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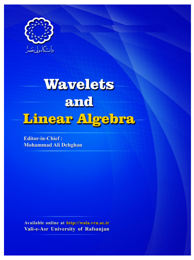

∗-frames for operators on Hilbert modules

Bahram Dastourian[a], Mohammad Janfada[a,∗]

[a]*Department of Pure Mathematics, Ferdowsi University of Mashhad, Mashhad, Islamic Republic of Iran*

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Abstract

K-frames which are generalization of frames on Hilbert spaces, were introduced to study atomic systems with respect to a bounded linear operator. In this paper, $*$-K-frames on Hilbert C^*-modules, as a generalization of K-frames, are introduced and some of their properties are obtained. Then some relations between $*$-K-frames and $*$-atomic systems with respect to an adjointable operator are considered and some characterizations of $*$-K-frames are given. Finally perturbations of $*$-K-frames are discussed.

1. Introduction and Preliminaries

In 1952, frames in Hilbert spaces were introduced by Duffin and Schaffer [7] to deal with some problems in nonharmonic Fourier analysis. Now frames play an important role not only in the theoretics but also in many kinds of applications, and have been widely applied in signal processing [13, 24], sampling [9, 10], coding and communications [25], filter bank theory [8], system

∗Corresponding author

Email addresses: `bdastorian@gmail.com` (Bahram Dastourian), `janfada@um.ac.ir` (Mohammad Janfada)

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modelling [4], and so on.
In contrast to frames, there exist systems of functions generating proper subspaces even though they do not belong to them. These families were considered by H.G. Feichtinger and T. Werther in [12] and called families of local atoms. In 2012, K-frames were introduced by Găvruţa [16] to study the atomic systems with respect to a bounded linear operator K in Hilbert spaces. K-frames are more general than ordinary frames in the sense that the lower frame bound only holds for the elements in the range of K, where K is a bounded linear operator in a separable Hilbert space. This generalization of frames allows to reconstruct elements from the range of a linear and bounded operator in a Hilbert space. In general, range is not a closed subspace (see [3, 5, 27, 28]). In other hand, the notion of frames for Hilbert spaces had been extended by Frank and Larson [15] to the Hilbert C^*-modules and some properties of these frames were also investigated in [14, 17, 18]. Next, Alijani and Dehghan [2] introduced the $*$-frames, as a generalization of frames in Hilbert C^*-modules. Hilbert C^*-modules are generalizations of Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than in the field of complex numbers. They appear naturally in a number of situations. Many useful techniques in Hilbert spaces are either not available or not known in Hilbert C^*-modules. For example, it is well-known that the Riesz representation theorem for bounded $\mathcal{A}$-linear mapping on Hilbert $\mathcal{A}$-module $\mathcal{H}$ is not valid but this is true for self-dual Hilbert modules [23]. Note that a Hilbert $\mathcal{A}$-module $\mathcal{H}$ is called self-dual if $\mathcal{H} \cong \mathcal{H}'$, where $\mathcal{H}'$ is the set of all bounded $\mathcal{A}$-linear maps from $\mathcal{H}$ to $\mathcal{A}$. Moreover, there exist closed subspaces in Hilbert C^*-modules that have no orthogonal complement [20] and there are bounded operators on Hilbert C^*-modules which do not have any adjoint [21]. Problems about frames and $*$-frames for Hilbert C^*-modules are more complicated than those for Hilbert spaces. This makes the study of the $*$-frames for Hilbert C^*-modules important and interesting. The paper is organized as follows. Section 2 introduces families of local $*$-atoms. Then $*$-atomic systems for an adjointable operator on a Hilbert C^*-module are presented and it is proved that a family of local $*$-atoms is a special case of a $*$-atomic systems for an appropriate adjointable operator. One of the main results of the paper is included in Section 3, where $*$-K-frames in modular spaces are studied. Some relations between $*$-K-frames and $*$-atomic systems with respect to an adjointable operator are obtained in this section. Some more properties of $*$-K-frames are given in Section 4. In particular, we construct some new $*$-K-frames by given $*$-K-frames. The final section discuss the perturbations of $*$-K-frames. In the following, we review some definitions and notions which will appear in the rest of the paper. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a Hilbert space $\mathcal{H}$ is called an atomic system for a bounded linear operator K on $\mathcal{H}$ if the following statements hold
(i) the series $\sum_{n\in\mathbb{N}} c_n x_n$ converges for all $c = (c_n) \in l^2$;
(ii) there exists a positive real number $\nu > 0$ such that for every $x \in \mathcal{H}$ there exists $a_x = (a_n) \in l^2$ such that $\|a_x\|_{l^2} \le \nu\|x\|$ and $Kx = \sum_{n\in\mathbb{N}} a_n x_n$.
Also a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$ is said to be a K-frame for $\mathcal{H}$ if there exist positive real numbers λ, μ such that

$$\lambda\|K^*x\|^2 \le \sum_{n\in\mathbb{N}} |\langle x, x_n\rangle|^2 \le \mu\|x\|^2, \quad (x \in \mathcal{H}).$$

