Wavelets and Linear Algebra 3 (2016) 61 - 67

Vali-e-Asr University of Rafsanjan

Wavelets and Linear Algebra

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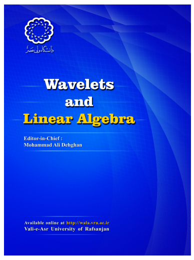

Some results on functionally convex sets in real Banach spaces

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Article Info

Article history:
Received 1 February 2016
Accepted 17 May 2016
Available online June 2016
Communicated by H. R. Afshin

Keywords:
F-convex set
convex set,

2000 MSC:
65F05, 46L05, 11Y50

Abstract

We use of two notions functionally convex (briefly, F–convex) and functionally closed (briefly, F–closed) in functional analysis and obtain more results. We show that if $\{A_\alpha\}_{\alpha\in I}$ is a family F–convex subsets with non empty intersection of a Banach space X, then $\bigcup_{\alpha\in I} A_\alpha$ is F–convex. Moreover, we introduce new definition of notion F–convexiy.

1. Introduction

In [5], M. Eshahgi, H. R. Reisi and A. R. Moazzen introduced two new notions in functional analysis. By defining functionally convex (briefly, F–convex) and functionally closed (briefly, F–

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closed) sets, they improved some basic theorems in functional analysis. Among other things, the Krein-Milman theorem has been generalized on finite dimensional Banach spaces. Hence, they have proved that, the set of extreme points of every bounded, F-convex and F-closed subset of a finite dimensional space is nonempty. Additionally, they partially proved the famous Chebyshev open problem (which asks whether or not every Chebyshev set in a Hilbert space is convex?). Hence, they have shown that, if A is a Chebyshev subset of a Hilbert space and the metric projection P_A is continuous, then A is F-convex

From now on, we suppose that all normed spaces and Banach spaces are real.

Definition 1.1. [5] In a normed space X, we say that $K(\subseteq X)$ is functionally convex (briefly, F–convex) if for every bounded linear transformation $T \in B(X, \mathbb{R})$, the subset $T(K)$ of $\mathbb{R}$ is convex.

Proposition 1.2. *[5] If T is a bounded linear mapping from a normed space X into a normed space Y, and K is F–convex in X, then $T(K)$ is F–convex in Y.*

Corollary 1.3. *[5] Let A, B be two F–convex subsets of a normed space X and λ be a real number, then*

$$A + B = \{a + b : a \in A, b \in B\}, \textit{ and } \quad \lambda A = \{\lambda.a : a \in A\}$$

are F–convex.

Proposition 1.4. *[5] Let A and B be F–convex subsets of a linear space X, which have nonempty intersection. Then $A \cup B$ is F–convex.*

Definition 1.5. [5] Let X be a normed space and let $A \subseteq X$. A is functionally closed (briefly, F–closed), if $f(A)$ is closed for all $f \in X^*$.

Note that every compact set is F–closed. Also, every closed subset of real numbers $\mathbb{R}$ is F–closed. In $X = \mathbb{R}^2$, the set $A = \{(x, y) : x, y \geq 0\}$ is (non-compact) F–closed whereas, the set $A = \mathbb{Z}\times\mathbb{Z}$ is closed but it is not F–closed (by taking $f(x, y) = x+ \sqrt{2}y$, the set $f(A)$ is not closed in $\mathbb{R}$). By taking $A = \{(x, y) : 1 \leq x^2 + y^2 \leq 4\}$ a nonconvex F–closed and F– convex set is obtained. Also, the set $B = \{(x, y) : x \in [0, \frac{\pi}{2}), y \geq \tan(x)\}$ is a closed convex set which is not F–closed. On the other hand, $A = \{(x, y) : 1 < x^2 + y^2 \leq 4\}$ is a non-compact and F–closed set. The two last examples show that weakly closed(weakly compact) and F–closed sets are different.

