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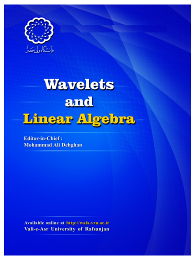

A note on λ-Aluthge transforms of operators

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Abstract

Let $A = U|A|$ be the polar decomposition of an operator A on a Hilbert space $\mathscr{H}$ and $\lambda \in (0, 1)$. The λ-Aluthge transform of A is defined by $\tilde{A}_\lambda := |A|^\lambda U|A|^{1-\lambda}$. In this paper we show that *i*) when $\mathscr{N}(|A|) = 0$, A is self-adjoint if and only if so is $\tilde{A}_\lambda$ for some $\lambda \neq \frac{1}{2}$. Also A is self-adjoint if and only if $A = \tilde{A}^*_\lambda$, *ii*) if A is normaloid and either $\sigma(A)$ has only finitely many distinct nonzero value or U is unitary, then from $A = c\tilde{A}_\lambda$ for some complex number c, we can conclude that A is quasinormal, *iii*) if A^2 is self-adjoint and any one of the $Re(A)$ or $-Re(A)$ is positive definite then A is self-adjoint, *iv*) and finally we show that

$$|||\,|A|^{2\lambda} + |A^*|^{2-2\lambda} \oplus 0\,||| \leq |||\,|A|^{2-2\lambda} \oplus |A|^{2\lambda}\,||| + |||\tilde{A}_\lambda \oplus (\tilde{A}_\lambda)^*|||$$

where $|||\cdot|||$ stands for some unitarily invariant norm. From that we conclude that $\||A|^{2\lambda} + |A^*|^{2-2\lambda}\| \leq \max(\|A\|^{2\lambda}, \|A\|^{2-2\lambda}) + \|\tilde{A}_\lambda\|$.

1. Introduction

In this paper $\mathbb{B}(\mathscr{H})$ denotes the algebra of all bounded linear operators on complex Hilbert space $\mathscr{H}$. An operator A is said to be self–adjoint if $A = A^*$. Let A be a bounded linear operator

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and $A = U|A|$ be the polar decomposition of A. So $U^*U|A| = |A|$ and $U|A|U^* = |A^*|$. We denote by $\mathscr{R}(A)$ and $\mathscr{N}(A)$ the range and the kernel of A, respectively; see [4]. For $0 < \lambda < 1$, the $\lambda-$Aluthge transform of A is defined by $\tilde{A}_\lambda = |A|^\lambda U|A|^{1-\lambda}$. This notation was first introduced by Aluthge in the case when $\lambda = \frac{1}{2}$ in [1] and it is a powerful tool in operator theory. We denote $\tilde{A}_{\frac{1}{2}}$ by $\tilde{A}$ and we call it the Aluthge transform of A. It follows easily from definition that $\|\tilde{A}_\lambda\| \leq \|A\|$. The sequence of the iterations of λ-Aluthge transform of A for $n > 1$ is defined in the obvious way, inductively, by $\tilde{A}_\lambda^{(n)} = \widetilde{\tilde{A}_\lambda^{(n-1)}}$. A surprising fact about this sequence is the convergence of their norms to the spectral radius of A; see [13]. A bounded linear operator A is said to be quasinormal if $|A|U = U|A|$ whenever $A = U|A|$ is the polar decomposition of A and to be p-hyponormal for some positive number p, if $(A^*A)^p \geq (AA^*)^p$. In the case when $p = 1$, A is called hyponormal. It is known that A is quasinormal if and only if $A = \tilde{A}$; see [5]. A class of p-hyponormal operators is contained in the greater class of operators named the class of normaloid operators. An operator A is said to be normaloid whenever $r(A) = \|A\|$ where $r(A)$ is the spectral radius of A. When A is normaloid it is easy to see that $\|A\| = \|\tilde{A}_\lambda\|$. In fact in this case the norms of iterations of the λ-Aluthge transform are the same as $r(A) = \|A\|$. An operator A is said to be qusiaffinity if $\mathscr{N}(A) = 0$ and A has dense range.