Frames are a special case of K-frames when K is the identity operator. It has been proved that a sequence $\{x_n\}_{n\in\mathbb{N}}$ is an atomic system for K if and only if it is a K-frame [16]. K-frames and their properties have recently been studied in [3, 5, 27, 28]. A nonzero element A in a unital C^*-algebra

$\mathcal{A}$ is called strictly nonzero if zero does not belong to $\sigma(A)$, where $\sigma(A)$ is the spectrum of the element A.
Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two C^*-algebras. Let $\mathcal{A}\otimes\mathcal{B}$ be the completion of $\mathcal{A}\otimes_{alg}\mathcal{B}$ with the spatial norm and the following operation and involution,

$$(A\otimes B)(C\otimes D)=AC\otimes BD \quad , \quad (A\otimes B)^*=A^*\otimes B^*, \quad A\otimes B, C\otimes D\in\mathcal{A}\otimes\mathcal{B}.$$

Then $\mathcal{A}\otimes\mathcal{B}$ is a C^*-algebra such that $\|A\otimes B\|=\|A\|\|B\|$, for $A\otimes B\in\mathcal{A}\otimes\mathcal{B}$. We note that if $A\in\mathcal{A}^+$ and $B\in\mathcal{B}^+$, then $A\otimes B\in(\mathcal{A}\otimes\mathcal{B})^+$. If A,B are hermitian elements of $\mathcal{A}$ and $A\leq B$, then for every positive element C of $\mathcal{B}$, we have $A\otimes C\leq B\otimes C$. For basic notations and theory of C*-algebras one can see [22]. Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{H}$ be an algebraic (left) $\mathcal{A}$-module. $\mathcal{H}$ is called a pre-Hilbert $\mathcal{A}$-module if there exists an $\mathcal{A}$-valued inner product $\langle .,.\rangle:\mathcal{H}\times\mathcal{H}\to\mathcal{A}$ that possesses the following properties:
(i) $\langle f,f\rangle\geq 0$, for all $f\in\mathcal{H}$ and $\langle f,f\rangle=0$ if and only if $f=0$;
(ii) $\langle Af+Bg,h\rangle=A\langle f,h\rangle+B\langle g,h\rangle$, for all $A,B\in\mathcal{A}$ and $f,g,h\in\mathcal{H}$;
(iii) $\langle f,g\rangle=\langle g,f\rangle^*$, for all $f,g\in\mathcal{H}$;
(iv) $\langle\mu f,g\rangle=\mu\langle f,g\rangle$, for all $\mu\in\mathbb{C}$ and $f,g\in\mathcal{H}$;
The mapping $f\mapsto\|f\|=\|\langle f,f\rangle\|^{\frac{1}{2}}$ defines a norm on $\mathcal{H}$. If a pre-Hilbert C*-module $\mathcal{H}$ is complete with respect to this norm, then $(\mathcal{H},\mathcal{A},\langle .,.\rangle)$ is called a Hilbert C*-module over $\mathcal{A}$ or, simply, a Hilbert $\mathcal{A}$-module. We write $\mathcal{H}$ or $(\mathcal{H},\langle .,.\rangle)$ instead of $(\mathcal{H},\mathcal{A},\langle .,.\rangle)$ when the $\mathcal{A}$-valued inner product and the C*-algebra are well known.
The C^*-algebra $\mathcal{A}$ itself can be recognized as a Hilbert $\mathcal{A}$-module with the inner product $\langle A,B\rangle=AB^*$, for any $A,B\in\mathcal{A}$. Also for a C^*-algebra $\mathcal{A}$ the standard Hilbert $\mathcal{A}$-module $\ell^2(\mathcal{A})$ is defined by

$$\ell^2(\mathcal{A})=\{\{A_j\}_{j\in\mathbb{N}}:\sum_{j\in\mathbb{N}}A_jA_j^* \text{ norm-converges in } \mathcal{A}\}$$

with $\mathcal{A}$-inner product $\langle\{A_j\}_{j\in\mathbb{N}},\{B_j\}_{j\in\mathbb{N}}\rangle=\sum_{j\in\mathbb{N}}A_jB_j^*$. Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules. A mapping $T:\mathcal{H}\to\mathcal{K}$ is called adjointable if there exists a mapping $S:\mathcal{K}\to\mathcal{H}$ such that $\langle Tf,g\rangle=\langle f,Sg\rangle$ for all $f\in\mathcal{H}$, $g\in\mathcal{K}$. The unique mapping S is denoted by T^* and is called the adjoint of T. It is well-known that T and T^* must be bounded linear $\mathcal{A}$-module mappings. The set of all adjointable operators from $\mathcal{H}$ to $\mathcal{K}$ is denoted by $Hom^*_{\mathcal{A}}(\mathcal{H},\mathcal{K})$. The algebra $Hom^*_{\mathcal{A}}(\mathcal{H})=Hom^*_{\mathcal{A}}(\mathcal{H},\mathcal{H})$ is indeed a C^*-algebra. For any $T\in Hom^*_{\mathcal{A}}(\mathcal{H})$, the inequality

$$\langle T(f),T(f)\rangle\leq\|T\|^2\langle f,f\rangle,$$

holds in $\mathcal{A}$, for every $f\in\mathcal{H}$ [19, 26].
Let $\mathcal{H}$ be a Hilbert C^*-module and $\mathcal{M}\subseteq\mathcal{H}$ be a closed submodule of a Hilbert module $\mathcal{H}$. The orthogonal complement $\mathcal{M}^\perp$ of $\mathcal{M}$ is defined by

$$\mathcal{M}^\perp=\{g\in\mathcal{H}:\langle f,g\rangle=0,\quad\forall f\in\mathcal{M}\}.$$

