$ is infinite dimensional, then by [18, Theorem 1], every element in $B(\mathcal{H})$ is a sum of five idempotents. So in any case $B(\mathcal{H})$ is generated by its idempotents.

Assume $k \geq 2$ and for each nest $\mathcal{N}$ with k elements on any complex Hilbert space $\mathcal{H}$ the nest algebra $Alg\mathcal{N}$ is generated by its idempotents.

Let $\mathcal{N} = \{\{0\} = N_0, N_1, \cdots, N_k = \mathcal{H}\}$ be a nest with $k+1$ elements, then $\mathcal{M}_1 = P_{N_1}(\mathcal{N})$ is a trivial nest on N_1 and $\mathcal{M}_2 = (I - P_{N_1})(\mathcal{N})$ is a finite nest with k elements on $N_1^{\perp}$. Denotes the subalgebra of $Alg\mathcal{N}$ generated by idempotents by $\mathcal{R}$. It follows from Remark 2.3 and induction hypothesis that $P_{N_1}(Alg\mathcal{N})P_{N_1} \cong Alg\mathcal{M}_1 = B(N_1)$ and $(I - P_{N_1})(Alg\mathcal{N})(I - P_{N_1}) \cong Alg\mathcal{M}_2$ are subalgebras of $Alg\mathcal{N}$ which are generated by their idempotents. Thus

$$P_{N_1}(Alg\mathcal{N})P_{N_1} \subseteq \mathcal{R} \quad \text{and} \quad (I - P_{N_1})(Alg\mathcal{N})(I - P_{N_1}) \subseteq \mathcal{R}.$$

Let $T \in Alg\mathcal{N}$. We have

$$(P_{N_1} + P_{N_1}T(I - P_{N_1}))^2 = P_{N_1} + P_{N_1}T(I - P_{N_1}).$$

Hence $P_{N_1} + P_{N_1}T(I - P_{N_1})$ is an idempotent, and so $P_{N_1} + P_{N_1}T(I - P_{N_1}) \in \mathcal{R}$. Since $P_{N_1} \in \mathcal{R}$, it follows that $P_{N_1}T(I - P_{N_1}) \in \mathcal{R}$ for all $T \in Alg\mathcal{N}$. Therefore,

$$P_{N_1}(Alg\mathcal{N})(I - P_{N_1}) \subseteq \mathcal{R}.$$

Hence

$$Alg\mathcal{N} = P_{N_1}(Alg\mathcal{N})P_{N_1} \dotplus P_{N_1}(Alg\mathcal{N})(I - P_{N_1}) \dotplus (I - P_{N_1})(Alg\mathcal{N})(I - P_{N_1}) \subseteq \mathcal{R},$$

and so $alg\mathcal{N} = \mathcal{R}$. This completes the proof of the lemma. □

Now, we are ready to prove Theorem 3.1.

Proof of Theorem 3.1: Let $Alg\mathcal{N}$ be a finite nest algebra on the complex Hilbert space $\mathcal{H}$. Since $Alg\mathcal{N}$ is generated by idempotents (by Lemma 3.3), it follows from [9] that the Banach algebra $Alg\mathcal{N}$ is a zero product determined algebra, and so it has property $\mathbb{B}$. On the other hand, by Lemma 3.2 the unital Banach algebra $Alg\mathcal{N}$ is weakly amenable. It now follows from Lemma 2.2 that $Alg\mathcal{N}$ is a zero Lie product determined Banach algebra. The proof of theorem is complete. □

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