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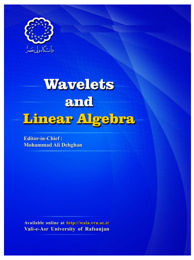

Inverse Young inequality in quaternion matrices

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ABSTRACT

Inverse Young inequality asserts that if $\nu > 1$, then

$$|zw| \geq \nu|z|^{\frac{1}{\nu}} + (1-\nu)|w|^{\frac{1}{1-\nu}},$$

for all complex numbers z and w, and equality holds if and only if $|z|^{\frac{1}{\nu}} = |w|^{\frac{1}{1-\nu}}$. In this paper the complex representation of quaternion matrices is applied to establish the inverse Young inequality for matrices of quaternions. Moreover, a necessary and sufficient condition for equality is given.

1. Introduction

Many of the most common inequalities of classic analysis have been extended to more general C^*-algebras. Since the C^*-algebras of matrices with elements from the complex field or from the quaternion skew field have applications in physics and mechanics, there is interest extending these inequalities for operators in these C^*-algebras.
An extension of the triangle inequality to $M_n(\mathbb{C})$ is made by Thompson in [10] and [11]. In

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[12], Thompson showed that a version of triangle inequality could be formulated for quaternion matrices. Two other important inequalities having generalisations in the matrix setting are the Young inequality and its special case, the arithmetic-geometric mean inequality. Generalization of the former to $M_n(\mathbb{C})$ and to $M_n(\mathbb{H})$ with analysing the case of equality, are respectively due to Ando [1] and to Zeng [13] while of the latter is due to Bhatia and Kittaneh [3]. One may also refer to [2, 4, 5] for more extensions of the Young inequality to some other C^*-algebras.
Young inequality refers to the following elementary, though fundamental, inequality between the moduli of any pair of complex numbers z, w:

$$|zw| \leq \frac{|z|^p}{p} + \frac{|w|^q}{q}\ , \tag{1.1}$$

where, p and q denote any positive real numbers with the property that

$$\frac{1}{p} + \frac{1}{q} = 1\ . \tag{1.2}$$

Furthermore, it is well known that

$$|zw| = \frac{|z|^p}{p} + \frac{|w|^q}{q} \quad \text{if and only if} \quad |w|^q = |z|^p\ . \tag{1.3}$$

Young inequality can also be written as

$$|zw| \leq \nu|z|^{\frac{1}{\nu}} + (1-\nu)|w|^{\frac{1}{1-\nu}}, \quad \text{where} \ \ \nu \in (0,1)\,. \tag{1.4}$$

A very close inequality to the Young inequality, which may be called the inverse Young inequality is

$$|zw| \geq \nu|z|^{\frac{1}{\nu}} + (1-\nu)|w|^{\frac{1}{1-\nu}}, \tag{1.5}$$

in which $\nu > 1$. Comparing (1.4) to (1.5) clarifies why we call (1.5) the inverse Young inequality. In [9] we established the following extension of (1.5) to complex matrices.

Theorem 1.1. *Let A and B be non singular $n \times n$ complex matrices and $\nu \in (1, \infty)$. Then there exists a unitary matrix U such that*

$$U^*|AB^*|\,U \geq \nu\,|A|^{\frac{1}{\nu}} + (1-\nu)|B|^{\frac{1}{1-\nu}}\ . \tag{1.6}$$

In addition, [9] contains the necessary and sufficient condition for equality in (1.6).

Theorem 1.2. *Equality holds in Theorem 1.1 if and only if $|A|^{\frac{1}{\nu}} = |B|^{\frac{1}{1-\nu}}$.*

Our main purpose in this paper is to extend the inverse Young inequality to quaternion matrices as a sequel to [9].