$\mathcal{M}^\perp$ is also a closed submodule of the Hilbert module $\mathcal{H}$. However, the equality $\mathcal{H}=\mathcal{M}\oplus\mathcal{M}^\perp$ is not fulfilled in general [20]. The closed submodule M of $\mathcal{H}$ is called orthogonally complemented if $\mathcal{H}=\mathcal{M}\oplus\mathcal{M}^\perp$. The following generalizations of the so-called Douglas's theorem [6] for Hilbert modules are frequently used in this paper.

Theorem 1.1. *[11] Suppose that $\mathcal{H}$, $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert modules over a C^*-algebra $\mathcal{A}$. Let $T \in Hom^*_{\mathcal{A}}(\mathcal{H}_1, \mathcal{H})$ and $S \in Hom^*_{\mathcal{A}}(\mathcal{H}_2, \mathcal{H})$. If $\overline{Rang(S^*)}$ is orthogonally complemented, then the following are equivalent:*
(i) $Rang(T) \subseteq Rang(S)$;
(ii) $\mu TT^* \leq SS^*$, for some positive real number $\mu > 0$;
(iii) There exists positive real number $\lambda > 0$ such that $\lambda\|T^* f\|^2 \leq \|S^* f\|^2$, for all $f \in \mathcal{H}$;
(iv) There exists an adjointable operator $Q : \mathcal{H}_1 \to \mathcal{H}_2$ such that $T = SQ$.

Theorem 1.2. *[29] Suppose that $\mathcal{H}$ and $\mathcal{H}_1$ are Hilbert module over a C*-algebra $\mathcal{A}$. Let $T \in Hom^*_{\mathcal{A}}(\mathcal{H})$ and $S : \mathcal{H}_1 \to \mathcal{H}$ is adjointable operator. If $Rang(S)$ is closed, then the following are equivalent:*
(i) $Rang(T) \subseteq Rang(S)$;
(ii) $\lambda TT^* f \leq SS^* f$, $f \in \mathcal{H}$, for some $\lambda > 0$;
(iii) There exists an adjointable operator $Q : \mathcal{H} \to \mathcal{H}_1$ such that $T = SQ$.

One can easily verify that each of the above conditions is also equivalent to the following condition,
(iv) There exists a positive real number $\mu > 0$ such that $\|T^* f\|^2 \leq \|S^* f\|^2$, for every $f \in \mathcal{H}$. Let $(\mathcal{H}, \mathcal{A}, \langle ., .\rangle_{\mathcal{A}})$ and $(\mathcal{K}, \mathcal{B}, \langle ., .\rangle_{\mathcal{B}})$ be two Hilbert C^*-modules. We denote by $\mathcal{H} \otimes \mathcal{K}$ the completion of $\mathcal{H} \otimes_{alg} \mathcal{K}$ with the module action

$$(A \otimes B)(f \otimes g) = Af \otimes Bg \qquad (A \in \mathcal{A},\ B \in \mathcal{B}, f \in \mathcal{H}, g \in \mathcal{K}),$$

and the following $\mathcal{A} \otimes \mathcal{B}$-valued inner product

$$\langle f_1 \otimes g_1, f_2 \otimes g_2\rangle = \langle f_1, f_2\rangle \otimes \langle g_1, g_2\rangle \ \ (f_1, f_2 \in \mathcal{H} \text{ and } g_1, g_2 \in \mathcal{K}).$$

It is well-known that $\mathcal{H} \otimes_{alg} \mathcal{K}$ with these operations is a Hilbert $\mathcal{A} \otimes \mathcal{B}$-module (see [19]).
If T_1 and T_2 are two adjointable operator on $(\mathcal{H}, \mathcal{A}, \langle ., .\rangle_{\mathcal{A}})$ and $(\mathcal{K}, \mathcal{B}, \langle ., .\rangle_{\mathcal{B}})$, respectively, then the tensor product T_1 and T_2 on $\mathcal{H} \otimes \mathcal{K}$ defined by $(T_1 \otimes T_2)(f \otimes g) = T_1 f \otimes T_2 g$, $f \otimes g \in \mathcal{H} \otimes \mathcal{K}$, is adjointable and its adjoint is $T_1^* \otimes T_2^*$. For more details one can see [19, 26].
Let $\mathcal{A}$ be a unital C^*-algebra and J be a finite or countable index set. A sequence $\{f_j\}_{j\in J}$ of elements in a Hilbert $\mathcal{A}$-module $\mathcal{H}$ is said to be a (standard) $*$-frame for $\mathcal{H}$ if there exist strictly nonzero elements A and B of $\mathcal{A}$ such that

$$A\langle f, f\rangle A^* \leq \sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle \leq B\langle f, f\rangle B^*,\ \ f \in \mathcal{H},$$

where the sum in the middle of the inequality is convergent in norm (see [2]). The elements A and B are called lower and upper $*$-frame bound, respectively. If the right side of this inequality holds then we say that $\{f_j\}_{j\in J}$ is a $*$-Bessel sequence. Trivially every frame for a Hilbert module is a $*$-frame. If $\mathcal{A} = \mathbb{C}$ then the $*$-frame $\{f_j\}_{j\in J}$ is indeed a frame for the Hilbert space $\mathcal{H}$. The following result was obtained independently by Arambašić [1] and Jing [17].