Recently the relationship between operators on a Hilbert space and their Aluthge transform have been investigated by many authors; see [2, 3, 5, 8, 9, 10, 12, 13, 14]. For example the authors of [8],[12] and [10] considered the problem that under what conditions, the normality of the Aluthge transform of an operator implies the normality of that operator. In the second section of this paper we express some results concerned with this issue with self-adjointness instead of normality. More precisely we prove that if $\tilde{A}_\lambda$ is self-adjoint for some $\lambda \neq \frac{1}{2}$ and $|A|$ is qusiaffinity, then A is self-adjoint. A is self-adjoint when $A = A^*$. We show that A is self-adjoint when $A = \tilde{A}^*$. Finally in this section we show that if $A = \lambda\tilde{A}$ for some complex number λ, then A is qusinormal. Also in this section we express some results about the problem that when can we conclude self-adjointness of an operator A from the self-adjointness of the second power of it i.e A^2.

In the third section we state an inequality for unitarily invariant norms. A norm $|||\cdot|||$ on appropriate norm ideal of $\mathbb{B}(\mathscr{H})$ is called unitarily invariant if it satisfies in invariance condition $|||UTV||| = |||T|||$ for all operator T and all unitary operators U and V. It follows easily from the basic properties of unitarily invariant norms that

i) $|||S||| = |||\,|S|\,||| = |||S^*|||$

ii) $|||S^*S||| = |||SS^*|||$

iii) $|||S \oplus T||| = \left|\left|\left|\begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix}\right|\right|\right|$.

The Schatten p-norms are the spatial case for such norms where $1 \leq p \leq \infty$. For $1 \leq p < \infty$ these norms are defined as follows. If T is a compact operator with decreasingly ordered singular values $s_j(T)$, which are the eigenvalues of $|T|$ in decreasing order and repeated according to the multiplicity, let

$$\|T\|_p = \left[\sum_{j=1}^{\infty} \left(s_j(T)\right)^p\right]^{\frac{1}{p}}.$$

This defines a norm, called Schatten p-norm, on C_p consisting of all operators T which $\|T\|_p < \infty$, called the Schatten p-class. In the case when $p = 2$, the $\|\cdot\|_2$ is said to be the Hilbert-Schmidt norm and C_2 is called the class of Hilbert-Schmidt operators. By convention, $\|T\|_\infty$ stands for the usual operator norm $\|T\|$. In this case C_∞ is $\mathbb{K}(\mathscr{H})$ the two-sided ideal of all compact operators on $\mathscr{H}$. If $A, B \in C_p$ then

$$\begin{aligned} \|A \oplus B\|_\infty &= \max\{\|A\|_\infty, \|B\|_\infty\}, \\ \|A \oplus B\|_p &= (\|A\|_p^p + \|B\|_p^p)^{\frac{1}{p}} \qquad (1 \le p < \infty). \end{aligned}$$

We refer the reader to [11] for further properties of the Schatten p-classes and unitarily invariant norms.

The following result of Hirzallah and Kittaneh is a version of well-known Young's inequality for Hilbert-Schmidt operators.

Proposition 1.1. *[7]Let S and T be positive operators and $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that $S^p, T^q \in C_2$. Then $ST \in C_2$ and*

$$\|ST\|_2^2 + \frac{1}{r^2}\|S^p - T^q\|_2^2 \le \left\| \frac{S^p}{p} + \frac{T^q}{q} \right\|_2^2,$$

where $r = \max\{p, q\}$.

In the last section using the fact that $|||AA^*||| = |||A^*A|||$ we show that

$$|||\,|A|^{2\lambda} + |A^*|^{2-2\lambda} \oplus 0\,||| \le |||\,|A|^{2-2\lambda} \oplus |A|^{2\lambda}\,||| + |||\tilde{A}_\lambda \oplus (\tilde{A}_\lambda)^*|||.$$

Due to it some inequalities are stated for Schatten p-norms as well.

2. Some conditions implying the self-adjointness of operators

We start this section with the following theorem. It states a relationship between operators and their Aluthge transforms.