2. Complex representation of quaternion matrices

The vector space over the real number field with a four elements basis $\{1, i, j, k\}$ satisfying the multiplication laws

$$\begin{array}{lll} i^2 = j^2 = k^2 = -1 & , \; ijk = -1 & \\ ij = -ji = k & , \; jk = -kj = i \;\; , & ki = -ik = j \end{array} \tag{2.1}$$

and with 1 acting as the unity element is known as the real quaternion algebra $\mathbb{H}$. Any element in $\mathbb{H}$ can be written as $q = a_0 + a_1 i + a_2 j + a_3 k$ where a_0, a_1, a_2, and a_3 are real numbers. The conjugate $\overline{q}$ and the modulus $|q|$ of q are defined to be $\overline{q} = a_0 - a_1 i - a_2 j - a_3 k$ and $|q| = \sqrt{q\overline{q}}$. Not surprisingly, the map $\langle \cdot, \cdot \rangle : \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ defined by $\langle q_1, q_2 \rangle = q_1 \overline{q_2}$ is an inner product on the right real vector space $\mathbb{H}$.
The multiplication rules (2.1) allow us to write every quaternion q in the form $q = z_1 + z_2 j$ where $z_1 = a_0 + a_1 i$ and $z_2 = a_2 + a_3 i$ are complex numbers. Therefore, for every $n \times n$ quaternion matrix A, there exsit unique matrices A_1 and A_2 such that $A = A_1 + A_2 j$. The matrix

$$\begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix} \in M_{2n}(\mathbb{C})$$

is called the *complex representation* of A and is denoted by $\Psi(A)$.
Various properties of the complex representations of real quaternion matrices can be proved easily (See [8] for a slightly different definition of the complex representation and proofs of the following statements.)

Theorem 2.1. *Let $A, B, C \in M_n(\mathbb{H})$ and $r \in \mathbb{R}$ be given then*

(a) $A = B$ if and only if $\Psi(A) = \Psi(B)$,
(b) $\Psi(A + B) = \Psi(A) + \Psi(B)$, $\quad \Psi(AC) = \Psi(A)\Psi(C)$, $\quad \Psi(rA) = \Psi(Ar) = r\Psi(A)$,
(c) $\Psi(A^*) = (\Psi(A))^*$,
(d) A is invertible if and only if $\Psi(A)$ is invertible and $\Psi(A^{-1}) = (\Psi(A))^{-1}$,
(e) A is normal, Hermitian or unitary if and only if $\Psi(A)$ is so.

Thus Ψ is an injective $*$-homomorphism from the real algebra $M_n(\mathbb{H})$ into the real algebra $M_{2n}(\mathbb{C})$.
A quaternion q is said to be a *right eigenvalue* of the quaternion matrix A if there exists a non zero vector $\xi \in \mathbb{H}^n$ such that $A\xi = \xi q$.
The spectrum of a quaternion matrix A is defined to be the set of all roots of the minimal polynomial annihilating A. [6] is a good reference for a complete discussion of right eigenvalues and the ways they differ from the spectrum. The following theorem characterizes the complex right eigenvalues of a quaternion matrix.

Theorem 2.2. *The complex right eigenvalues of a quaternion matrix A are exactly the eigenvalues of its complex representation $\Psi(A)$.*

Proof. Express A as $A = A_1 + A_2 j$ where $A_1, A_2 \in M_n(\mathbb{C})$. Let λ be a complex number. Then the equation $A\xi = \xi\lambda$ for a nonzero vector $\xi = \xi_1 + \xi_2 j \in \mathbb{H}^n$, where $\xi_1, \xi_2 \in \mathbb{C}^n$, is equivalent to the equation

$$A_1\xi_1 - A_2\overline{\xi_2} + (A_1\xi_2 + A_2\overline{\xi_1})j = \xi_1\lambda + \xi_2\overline{\lambda}j\,.$$

This in turn is the same as the system

$$\begin{cases} A_1\xi_1 - A_2\overline{\xi_2} = \xi_1\lambda \\ A_1\xi_2 + A_2\overline{\xi_1} = \xi_2\overline{\lambda}\,. \end{cases}$$

Taking the complex conjugate of the second equation of the system and multiplying it with -1, we obtain the equivalent system

$$\begin{cases} A_1\xi_1 - A_2\overline{\xi_2} = \xi_1\lambda \\ -\overline{A_1}\,\overline{\xi_2} - \overline{A_2}\xi_1 = -\overline{\xi_2}\lambda\,. \end{cases}$$

which can be written in the matrix form as

$$\begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix}\begin{bmatrix} \xi_1 \\ -\overline{\xi_2} \end{bmatrix} = \begin{bmatrix} \xi_1 \\ -\overline{\xi_2} \end{bmatrix}\lambda\,.$$

Therefore λ is a right eigenvalue of A if and only if λ is an eigenvalue of $\Psi(A)$. □

Remark 2.3. As a complementary result to Theorem 2.2, we mention that if r is a real right eigenvalue of A then the geometric multiplicity of r as an eigenvalue of $\Psi(A)$ is twice that for as a right eigenvalue of A, see [6].