Lemma 1.3. *$\{f_j\}_{j\in J}$ is a frame of a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$ with the frame bounds A, B, respectively, if and only if*

$$A\|f\|^2 \leq \|\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle\| \leq B\|f\|^2, \quad (f \in \mathcal{H}).$$

Throughout the paper, we assume that $\mathcal{H}$ is a finitely or countably generated Hilbert C^*-modules over a unital C*-algebra $\mathcal{A}$ with the unit element $1_{\mathcal{A}}$. By $\mathcal{Z}(\mathcal{A})$ we denote the center of the C*-algebra $\mathcal{A}$ and J is applied for a finite or countably infinite index set.

2. Local ∗-Atoms and ∗-Atomic Systems

Let $\mathcal{H}$ be a finitely or countably generated Hilbert C^*-modules over a unital C*-algebra $\mathcal{A}$. In this section first a family of local ∗-atoms for $\mathcal{H}$ is introduced and then a generalization of local ∗-atoms, namely ∗-atomic system for an adjointable operator over $\mathcal{H}$, is studied.

Definition 2.1. Suppose that $\{f_j\}_{j\in J} \subseteq \mathcal{H}$ is a ∗-Bessel sequence and $\mathcal{H}_0$ is a closed submodule of $\mathcal{H}$. The sequence $\{f_j\}_{j\in J}$ is called a family of local ∗-atoms for $\mathcal{H}_0$ if there exists a sequence $\{c_j\}_{j\in J}$ of linear operators $c_j : \mathcal{H}_0 \to \mathcal{A}$ such that for every $f \in \mathcal{H}_0$
(i) there exists strictly nonzero element $C \in \mathcal{A}$ with $\sum_{j\in J}(c_j(f))(c_j(f))^* \leq C\langle f, f\rangle C^*$,
(ii) $f = \sum_{j\in J} c_j(f) f_j$.

Trivially every ∗-frame for a Hilbert module $\mathcal{H}$ is a family of local ∗-atoms for $\mathcal{H}_0 = \mathcal{H}$.

Proposition 2.2. *Let $\mathcal{H}_0$ be an orthogonally complemented submodule of $\mathcal{H}$. Suppose that $\{f_j\}_{j\in J} \subseteq \mathcal{H}$ is a family of local ∗-atoms for $\mathcal{H}_0$ then $\{P_{\mathcal{H}_0} f_j\}_{j\in J}$ is a ∗-frame for $\mathcal{H}_0$, where $P_{\mathcal{H}_0}$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_0$.*

Proof. Since $\{f_j\}_{j\in J}$ is a ∗-Bessel sequence, it is enough to show that $\{P_{\mathcal{H}_0} f_j\}_{j\in J}$ has a lower ∗-frame bound. Let $\{c_j\}_{j\in J}$ and C be as in Definition 2.1. For every $f \in \mathcal{H}_0$ we have

$$\begin{aligned}
\|f\|^4 &= \|\langle \sum_{j\in J} c_j(f) f_j, f\rangle\|^2 \\
&= \|\sum_{j\in J} c_j(f)\langle f_j, f\rangle\|^2 \\
&\leq \|\sum_{j\in J}(c_j(f))(c_j(f))^*\| \, \|\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle\| \\
&\leq \|C\|^2\|f\|^2\|\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle\| \\
&= \|C\|^2\|f\|^2\|\sum_{j\in J}\langle P_{\mathcal{H}_0} f, f_j\rangle\langle f_j, P_{\mathcal{H}_0} f\rangle\| \\
&= \|C\|^2\|f\|^2\|\sum_{j\in J}\langle f, P_{\mathcal{H}_0} f_j\rangle\langle P_{\mathcal{H}_0} f_j, f\rangle\|.
\end{aligned}$$

Hence by Lemma 1.3

$$\frac{1_{\mathcal{A}}}{\|C\|}\langle f, f\rangle\left(\frac{1_{\mathcal{A}}}{\|C\|}\right)^* \leq \sum_{j\in J}\langle f, P_{\mathcal{H}_0} f_j\rangle\langle P_{\mathcal{H}_0} f_j, f\rangle.$$

So $\{P_{\mathcal{H}_0} f_j\}_{j\in J}$ is a ∗-frame for $\mathcal{H}_0$ with the lower frame bound $\frac{1_{\mathcal{A}}}{\|C\|}$. Note that $P_{\mathcal{H}_0} f = f$, since $f \in \mathcal{H}_0$. □

Here $*$-atomic system for an adjointable operator is introduced. It will be proved that a family of local $*$-atomes for a submodule $\mathcal{H}_0$ is indeed $*$-atomic system for the operator $P_{\mathcal{H}_0}$, the orthogonal projection of $\mathcal{H}$ into $\mathcal{H}_0$.