Theorem 2.1. *Let A be an operator with $\mathscr{N}(|A|) = 0$ and let $\lambda \in (0, 1) - \{\frac{1}{2}\}$. Then $\tilde{A}_\lambda$ is self adjoint if and only if so is A.*

Proof. Let $A = U|A|$ be the polar decomposition and assume that $\tilde{A}_\lambda$ is self-adjoint and assume that $\lambda < 1 - \lambda$. Thus $\tilde{A}_\lambda = \tilde{A}_\lambda^*$ that is

$$|A|^\lambda U |A|^{1-\lambda} = |A|^{1-\lambda} U^* |A|^\lambda$$

hence

$$|A|^\lambda (U|A|^{1-2\lambda} - |A|^{1-2\lambda} U^*) |A|^\lambda = 0.$$

since $\mathscr{N}(|A|) = 0$ we have that

$$U|A|^{1-2\lambda} = |A|^{1-2\lambda} U^* \tag{2.1}$$

i.e. $U|A|^{1-2\lambda}$ is self–adjoint. This means that $U|A|^{1-2\lambda} = |A|^{1-2\lambda} U$ because $U|A|^{1-2\lambda}$ is the polar decomposition of a self-adjoint operator and the operators engaged in the polar decomposition of a normal operator (and consequently in a self-adjoint operator) commute; see [4]. This and (2.1) imply that $|A|^{1-2\lambda} U^* = |A|^{1-2\lambda} U$. So $U = U^*$ and this yields that U is unitary and A is self adjoint. When $\lambda > 1 - \lambda$ the result is obtained in the similar fashion. □

Remark 2.2. If $\lambda = \frac{1}{2}$, then the preceding Theorem is not valid. In fact there exist an operator A which its $\frac{1}{2}$-Aluthge transform is self adjoint but A is not self-adjoint. For instance see the Example 2.12 in [8].

Theorem 2.3. *Let A be an operator with $A = \widetilde{A}^*_\lambda$ for some $\lambda \in (0, 1)$. Then A is self-adjoint.*

Proof. Let $A = U|A|$ be the polar decomposition, thus $U|A| = |A|^{1-\lambda}U^*|A|^\lambda$ which implies that

$$(U|A|^{1-\lambda} - |A|^{1-\lambda}U^*)|A|^\lambda = 0.$$

Thus $U|A|^{1-\lambda} = |A|^{1-\lambda}U^*$ on $\overline{\mathscr{R}(|A|)}$. $\xi \in \mathscr{R}(|A|)^\perp = \mathscr{N}(|A|)$, hence $|A|^{1-\lambda}\xi = 0$ which yields $U|A|^{1-\lambda}\xi = 0$. We are going to show that $|A|^{1-\lambda}U^*\xi = 0$. Let $\zeta \in \mathscr{H}$ and $\zeta = \gamma + \eta$ which $\gamma \in \overline{\mathscr{R}(|A|)}$ and $\eta \in \mathscr{N}(|A|)$. Hence

$$\begin{aligned}
\langle |A|^{1-\lambda}U^*\xi, \zeta\rangle &= \langle |A|^{1-\lambda}U^*\xi, \gamma + \eta\rangle \\
&= \langle |A|^{1-\lambda}U^*\xi, \gamma\rangle + \langle |A|^{1-\lambda}U^*\xi, \eta\rangle \\
&= \langle |A|^{1-\lambda}U^*\xi, \gamma\rangle + \langle U^*\xi, |A|^{1-\lambda}\eta\rangle \\
&= \langle |A|^{1-\lambda}U^*\xi, \gamma\rangle \qquad \text{since} \quad |A|^{1-\lambda}\eta = 0 \\
&= \langle \xi, U|A|^{1-\lambda}\gamma\rangle \\
&= \langle \xi, |A|^{1-\lambda}U^*\gamma\rangle \qquad \text{since} \quad U|A|^{1-\lambda}\gamma = |A|^{1-\lambda}U^*\gamma \quad \text{on} \overline{\mathscr{R}(|A|)} \\
&= \langle U|A|^{1-\lambda}\xi, \gamma\rangle = 0
\end{aligned}$$

thus $|A|^{1-\lambda}U^*\xi = 0$. Therefore $U|A|^{1-\lambda}$ is self-adjoint which ensures that $U|A|^{1-\lambda} = |A|^{1-\lambda}U$. Hence $U^*|A|^{1-\lambda} = |A|^{1-\lambda}U^*$. Thus A is self-adjoint. □