A useful result concerning the spectra of Hermitian quaternion matrices is:

Theorem 2.4. *If $A \in M_n(\mathbb{H})$ is Hermitian then every right eigenvalue of A is real.*

Proof. The complex representation $\Psi(A)$ of A is Hermitian by Theorem 2.1. Now, Theorem 2.2 implies that right eigenvalues of A are all real. □

A different proof of Theorem 2.4 using the Spectral Theorem for quaternion matrices can be found in [6]. If a quaternion matrix A is Hermitian we will always arrange its eigenvalues (which by Theorem 2.4 are all real numbers) in non increasing order.

3. Main results

A quaternion matrix A is said to be positive semidefinite if, for each $\xi \in \mathbb{H}^n$, $\langle A\xi, \xi\rangle \geq 0$ where the quaternion inner product on $\mathbb{H}^n$ is defined for $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ and $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$ as

$$\langle \xi, \eta\rangle = \sum_{i=1}^{n} \xi_i\overline{\eta_i}.$$

The set $P_n(\mathbb{H})$ of positive semidefinite quaternion matrices is a closed convex cone in the set $H_n(\mathbb{H})$ of all Hermitian quaternion matrices and therefore the relation

$$A \geq B \quad \text{if and only if} \quad A - B \in P_n(\mathbb{H})\,.$$

defines a partial order on $H_n(\mathbb{H})$.
Two useful equivalent conditions (the same as those for the complex case) for a quaternion matrix A to be positive semidefinite are listed in the following lemma.

Lemma 3.1. *Let A be a quaternion matrix.Then the following statements are equivalent.*

(a) A is positive semidefinite,
(b) A is Hermition and $\sigma(A) \subseteq \mathbb{R}_0^+$,
(c) $A = B^*B$ *for some* $B \in M_n(\mathbb{H})$.

[6] (resp. [7]) contains a proof that (a) and (b) (resp. (a) and (c)) are equivalent.

The following proposition gives another such condition.

Proposition 3.2. *A quaternion matrix A is positive semidefinite if and only if its complex representation* $\Psi(A)$ *is positive semidefinite.*

Proof. By (e) of Theorem 2.1, A is Hermitian if and only if $\Psi(A)$ is Hermitian and by Theorem 2.2, the spectra of A is contained in $\mathbb{R}_0^+$ if and only if the spectra of $\Psi(A)$ is contained in $\mathbb{R}_0^+$. □

A straight consequence of Proposition 3.2 and the linearity property of Ψ is the following.

Corollary 3.3. *Let A and B be Hermition quaternion matrices. Then* $A \geq B$ *if and only if* $\Psi(A) \geq \Psi(B)$.

For every $A \in M_n(\mathbb{H})$, A^*A is positive semidefinite by Lemma 3.1. It therefore, via the Spectral Theorem for quaternion matrices (see for example [6]), has a unique positive square root $(A^*A)^{1/2}$ which we denote by $|A|$ and call it the modulus of A.

Lemma 3.4. *For each* $A \in M_n(\mathbb{H})$, $\Psi(|A|) = |\Psi(A)|$.

Proof. By definition of the matrix modulus and the fact that Ψ is a $*$-homomorphism we have

$$|\Psi(A)|^2 = (\Psi(A))^*\Psi(A) = \Psi(A^*)\Psi(A) = \Psi(A^*A) = \Psi(|A|^2) = (\Psi(|A|))^2 .$$

Since both $|\Psi(A)|$ and $\Psi(|A|)$ are positive semidefinite complex matrices, we can take square roots to obtain $\Psi(|A|) = |\Psi(A)|$. □

The following easy proposition plays an important role in the proof of our main results.

Proposition 3.5. *Let A and B be quaternion matrices. Then*

(a) $\Psi(|A|^r) = |\Psi(A)|^r$ *for each non negative real number r,*
(b) If A is invertible then $\Psi(|A|^r) = |\Psi(A)|^r$ *for each real number r,*
(c) $\Psi(|AB^*|) = |\Psi(A)(\Psi(B))^*|$.