Definition 2.3. Let $K \in Hom^*_{\mathcal{A}}(\mathcal{H})$. We say that $\{f_j\}_{j\in J} \subseteq \mathcal{H}$ is a $*$-atomic system for K if $\{f_j\}_{j\in J}$ is a $*$-Bessel sequence and there exists a strictly nonzero $C \in \mathcal{A}$ such that for all $f \in \mathcal{H}$ there is $a_f = \{A_j\}_{j\in J} \in \ell^2(\mathcal{A})$ such that $K(f) = \sum_{j\in J} A_j f_j$ and $\langle a_f, a_f\rangle \leq C\langle f, f\rangle C^*$.

Example 2.4. (See [2]) Let ℓ^∞ be the unital C^*-algebra of all bounded complex-valued sequences. Let c_0 be the set of all sequences converging to zero. Then c_0 is a Hilbert ℓ^∞-module with ℓ^∞-valued inner product $\langle u, v\rangle = \{u_i\overline{v_i}\}_{i\in\mathbb{N}}$, for $u, v \in c_0$. For any $j \in \mathbb{N}$, let $f_j = \{f^j_i\}_{i\in\mathbb{N}} \in c_0$ be defined by

$$f^j_i = \begin{cases} \frac{1}{3} + \frac{1}{i} & i = j \\ 0 & i \neq j \end{cases}, \quad j \in \mathbb{N}$$

and define $K : c_0 \to c_0$ by $K(u) = \{\sum_{j\in\mathbb{N}} u_i|f^j_i|^2\}_{i\in\mathbb{N}}$, $u = \{u_i\}_{i\in\mathbb{N}} \in c_0$. Then $K \in Hom^*_{\ell^\infty}(c_0)$. So $\{f_j\}_{j\in\mathbb{N}}$ is a $*$-atomic system for c_0. Note that in this case $a_f = \{u\overline{f_j}\}_{j\in\mathbb{N}} \in \ell^2(\ell^\infty)$ and $C = \{\frac{1}{3} + \frac{1}{i}\}_{i\in\mathbb{N}} \in \ell^\infty$. Indeed

$$K(u) = \{\sum_{j\in\mathbb{N}} u_i|f^j_i|^2\}_{i\in\mathbb{N}} = \sum_{j\in\mathbb{N}}\{u_i|f^j_i|^2\}_{i\in\mathbb{N}} = \sum_{j\in J}\{u_i\overline{f^j_i}\}_{i\in\mathbb{N}}f_j = \sum_{j\in J} u\overline{f_j}f_j$$

and

$$\langle a_f, a_f\rangle = \sum_{j\in J} u\overline{f_j}f_j\overline{u} = \{|u_i|^2(\frac{1}{3} + \frac{1}{i})^2\}_{i\in\mathbb{N}} = \{\frac{1}{3} + \frac{1}{i}\}_{i\in\mathbb{N}}\langle u, u\rangle\{\frac{1}{3} + \frac{1}{i}\}_{i\in\mathbb{N}}.$$

The following proposition shows that a family of local $*$-atomes is a special case of $*$-atomic system . The proof is straightforward.

Proposition 2.5. *Let $\mathcal{H}_0$ be an orthogonally complemented submodule of $\mathcal{H}$ and $\{f_j\}_{j\in J} \subseteq \mathcal{H}$ be a $*$-Bessel sequence then the following are equivalent.*
(i) $\{f_j\}_{j\in J}$ is a family of local $*$-atoms for $\mathcal{H}_0$.
(ii) $\{f_j\}_{j\in J}$ is a $*$-atomic system for $P_{\mathcal{H}_0}$, where $P_{\mathcal{H}_0}$ is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_0$.

3. $*$-Atomic Systems and $*$-K-Frames

Let $\mathcal{H}$ be a finitely or countably generated Hilbert C^*-modules over a unital C*-algebra $\mathcal{A}$ and $K \in Hom^*_{\mathcal{A}}(\mathcal{H})$. In this section a $*$-K-frame for $\mathcal{H}$ is introduced and its relations with $*$-atomic system for the operator K is discussed. Next some characterizations of $*$-K-frames are obtained.

Definition 3.1. Let $K \in Hom^*_{\mathcal{A}}(\mathcal{H})$. A sequence $\{f_j\}_{j\in J} \subseteq \mathcal{H}$ is called a $*$-K-frame (or $*$-frame for the operator K) if there exist strictly nonzero $A, B \in \mathcal{A}$ such that

$$A\langle K^*f, K^*f\rangle A^* \leq \sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle \leq B\langle f, f\rangle B^*, \tag{3.1}$$

for every $f \in \mathcal{H}$, where the sum in the middle of the inequality is convergent in norm.