Theorem 2.4. *Let $A = P + iQ$ be the Cartesian decomposition of an operator A and $P > 0$ or $P < 0$. If A^2 is self-adjoint then so is A.*

Proof. First we assume that $P > 0$. Obviously $A^2 = P^2 - Q^2 + i(PQ + QP)$ is the Cartesian decomposition. Since A^2 is self-adjoint we have that $PQ + QP = 0$. Thus

$$PQ = -QP \tag{2.2}$$

which implies that $P^2Q = QP^2$. Hence the positivity of P ensures that $PQ = QP$. So by (2.2) $QP = -QP$. Therefore $QP = 0$. Since P is invertible we reach to $Q = 0$ i.e. A is self-adjoint.

For the case when $P < 0$ we have that $-P > 0$, $-A = -P - iQ$ is Cartesian decomposition and $(-A)^2 = A^2$ is self-adjoint. hence $-Q = 0$ which implies that $Q = 0$. It follows that A is self-adjoint. □

Corollary 2.5. *Let $A = P + iQ$ be the Cartesian decomposition of operator A such that*

i) *either $P > 0$ or $P < 0$,*

ii) *either $Q > 0$ or $Q < 0$ and*

iii) *A^2 is self-adjoint.*

Then $A = 0$.

Proof. First assume that $P > 0$ and $Q > 0$. Thus $-iA = -iP + Q$ is the Cartesian decomposition and satisfies in the conditions of the preceding theorem. Thus $P = 0$. on the other hand since A itself satisfies in the condition of this theorem we have that $Q = 0$ which implies that $A = 0$. In the other cases the result is concluded in the similar manner. □

Remark 2.6. In the preceding theorem the condition that P is invertible is essential. For example consider 2 by 2 matrix $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ which is not self-adjoint, has the Cartesian decomposition $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + i\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and A^2 is self-adjoint. The following proposition asserts a condition on the A under which in this case the Theorem 2.4 is valid.

Proposition 2.7. *Let $A = P + iQ$ be the Cartesian decomposition of an operator A and $P \geq 0$ or $P \leq 0$. If A^2 is positive, then so is A.*

Proof. First we assume that $P \geq 0$. Obviously $A^2 = P^2 - Q^2 + i(PQ + QP)$ is the Cartesian decomposition. Since A^2 is positive we have that $PQ + QP = 0$ and $P^2 \geq Q^2$. So

$$PQ = -QP \tag{2.3}$$

which implies that $P^2Q = QP^2$. Hence by the positivity of P we have that $PQ = QP$. So by (2.3) $QP = -QP$ which yields $QP = 0$. Thus $Q = 0$ on $\overline{\mathscr{R}(P)}$. If $\xi \in \mathscr{N}(P)$, since $P^2 \geq Q^2$, we have that $Q\xi = 0$ which is $Q = 0$. In the case when $P \leq 0$ the proof is accomplished in the similar way. □

Proposition 2.8. *Let $A = U|A|$ be the polar decomposition and $A = \lambda\tilde{A}$ for some complex number λ. Then $U|A|^{\frac{1}{2}} = \lambda|A|^{\frac{1}{2}}U$ and $\lambda \geq 1$.*

Proof. Since $A = \lambda\tilde{A}$ we have that $U|A| = \lambda|A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$ which implies that $(U|A|^{\frac{1}{2}} - \lambda|A|^{\frac{1}{2}}U)|A|^{\frac{1}{2}} = 0$. So $U|A|^{\frac{1}{2}} - \lambda|A|^{\frac{1}{2}}U = 0$ on the $\overline{\mathscr{R}(|A|)}$. Now let $\xi \in \mathscr{N}(|A|)$. Thus $U|A|^{\frac{1}{2}}\xi = 0$ and $|A|^{\frac{1}{2}}U\xi = 0$ because $\mathscr{N}(|A|) = \mathscr{N}(U)$. Hence $|A|^{\frac{1}{2}} = \lambda U^*|A|^{\frac{1}{2}}U$ which yields that $\lambda > 0$ because $|A|^{\frac{1}{2}}$ and $U^*|A|^{\frac{1}{2}}U$ are positive. On the other hand we have that $\|A\| = \lambda\|\tilde{A}\| \leq \lambda\|A\|$ whence $\lambda \geq 1$. □