Proof. Note first that

$$(\Psi(|A|^{1/2}))^2 = \Psi(|A|^{1/2}) \cdot \Psi(|A|^{1/2}) = \Psi(|A|^{1/2} \cdot |A|^{1/2}) = \Psi(|A|) = |\Psi(A)|,$$

where in the last equality we made use of Lemma 3.4. This gives us $\Psi(|A|^{1/2}) = |\Psi(A)|^{1/2}$. Using induction on n and the fact that Ψ is a homomorphism shows that for each natural number n,

$$\Psi(|A|^{\frac{1}{2^n}}) = |\Psi(A)|^{\frac{1}{2^n}}$$

and a second induction implies that

$$\Psi(|A|^{\frac{m}{2^n}}) = |\Psi(A)|^{\frac{m}{2^n}}, \quad (m, n \in \mathbb{N}).$$

Since the set of all $\frac{m}{2^n}$ is dense in the set of all nonnegative real numbers, we see that for each nonnegative real r, $\Psi(|A|^r) = |\Psi(A)|^r$, proving (a).
If A is invertible then by Theorem 2.1, $\Psi(|A|^{-1}) = |\Psi(A)|^{-1}$. Applying (a) then we have for each $r < 0$,

$$\Psi(|A|^r) = \Psi((|A|^{-r})^{-1}) = (\Psi(|A|^{-r}))^{-1} = (|\Psi(A)|^{-r})^{-1} = |\Psi(A)|^r$$

proving (b).
(c) follows as a combination of Ψ being a $*$-homomorphism and Lemma 3.4. □

We are now in a position to prove a generalisation of the inverse Young inequality (1.1) for quaternion matrices.

Theorem 3.6. *For each pair of invertible $n \times n$ quaternion matrices A and B and each $\nu > 1$, a unitary $n \times n$ quaternion matrix U exists such that*

$$U|AB^*|U^* \geq \nu\, |A|^{\frac{1}{\nu}} + (1-\nu)\, |B|^{\frac{1}{1-\nu}}. \tag{3.1}$$

Proof. By the Spectral Theorem for complex matrices $\Psi(A)$ and $\Psi(B)$, there exist unitary matrices $V, W \in M_{2n}(\mathbb{C})$ such that

$$V|\Psi(A)\Psi(B)^*|V^* = \Delta \quad \text{and} \quad W\Big(\nu\, |\Psi(A)|^{\frac{1}{\nu}} + (1-\nu)\, |\Psi(B)|^{\frac{1}{1-\nu}}\Big)W^* = \Gamma,$$

where Δ and Γ are diagonal matrices and where the diagonal entries of Δ (resp. Γ) are the eigenvalues of $|\Psi(A)\Psi(B)^*|$ (resp. $\nu\, |\Psi(A)|^{\frac{1}{\nu}} + (1-\nu)\, |\Psi(B)|^{\frac{1}{1-\nu}}$). We mention here that the eigenvalues of both matrices are arranged in non inceasing order.
Note that by Proposition 3.5,

$$|\Psi(A)\Psi(B)^*| = \Psi(|AB^*|)$$

and

$$\nu\, |\Psi(A)|^{\frac{1}{\nu}} + (1-\nu)\, |\Psi(B)|^{\frac{1}{1-\nu}} = \Psi\Big(\nu\, |A|^{\frac{1}{\nu}} + (1-\nu)\, |B|^{\frac{1}{1-\nu}}\Big).$$

As a consequence of Theorem 2.2 and the remark following it we see that if $\Delta = \Delta_1 \oplus \cdots \oplus \Delta_n$ and $\Gamma = \Gamma_1 \oplus \ldots \oplus \Gamma_n$, where for each $i = 1, 2, \ldots, n$, $\Delta_i = \text{diag}\{\delta_i, \ldots, \delta_i\}$ and $\Gamma_i = \text{diag}\{\gamma_i, \ldots, \gamma_i\}$, then

$$\sigma(|AB^*|) = \{\delta_1, \ldots, \delta_n\} \text{ and } \sigma\Big(\nu\, |A|^{\frac{1}{\nu}} + (1-\nu)\, |B|^{\frac{1}{1-\nu}}\Big) = \{\gamma_1, \ldots, \gamma_n\} \tag{3.2}$$