The elements A and B are called the lower and the upper $*$-K-frame, respectively. Suppose that $\{f_j\}_{j\in J}$ is a $*$-K-frame of $\mathcal{H}$. The operator $T : \ell^2(\mathcal{A}) \to \mathcal{H}$ defined by $T(\{A_j\}_{j\in J}) = \sum_{j\in J} A_j f_j$ is called the synthesis operator. T is adjointable and $T^* : \mathcal{H} \to \ell^2(\mathcal{A})$ is given by $T^*(f) = \{\langle f, f_j\rangle\}_{j\in J}$. T^* is called the analysis operator. The operator $S : \mathcal{H} \to \mathcal{H}$ defined by $S(f) = TT^*(f) = \sum_{j\in J}\langle f, f_j\rangle f_j$ is called the $*$-K-frame operator of $\{f_j\}_{j\in J}$. The operator S is not invertible in general even on Hilbert spaces (see [16]). Although, with $K = I$, S is invertible and $\{S^{-1} f_j\}_{j\in J}$ is a $*$-frame (see [2]).

Note that every $*$-frame is a $*$-K-frame, for any $K \in Hom^*_{\mathcal{A}}(\mathcal{H})$. Indeed for any $K \in Hom^*_{\mathcal{A}}(\mathcal{H})$ the inequality

$$\langle K^* f, K^* f\rangle \leq \|K\|^2 \langle f, f\rangle, \quad f \in \mathcal{H} \tag{3.2}$$

holds. Now if $\{f_j\}_{j\in J}$ is a $*$-frame with bounds A and B then by (3.2) and the fact that for $A, B \in \mathcal{A}$ the inequality $A \leq B$ implies that $CAC^* \leq CBC^*$, for any $C \in \mathcal{A}$, we have

$$(A\|K\|^{-1})\langle K^* f, K^* f\rangle (A\|K\|^{-1})^* \leq A\langle f, f\rangle A^* \leq \sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle \leq B\langle f, f\rangle B^*.$$

Therefore $\{f_j\}_{j\in J}$ is a $*$-K-frame with $*$-frame bounds $A\|K\|^{-1}$ and B.

Lemma 3.2. *If $\{f_j\}_{j\in J}$ is a $*$-K-frame with $*$-frame bounds A and B then*

$$\|AK^* f\|^2 \leq \|\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle\| \leq \|Bf\|^2, \quad (f \in \mathcal{H}). \tag{3.3}$$

Conversely, if (3.3) holds, for some $A, B \in \mathcal{Z}(\mathcal{A})$ and $\overline{Rang(U)}$ of the operator $U : \mathcal{H} \to \ell^2(\mathcal{A})$ defined by $Uf = \{\langle f, f_j\rangle\}_{j\in J}$, is orthogonally complemented, then $\{f_j\}_{j\in J}$ is a $*$-K-frame.

Proof. $\Rightarrow$) It is obvious.
$\Leftarrow$) For every $f \in \mathcal{H}$,

$$\|Uf\|^2 = \|\langle Uf, Uf\rangle\| = \|\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle\| \leq \|Bf\|^2 \leq \|B\|^2\|f\|^2,$$

so $\|Uf\| \leq \|B\|\|f\|$. Therefore U is bounded. Also it is not hard to see that U is adjointable and its adjoint is $U^*(\{A_j\}_{j\in J}) = \sum_{j\in J} A_j f_j$, for every $\{A_j\}_{j\in J} \in \ell^2(\mathcal{A})$. For $B \in \mathcal{Z}(\mathcal{A})$, the mapping $Q_B : \mathcal{H} \to \mathcal{H}$ defined by $Q_B f = Bf$ has the adjoint Q_{B^*}, since

$$\langle Q_B f, g\rangle = \langle Bf, g\rangle = B\langle f, g\rangle = \langle f, g\rangle B = \langle f, B^* g\rangle = \langle f, Q_{B^*} g\rangle, \quad (f, g \in \mathcal{H}).$$

Therefore (3.3) is equivalent to

$$\|AK^* f\|^2 \leq \|Uf\|^2 \leq \|Q_B f\|^2 \quad (f \in \mathcal{H}).$$

By Theorem 1.1, there exist $\lambda, \mu > 0$ such that for every $f \in \mathcal{H}$,

$$\sqrt{\lambda}A\langle K^* f, K^* f\rangle(\sqrt{\lambda}A)^* \leq \sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle \leq \sqrt{\mu}B\langle f, f\rangle(\sqrt{\mu}B)^*.$$

which completes the proof. □

By a similar argument to the proof of Lemma 3.2 one may prove the following lemma by applying Theorem 1.2.

Lemma 3.3. *If $\{f_j\}_{j\in J}$ is a $*$-K-frame with $*$-frame bounds A and B then*

$$\|AK^* f\|^2 \leq \|\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle\| \leq \|Bf\|^2, \quad (f \in \mathcal{H}). \tag{3.4}$$

Conversely, if (3.4) holds for some $A, B \in \mathcal{Z}(\mathcal{A})$ and the rang of the operator $U : \mathcal{H} \to \ell^2(\mathcal{A})$, defined by $Uf = \{\langle f, f_j\rangle\}_{j\in J}$, is closed then $\{f_j\}_{j\in J}$ is a $*$-K-frame.

In the following theorem we show that under some conditions an $*$-atomic system for an operator K is indeed a $*$-K-frame and vice versa.