Corollary 2.9. *Let A be a non-zero operator such that A be either normaloid or $\sigma(A)$ has precisely n distinct nonzero values for some positive integer n and $A = \lambda\tilde{A}$. Then A is quasinormal.*

Proof. In the case when A is normaloid we have that $\|A\| = \lambda\|\tilde{A}\| = \lambda\|A\|$ which implies that $\lambda = 1$. If $\sigma(A)$ has n distinct non zero values then $\lambda^n = 1$ by [14, Theorem 2.4] which yields that $\lambda = 1$ because λ is positive. □

Proposition 2.10. *Let $A \neq 0$ and U two operators and U be unitary and $AU = \lambda UA$ for some complex number,then $|\lambda| = 1$. In addition If A is self-adjoint then $\lambda = \pm 1$ and if A is positive , then $\lambda = 1$.*

Proof. We have that $\|AU\| = |\lambda|\|UA\|$. Thus $\|A\| = |\lambda|\|A\|$ which implies that $|\lambda| = 1$. If A is self adjoint then $A = \lambda UAU^*$. Let $\xi \in \mathscr{H}$ such that $\langle A\xi, \xi\rangle$ is a non-zero element of $\mathbb{R}$. Therefore $\langle A\xi, \xi\rangle = \lambda\langle UAU^*\xi, \xi\rangle \in \mathbb{R}$ which implies that $\lambda \in \mathbb{R}$. Thus $\lambda = \pm 1$. If A is positive then $AU = -UA$ implies that $A^2U = UA^2$ thus $AU = UA$ by positivity of A i.e. $\lambda = 1$. □

Theorem 2.11. *Let $A = U|A|$ be the polar decomposition with U unitary and $A = \lambda\tilde{A}$ for some complex number λ, then A is normal.*

Proof. By Proposition 2.8 we see that $U|A|^{\frac{1}{2}} = \lambda|A|^{\frac{1}{2}}U$ and by Proposition 2.10 we have that $\lambda = 1$ Thus $U|A|^{\frac{1}{2}} = |A|^{\frac{1}{2}}U$ which implies that A is normal because U is unitary. □

3. A norm inequality for the Aluthge transform of operators

The following theorem expresses an inequality between unitarily invariant norms of operators and their Aluthge transform.

Theorem 3.1. *Let $A = U|A|$ be polar decomposition of A and $\lambda \in (0, 1)$. Then*

$$|||\,|A|^{2\lambda} + |A^*|^{2-2\lambda} \oplus 0\,||| \leq |||\,|A|^{2-2\lambda} \oplus |A|^{2\lambda}\,||| + |||\tilde{A}_\lambda \oplus (\tilde{A}_\lambda)^*|||$$

for any unitarily invariant norm $|||\cdot|||$.

Proof. Let $S = \begin{pmatrix} U|A|^{1-\lambda} & |A|^\lambda \\ 0 & 0 \end{pmatrix}$. Thus we have that

$$SS^* = \begin{pmatrix} U|A|^{2-2\lambda}U^* + |A|^{2\lambda} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} |A^*|^{2-2\lambda} + |A|^{2\lambda} & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$S^*S = \begin{pmatrix} |A|^{2-2\lambda} & (\tilde{A}_\lambda)^* \\ \tilde{A}_\lambda & |A|^{2\lambda} \end{pmatrix} = \begin{pmatrix} |A|^{2-2\lambda} & 0 \\ 0 & |A|^{2\lambda} \end{pmatrix} + \begin{pmatrix} 0 & (\tilde{A}_\lambda)^* \\ \tilde{A}_\lambda & 0 \end{pmatrix}$$

because $U^*U|A|^{1-\lambda} = |A|^{1-\lambda}$ and $U|A|^{2-2\lambda}U^* = |A^*|$. But for every unitarily invariant norm $|||\cdot|||$ we have $|||S^*S||| = |||SS^*|||$. Hence

$$\begin{aligned} |||\,|A|^{2\lambda} + |A^*|^{2-2\lambda} \oplus 0\,||| &= \left|\left|\left| \begin{pmatrix} |A|^{2-2\lambda} & (\tilde{A}_\lambda)^* \\ \tilde{A}_\lambda & |A|^{2\lambda} \end{pmatrix} \right|\right|\right| \\ &\leq |||\,|A|^{2-2\lambda} \oplus |A|^{2\lambda}\,||| + |||\tilde{A}_\lambda \oplus (\tilde{A}_\lambda)^*|||. \end{aligned}$$