Remark 1.6. Note that we can not reduce definition of F–convexity to a basis of X^*, in the sence that a set in X is F–convex whenever its image under elements of a basis is convex. For instance, by taking the Euclidean space $\mathbb{R}^2$ and the set

$$\begin{aligned} A = \{(0,\alpha) : \alpha \in \mathbb{R} - \mathbb{Q} \cap [-\sqrt{2}, 1]\} \cup \{(\beta, 1) : \beta \in \mathbb{R} - \mathbb{Q} \cap [0, \sqrt{2}]\} \\ \cup \{(r, -\sqrt{2}) : r \in \mathbb{Q} \cap [0, \sqrt{2}]\} \cup \{(\sqrt{2}, s) : s \in \mathbb{Q} \cap \{[-\sqrt{2}, 1]\} \\ \cup \{(0, 1), (0, \sqrt{2}), (\sqrt{2}, -\sqrt{2}), (\sqrt{2}, 1)\} \end{aligned}$$

$p_x(x, y) = x$ and $p_y(x, y) = y$, projections on axis, is a base for $X = \mathbb{R}^2$ and $P_x(A) = [0, 1]$ also, $p_y(A) = [-\sqrt{2}, 1]$ but $f(x, y) = x + y$ is an element of X^* and $f(A)$ is not convex.

In [5], we prove the following theorem, which help us to find a big class of F–convex sets.

Theorem 1.7. *Every arcwise connected subset of a normed space X is F–convex.*

Remark 1.8. The converse of the above theorem is not valid. Hence, by taking $S = \{(x, \sin(\frac{1}{x}) : 0 < x \leq 1\}$, the set $\overline{S}$ which is called the sine's curve of topologist is connected and so for any linear functional $f \in (\mathbb{R} \times \mathbb{R})^*$, the set $f(\overline{S})$ is an interval. Thus, $\overline{S}$ is an F–convex set which is not arcwise connected.

2. Main Results

In this section, we show, how construct new subset F–convex one of given ones.

Proposition 2.1. *Let A, B be subsets of Banach space X. If A is F–convex and $A \subset B \subset \overline{A}$ then, B is F–convex.*

Proof. For every $f \in X^*$, we have $f(A) \subseteq f(B) \subseteq f(\overline{A}) \subseteq \overline{f(A)}$. Hence, by assumption, $f(\overline{A})$ is an interval. This completes the proof. □

Remark 2.2. In contrary the case of convex sets, interior of an F–convex set, necessarily is not F–convex. For instance, take $X = \mathbb{R} \times \mathbb{R}$ and let $B = \{(x, y) : x^2 + y^2 \leq 1\}$. Then if A is all elements surrounded by B and $B + \frac{1}{2}$ is F–convex, but the interior of A is not F–convex. Since, by taking f as projection on x-axis we have $f(A^\circ) = (-\frac{1}{2}, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{2})$, which is not convex.

Theorem 2.3. *Let $\{A_\alpha\}_{\alpha \in I}$ be collection of F–convex subsets in Banach space X. If $\bigcap_{\alpha \in I} A_\alpha \neq \phi$ then, $\bigcup_{\alpha \in I} A_\alpha$ is F–convex.*

Proof. For each $f \in X^*$ and $\alpha \in I$, we know, $f(A_\alpha)$ is an interval and $\bigcap_{\alpha \in I} f(A_\alpha) \neq \phi$. Thus, $f(\bigcup_{\alpha \in I} A_\alpha) = \bigcup_{\alpha \in I} f(A_\alpha)$ is convex. □

We know that, if $\{A_\alpha\}_{\alpha \in I}$ be a collection of connected subsets in X, A is connected and $A \bigcap A_\alpha \neq \phi$ for all $\alpha \in I$, then $A \bigcup (\bigcup_{\alpha \in I} A_\alpha)$ is connected. Now, we have the following theorem;

Theorem 2.4. *Let $\{A_\alpha\}_{\alpha \in I}$ be a collection of F–convex subsets in Banach space X. If A is F–convex and $A \bigcap A_\alpha \neq \phi$ for evrey $\alpha \in I$, then $A \bigcup (\bigcup_{\alpha \in I} A_\alpha)$ is F–convex.*