Theorem 3.4. *Let $K \in Hom^*_{\mathcal{A}}(\mathcal{H})$ and $\{f_j\}_{j\in J} \subseteq \mathcal{H}$ be a $*$-Bessel sequence. Suppose that $T : \ell^2(\mathcal{A}) \to \mathcal{H}$ is defined by $T(\{A_j\}_{j\in J}) = \sum_{j\in J} A_j f_j$ and $\overline{Rang(T^*)}$ is orthogonally complemented. Then $\{f_j\}_{j\in J}$ is a $*$-atomic system for K if and only if $\{f_j\}_{j\in J}$ is a $*$-K-frame. Moreover, in this case if $\mathcal{A}$ is finite dimensional then there exists another $*$-Bessel sequence $\{h_j\}_{j\in J}$ such that for all $f \in \mathcal{H}$,*

$$K(f) = \sum_{j\in J}\langle f, h_j\rangle f_j.$$

Proof. Suppose that $\{f_j\}_{j\in J}$ is a $*$-atomic system for K. For any $f \in \mathcal{H}$ we have

$$\|K^* f\|^2 = \sup_{\|g\|=1}\|\langle g, K^* f\rangle\|^2 = \sup_{\|g\|=1}\|\langle Kg, f\rangle\|^2.$$

By definition of $*$-atomic system, there is $b_g = \{B_j\}_{j\in J} \in \ell^2(\mathcal{A})$ such that $K(g) = \sum_{j\in J} B_j f_j$ and $\langle b_g, b_g\rangle \leq C\langle g, g\rangle C^*$, for some strictly nonzero $C \in \mathcal{A}$. Thus

$$\begin{aligned}
\|K^* f\|^2 &= \sup_{\|g\|=1}\|\langle\sum_{j\in J} B_j f_j, f\rangle\|^2 \\
&= \sup_{\|g\|=1}\|\sum_{j\in J} B_j\langle f_j, f\rangle\|^2 \\
&\leq \sup_{\|g\|=1}\|\sum_{j\in J} B_j B_j^*\|\|\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle\| \\
&\leq \|C\|^2\|\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle\|.
\end{aligned}$$

Note that the last inequality holds by the fact that in a C^*-algebra $\mathcal{A}$, if $A, B \in \mathcal{A}$ and $0 \leq A \leq B$ then $\|A\| \leq \|B\|$. Hence

$$\frac{1}{\|C\|^2}\|K^* f\|^2 \leq \|\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle\|, \quad (f \in \mathcal{H}).$$

So by Theorem 1.1 there exists a positive real number $\mu > 0$ such that $\mu KK^* \leq TT^*$. Therefore

$$(\sqrt{\mu}1_{\mathcal{A}})\langle K^* f, K^* f\rangle(\sqrt{\mu}1_{\mathcal{A}})^* \leq \sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle(f \in \mathcal{H}).$$

For the converse, suppose that there exist strictly nonzero $A, B \in \mathcal{A}$ such that (3.1) holds. So for each $f \in \mathcal{H}$, $\|A^{-1}\|^{-2}\|K^* f\|^2 \leq \|T^* f\|^2$. Thus by Theorem 1.1, there exists an adjointable operator $Q : \mathcal{H} \to \ell^2(\mathcal{A})$ such that $K = TQ$. For any $f \in \mathcal{H}$ let $a_f := Qf = \{A_j\}_{j\in J}$. Thus $K(f) = T(Q(f))$. But by adjointability of Q we have

$$\langle a_f, a_f\rangle = \langle Qf, Qf\rangle \leq \|Q\|^2\langle f, f\rangle = (\|Q\|1_{\mathcal{A}})\langle f, f\rangle(\|Q\|1_{\mathcal{A}})^*$$

which implies that $\langle a_f, a_f\rangle \leq C\langle f, f\rangle C^*$ with $C = \|Q\|1_{\mathcal{A}}$.
Now let $\mathcal{A}$ be finite dimensional. We know for each $f \in \mathcal{H}$, $\|A^{-1}\|^{-2}\|K^* f\|^2 \leq \|T^* f\|^2$. Thus by Theorem 1.1, there exists an adjointable operator $Q : \mathcal{H} \to \ell^2(\mathcal{A})$ such that $K = TQ$. In this case $\ell^2(\mathcal{A})$ is self dual so there exists $\{h_j\}_{j\in J}$ for which $Q(f) = \{\langle f, h_j\rangle\}_{j\in J}$. Thus

$$K(f) = T(Q(f)) = T(\{\langle f, h_j\rangle\}_{j\in J}) = \sum_{j\in J}\langle f, h_j\rangle f_j.$$

and

$$\sum_{j\in J}\langle f, h_j\rangle\langle h_j, f\rangle = \langle Qf, Qf\rangle \leq \|Q\|^2\langle f, f\rangle = (\|Q\|1_{\mathcal{A}})\langle f, f\rangle(\|Q\|1_{\mathcal{A}})^*.$$