□

Corollary 3.2. *Let $0 \leq p < \infty$ and $\lambda \in (0, 1)$. Then for operator A the following inequality is held,*

$$\|\,|A|^{2\lambda} + |A^*|^{2-2\lambda}\|_p \leq \left(\|\,|A|^{2-2\lambda}\|_p^p + \|\,|A|^{2\lambda}\|_p^p\right)^{\frac{1}{p}} + 2^{\frac{1}{p}}\|\tilde{A}_\lambda\|^p_. \tag{3.1}$$

In particular if $p = 2$ and $\lambda = \frac{1}{2}$ then for subtraction we have that

$$\|\,|A| - |A^*|\,\|_2^2 \leq 2\|A\|_2^2 - 2\|\tilde{A}\|_2^2 + 2\sqrt{2}\|A\|_2\|\tilde{A}\|_2. \tag{3.2}$$

Proof. Inequality (3.1) is obvious due to the fact that $\left\|\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right\|_p = \left(\|A\|_p^p + \|B\|_p^p\right)^{\frac{1}{p}}$ for any operator A and B. For (3.2) we use some idea of Antezana et al for [3, Proposition 3.8]. Using the identity equation $\|\tilde{A}\|_2 = \||A|^{\frac{1}{2}}|A^*|^{\frac{1}{2}}\|_2$ and Proposition 1.1 with $S = |A|^{\frac{1}{2}}$, $T = |A^*|^{\frac{1}{2}}$ and $p = q = 2$ we get

$$\|\tilde{A}\|_2^2 + \frac{1}{4}\||A| - |A^*|\|_2^2 \le \frac{1}{4}\||A| + |A^*|\|_2^2 \le \frac{1}{2}\|A\|_2^2 + \frac{1}{2}\|\tilde{A}\|_2^2 + \frac{\sqrt{2}}{2}\|A\|_2\|\tilde{A}\|_2$$

where the last inequality is (3.1) for $p = 2$ and $\lambda = \frac{1}{2}$. □

Remark 3.3. For $p \ge 1$ and $\lambda \in (0,1)$, define $\mathcal{S}_{p,\lambda} := \{A \in \mathbb{B}(\mathscr{H}); \|\tilde{A}_\lambda\|_p \le (2^{1-\frac{1}{p}} - 1)\|A\|_p\}$. A routine computation shows that the inequality (3.1) is sharper than triangle inequality on $\mathcal{S}_{p,\lambda}$. In particular The inequality (3.2) is sharper than inequality Proposition 3.8 in [3] on set $\mathcal{S}_{2,\frac{1}{2}}$. In [3] the following inequality was obtained

$$\frac{1}{4}\||A| - |A^*|\|_2^2 \le \|A\|_2^2 - \|\tilde{A}\|_2^2.$$

The combination of this inequality and inequality (3.2) brings us to the following one

$$\||A| - |A^*|\|_2^2 \le \min\{2\|A\|_2^2 - 2\|\tilde{A}\|_2^2 + 2\sqrt{2}\|A\|_2\|\tilde{A}\|_2, 4(\|A\|_2^2 - \|\tilde{A}\|_2^2)\}.$$

The most interesting case in the Theorem 3.1 is related to the case when the norm $|||\cdot|||$ is the usual operator norm.

Corollary 3.4. *For operator A we have that*

$$\||A|^{2\lambda} + |A^*|^{2-2\lambda}\| \le \max(\|A\|^{2\lambda}, \|A\|^{2-2\lambda}) + \|\tilde{A}_\lambda\|.$$

Remark 3.5. Since $\|\tilde{A}_\lambda\| \le \|A\|$, we see that the inequality Corollary 3.4 is a refinement of the triangle inequality for this case. It has to be stated that if U is unitary, then this inequality is obtained from the one of Kittaneh [6, Theorem 3.2].

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