Theorem 1.1 implies that $\Delta \geq \Gamma$, hence for each $i = 1, 2, \ldots, n$, $\Delta_i \geq \Gamma_i$ and consequently

$$diag\{\delta_1, \ldots, \delta_n\} \geq diag\{\gamma_1, \ldots, \gamma_n\} \tag{3.3}$$

The relation in (3.2) combined with (3.3) finally ensures us of the existence of a unitary $U \in M_n(\mathbb{H})$ such that (3.1) holds. □

Theorem 3.7. *Let A and B be invertible $n \times n$ quaternion matrices and let $\nu > 1$. The following statements are equivalent*

(a) There is a unitary $n \times n$ quaternion matrix U such that

$$U|AB^*|U^* = \nu\, |A|^{\frac{1}{\nu}} + (1-\nu)\, |B|^{\frac{1}{1-\nu}}\, .$$

(b) $|A|^{\frac{1}{\nu}} = |B|^{\frac{1}{1-\nu}}$.

Proof. Suppose first that (a) holds. Then by Theorem 2.1 (a) and Proposition 3.5 we have

$$\Psi(U)|\Psi(A)\Psi(B)^*|\Psi(U)^* = \nu\, |\Psi(A)|^{\frac{1}{\nu}} + (1-\nu)\, |\Psi(B)|^{\frac{1}{1-\nu}}\, .$$

By Theorem 1.2, it implies that $|\Psi(A)|^{\frac{1}{\nu}} = |\Psi(B)|^{\frac{1}{1-\nu}}$ or equivalently that $\Psi(|A|^{\frac{1}{\nu}}) = \Psi(|B|^{\frac{1}{1-\nu}})$.Theorem 2.1 (a) then implies that $|A|^{\frac{1}{\nu}} = |B|^{\frac{1}{1-\nu}}$, proving that (a) implies (b).
Assume conversely that $|A|^{\frac{1}{\nu}} = |B|^{\frac{1}{1-\nu}}$. Then $\Psi(|A|^{\frac{1}{\nu}}) = \Psi(|B|^{\frac{1}{1-\nu}})$ which, by Proposition 3.5, implies that $|\Psi(A)|^{\frac{1}{\nu}} = |\Psi(B)|^{\frac{1}{1-\nu}}$. Thus

$$\nu|\Psi(A)|^{\frac{1}{\nu}} + (1-\nu)|\Psi(B)|^{\frac{1}{1-\nu}} = |\Psi(A)|^{\frac{1}{\nu}}\, . \tag{3.4}$$

Let $\Psi(A) = V|\Psi(A)|$ and $\Psi(B) = W|\Psi(B)|$ be the polar decompositions of $\Psi(A)$ and $\Psi(B)$ respectively, where V and W are unitary matrices in $M_{2n}(\mathbb{C})$. Then the proof of Proposition 4.1 in [4] demonstrates that

$$|\Psi(A)\Psi(B)^*| = W\Big||\Psi(A)|.|\Psi(B)|\Big|W^* = W|\Psi(A)|^{\frac{1}{\nu}}W^*.$$

By $|\Psi(A)|^{\frac{1}{\nu}} = |\Psi(B)|^{\frac{1}{1-\nu}}$, we have

$$|\Psi(A)\Psi(B)^*| = W|\Psi(A)|^{\frac{1}{\nu}}W^*.$$

Therefore

$$W^*|\Psi(A)\Psi(B)^*|W = |\Psi(A)|^{\frac{1}{\nu}}\, . \tag{3.5}$$

(3.4) combined with (3.5) now implies that

$$W^*|\Psi(A)\Psi(B)^*|W = \nu|\Psi(A)|^{\frac{1}{\nu}} + (1-\nu)|\Psi(B)|^{\frac{1}{1-\nu}}\, .$$

Using the notation from the proof of Theorem 3.6, we see that $\Delta = \Gamma$ which in turn implies that $\{\delta_1, \cdots, \delta_n\} = \{\gamma_1, \cdots, \gamma_n\}$. In light of (3.2), this implies that (a) holds. Thus (b) also implies (a). □

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