Proof. For evrey $f \in X^*$ and all $\alpha \in I$, $f(A_\alpha)$ and $f(A)$ are intervals such that $f(A) \cap f(A_\alpha) \neq \phi$. Therefore, $f(A \bigcup (\bigcup_{\alpha \in I} A_\alpha)) = \bigcup_{\alpha \in I} f(A_\alpha) \bigcup f(A)$ is interval for evrey $f \in X^*$. So, $A \bigcup (\bigcup_{\alpha \in I} A_\alpha)$ is F–convex. □

We know that, if $\{A_n\}_{n \in \mathbb{N}}$ be a collection of connected subsets in X such that $A_n \cap A_{n+1} \neq \phi$ for all $n \in \mathbb{N}$, then $\bigcup_{n \in \mathbb{N}} A_n$ is connected. Now, we have the following theorem;

Theorem 2.5. *Let $\{A_n\}_{n \in \mathbb{N}}$ be a collection of F–convex subsets in Banach space X. If $A_n \cap A_{n+1} \neq \phi$ for evrey $n \in \mathbb{N}$, then $\bigcup_{n \in \mathbb{N}} A_n$ is F–convex.*

Proof. For evrey $f \in X^*$ and all $n \in \mathbb{N}$, $f(A_n)$ is interval and $f(A_n) \cap f(A_{n+1}) \neq \phi$. Therefore, $f(\bigcup_{n \in \mathbb{N}} A_n) = \bigcup_{n \in \mathbb{N}} f(A_n)$ is interval for evrey $f \in X^*$. So, $\bigcup_{n \in \mathbb{N}} A_n$ is F–convex. □

Let A be a subset of linear space X. We define an equivalence relation on A as: $x \sim y$ if and only if both lie in a F–convex subset of A. The relation $\sim$ actually is an equivalence relation. For transitivity, note that if $x \sim y$ and $y \sim z$ then there are weakly convex subsets A and B such that $x, y \in A$ and $y, z \in B$. Proposition 1.4 asserts that $A \cup B$ is F–convex subset of X and so $x \sim z$.

Theorem 2.6. *Let $(X_i, \|.\|_i)$ be norm linear spaces, then $A_i \subset X_i$ are F–convex if and only if, $\prod_{i=1}^n A_i$ is F–convex in $\prod_{i=1}^n X_i$ equepted by the norm*

$$\|(x_1, x_2, \cdots, x_n)\| = \Big\{ \sum_{i=1}^n \|x_i\|_i^2 \Big\}^{\frac{1}{2}}.$$

Proof. We Know that

$$(\prod_{i=1}^n X_i)^* = \oplus_{i=1}^n X_i^*.$$

So, for every $g \in (\prod_{i=1}^n X_i)^*$ there are uniqe $f_i \in X_i^*, i = 1, 2, \cdots, n$ such that, $g = \sum_{i=1}^n f_i$. Now we have

$$g(\prod_{i=1}^n A_i) = \sum_{i=1}^n f_i(A_i).$$

Since, every A_i is F–convex so, $f_i(A_i)$ and their sum is an interval. Conversly, for every $f_i \in X_i^*$, taking $g = 0 + 0 + \cdots + f_i + \cdots + 0$, we have $f(A_i) = g(\prod_{i=1}^n A_i)$ so, A_i is F–convex. □

Theorem 2.7. *Let Y be a subspace of the norm linear space X. If $A \subset Y$ is F–convex then, A is F–convex in X.*

Proof. Let Y be a subspace of X. There exists subspace $Y^\perp$ of X such that $X = Y \oplus Y^\perp$. Thus, for evrey $f \in X^*$ we have, $f|_Y \in Y^*$. Now, if A is F–convex in Y, Therefore, $f(A) = f|_Y(A) + f(Y^\perp)$. By assumption, $f|_Y(A)$ is F–convex also, since $Y^\perp$ is a subspace, so $Y^\perp$ is F–convex in X. Thus, By using 1.3 $f(A)$ is F–convex in X. □

Definition 2.8. Let A be a subset of linear space X. Let $\frac{A}{\sim} = \{A_\alpha\}_{\alpha \in I}$ be the set of all equivalence classes. For each $\alpha \in I, A_\alpha$ is called F–convex component of A.

Theorem 2.9. *Let A be a subset of linear space X. The F–convex components of A are disjoint F–convex subsets of A whose their union is A, such that any non empty F–convex subset of A contains only one of them.*

Proof. Being equivalence classes, the F–convex component of A are disjoint and their union is A. Each F–convex subset of A contains only one of them. For if, A intersects the components A_1, A_2 of A say, in points x_1, x_2 respectively, then $x_1 \sim x_2$. this means $A_1 = A_2$. To show the F–convex component B is F–convex, choose a point x of B. For each $y \in B$, we know that $x_1 \sim x_2$, so there is a F–convex subset A_y containing x, y. By the result just proved $A_y \subset A$. thus, $B = \bigcup_{y \in A} A_y$. Since subsets A_y are F–convex and the point x is in their intersection, by 2.3 B is F–convex. □

Remark 2.10. Let A be a subset of linear space X. A is F–convex if and only if it has one F–convex component.

In the following theorem, for a subset A of a Banach space X, a necessary and sufficient condition for F–convexity is proved.

Theorem 2.11. *Let X be a Banach space, $A \subseteq X$ is F–convex if and only if*

$$co(A) \subseteq \bigcap_{f \in X^*} A + Ker(f).$$

Proof. The set $A \subseteq X$ is F–convex iff for all $f \in X^*$, the element $\sum_{i=1}^{n} \lambda_i f(a_i)$ belongs to $f(A)$ which, $\lambda_i \geq 0$, $a_i \in A$ and $\sum_{i=1}^{n} \lambda_i = 1$. This is equivalent that for all $f \in X^*$, there is $a \in A$ such that $a - \sum_{i=1}^{n} \lambda_i a_i \in Ker(f)$. □

Remark 2.12. Note that in special case $X = \mathbb{R}$, since every nonzero functional is one to one so we have $\bigcap_{f \in X^*} A + Ker(f) = A$. Thus $A \subseteq \mathbb{R}$ is F–convex iff $co(A) \subseteq A$. Also, we have $A \subseteq co(A)$. Then we obtain $A \subseteq \mathbb{R}$ is F–convex iff A is convex.

Let X be a vector space. A hyperplane in X (through $x_0 \in X$) is a set of the form $H = x_0 + Ker(f) \subseteq X$, where f is a non-zero linear functional on X. Equivalently, $H = f^{-1}(\gamma)$, where $\gamma = f(x_0)$. So, we have

$$\bigcap_{f \in X^*} A + Ker(f) = \bigcap_{f \in X^*} \bigcup_{a \in A} a + Ker(f) = \bigcap_{f \in X^*} f^{-1}(f(A)).$$

Hence, $A \subseteq X$ is F–convex if and only if

$$co(A) \subseteq \bigcap_{f \in X^*} f^{-1}(f(A)).$$

Proposition 2.13. *Let A be a subset of Banach space X. The set $U = \bigcap_{B \in \Gamma} \bigcap_{f \in X^*} f^{-1}(f(B))$ is F–convex, where $\Gamma = \{B : A \subseteq B, \quad B \text{ is F–convex}\}$.*

Proof. By discussion ago, we have $co(B) \subseteq \bigcap_{f \in X^*} f^{-1}(f(B))$. Intersecting on all $B \in \Gamma$, implies that

$$co(A) = \bigcap_{B \in \Gamma} co(B) \subseteq U \subseteq \bigcap_{f \in X^*} f^{-1}(f(co(A)))$$

On the other hand, for every $g \in X^*$,

$$g(co(A)) \subseteq g(U) \subseteq g(g^{-1}(g(co(A)))) \subseteq g(co(A))$$

Hence, for every $g \in X^*$, $g(U) = g(co(A))$. So U is F−convex. □

Theorem 2.14. *[3] If K_1 and K_2 are disjoint closed convex subsets of a locally convex linear topological space X, and if K_1 is compact, then there exist constants c and $\epsilon > 0$, and a continuous linear functional f on X, such that*

$$f(K_2) \leq c - \epsilon < c \leq f(K_1).$$

Lemma 2.15. *[5] If A is a subset of a Banach space X, then*

$$\bigcap_{f\in X^*} f^{-1}(f(A)) \subseteq \overline{co}(A)$$

Corollary 2.16. *[5] Let A be an F–closed subset of a Banach space X. Then A is F–convex if and only if*

$$\overline{co}(A) = \bigcap_{f\in X^*} f^{-1}(f(A)).$$

Corollary 2.17. *A compact subset A in a Banach space X is convex if and only if A is F–convex and X^* separates A and every element of $X - A$.*

Proof. If A is a compact convex subset of X, then by Theorem 2.14, the assertion holds. Conversely, assume that A is a compact F– convex subset of X. Hence, $\overline{co}(A) = \bigcap_{f\in X^*} f^{-1}(f(A))$. On the other hand, there is $f \in X^*$ such that for every $x \in X - A$, we have $f(A) < f(x)$. This implies that x is outside of $f^{-1}(f(A))$. Thus $f^{-1}(f(A)) = A$ and $\overline{co}(A) = A$. □

Remark 2.18. If X is a Hilbert space, then by Riesz representation theorem for every $f \in X^*$, there exists a unique $z \in X$ such that for all $x \in X$, $f(x) =< x, z >$, the inner product of x and z. Then

$$Ker(f) = \{x \in X :< x, z >= 0\} \doteq z^{\perp}.$$

In this case, we have

$$\bigcap_{f\in X^*} f^{-1}(f(A)) = \bigcap_{f\in X^*} A + Ker(f) = \bigcap_{z\in X} A + z^{\perp}. \tag{2.1}$$

Thus, in a Hilbert space X, every F–closed subset A of X is F–convex iff

$$\overline{co}(A) = \bigcap_{z\in X} A + z^{\perp}.$$

Corollary 2.19. *Let A and B be F–closed and F–convex subsets of a Banach space X which have nonempty intersection. Then*

$$\overline{co}(A \cup B) = \overline{co}(A) \cup \overline{co}(B).$$

Proof. By Proposition 1.4, $A \cup B$ is F–convex. Then we have

$$\begin{aligned} \overline{co}(A \cup B) &= \bigcap_{f\in X^*} f^{-1}(f(A \cup B)) \\ &= \Big(\bigcap_{f\in X^*} f^{-1}(f(A))\Big)\bigcup\Big(\bigcap_{f\in X^*} f^{-1}(f(A))\Big) \\ &= \overline{co}(A) \cup \overline{co}(B). \end{aligned}$$

□

Corollary 2.20. *Let A and B be F–closed and F–convex subsets of a Banach space X. Then*

$$\overline{co}(A + B) = \overline{co}(A) + \overline{co}(B).$$

Proof. Obviously, we have

$$\overline{co}(A + B) \subseteq \overline{co}(A) + \overline{co}(B).$$

Let x be an arbitrary element of $\overline{co}(A) + \overline{co}(B)$. Then there are $x_1 \in \overline{co}(A)$ and $x_2 \in \overline{co}(B)$ such that $x = x_1 + x_2$. Then for every $f \in X^*$, we have $f(x_1) \in f(A)$ and $f(x_2) \in f(B)$. This implies that $f(x_1 + x_2) \in f(A + B)$ and hence, $x \in f^{-1}(f(A + B))$. It follows that

$$\overline{co}(A) + \overline{co}(B) \subseteq \bigcap_{f \in X^*} f^{-1}(f(A + B)) = \overline{co}(A + B).$$

Note that if A and B are F–convex and F–closed then, $A + B$ is F–closed. □

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