Hence $\{h_j\}_{j\in J}$ is a $*$-Bessel sequence. □

Corollary 3.5. *Let $K \in Hom^*_{\mathcal{A}}(\mathcal{H})$ and $\{f_j\}_{j\in J} \subseteq \mathcal{H}$ be a $*$-Bessel sequence with bound B. Suppose that $T : \ell^2(\mathcal{A}) \to \mathcal{H}$ is defined by $T(\{A_j\}_{j\in J}) = \sum_{j\in J} A_j f_j$ and $\overline{Rang(T^*)}$ is orthogonally complemented. Then the following are equivalent.*
(i) $\{f_j\}_{j\in J}$ is a $*$-atomic system for K such that the strictly nonzero C in the definition of $*$-atomic system is in $\mathcal{Z}(\mathcal{A})$;
(ii) There exists positive real numbers $\mu > 0$ such that for any $f \in \mathcal{H}$,

$$(\frac{\mu}{C})\langle K^* f, K^* f\rangle(\frac{\mu}{C})^* \leq \sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle \leq B\langle f, f\rangle B^*.$$

Proof. The proof obtains by Lemma 3.2 and Theorem 3.4. □

The following characterization of $*$-atomic systems can be proved by applying Theorem 1.1.

Proposition 3.6. *Let $K \in Hom^*_{\mathcal{A}}(\mathcal{H})$ and $\{f_j\}_{j\in J} \subseteq \mathcal{H}$ be a $*$-Bessel sequence. Suppose that $T : \ell^2(\mathcal{A}) \to \mathcal{H}$ is defined by $T(\{A_j\}_{j\in J}) = \sum_{j\in J} A_j f_j$ and $\overline{Rang(T^*)}$ is orthogonally complemented. Then $\{f_j\}_{j\in J}$ is a $*$-atomic system if and only if $Rang(K) \subseteq Rang(T)$.*

4. Some more Properties of $*$-K-frames

Let $\mathcal{H}$ be a finitely or countably generated Hilbert C^*-modules over a unital C*-algebra $\mathcal{A}$ and $K \in Hom^*_{\mathcal{A}}(\mathcal{H})$. In this section first by using a $*$-K-frame and some elements of $Hom^*_{\mathcal{A}}(\mathcal{H})$, new $*$-frames for some adjointable operators are constructed. Next the tensor product of two $*$-K-frames and $*$-L-frames are considered.

Proposition 4.1. *Let $K, L \in Hom^*_{\mathcal{A}}(\mathcal{H})$ and $\{f_j\}_{j\in J}$ be a $*$-K-frame with the $*$-K-frame bounds A, B, then*
(i) *If $V : \mathcal{H} \to \mathcal{H}$ is a co-isometry such that $KV = VK$ then $\{Vf_j\}_{j\in J}$ is a $*$-K-frame with the same $*$-K-frame bounds.*
(ii) *$\{Lf_j\}_{j\in J}$ is a $*$-LK-frame with the $*$-frame bounds A and $B\|L\|$, respectively.*
(iii) *For any $n \in \mathbb{N}$, $\{L^n f_j\}_{j\in J}$ is a $*$-L^nK-frame.*
(iv) *If $Rang(L) \subseteq Rang(K)$ and K has closed range then $\{f_j\}_{j\in J}$ is also a $*$-L-frame.*

Proof. Form

$$A\langle K^*f, K^*f\rangle A^* \leq \sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle \leq B\langle f, f\rangle B^*, \quad (f \in \mathcal{H}).$$

we get

$$\sum_{j\in J}\langle f, Vf_j\rangle\langle Vf_j, f\rangle \leq B\langle V^*f, V^*f\rangle B^* = B\langle f, f\rangle B^*, \quad (f \in \mathcal{H}).$$

On the other hand V is a co-isometry so for any $f \in \mathcal{H}$,

$$\begin{aligned}\sum_{j\in J}\langle f, Vf_j\rangle\langle Vf_j, f\rangle &\geq A\langle K^*V^*f, K^*V^*f\rangle A^* \\ &= A\langle V^*K^*f, V^*K^*f\rangle A^* \\ &= A\langle K^*f, K^*f\rangle A^*,\end{aligned}$$

which proves (i).
For proving (ii), one may see that for any $f \in \mathcal{H}$,

$$\begin{aligned}A\langle (LK)^*f, (LK)^*f\rangle A^* = A\langle K^*L^*f, K^*L^*f\rangle A^* &\leq \sum_{j\in J}\langle f, Lf_j\rangle\langle Lf_j, f\rangle \\ &\leq B\langle L^*f, L^*f\rangle B^* \\ &\leq (B\|L\|)\langle f, f\rangle(B\|L\|)^*.\end{aligned}$$

(iii) Is trivial by applying (ii).
For proving (iv), if A and B are the $*$-K-frame bounds of $\{f_j\}_{j\in J}$ then by the facts that $Rang(K)$ is closed and $Rang(L) \subseteq Rang(K)$ and applying Theorem 1.2, there exists a positive real number $\lambda > 0$ such that for all $f \in \mathcal{H}$, $\lambda LL^*f \leq KK^*f$. Thus for any $f \in H$,

$$(\sqrt{\lambda}A)\langle L^*f, L^*f\rangle(\sqrt{\lambda}A)^* \leq A\langle K^*f, K^*f\rangle A^* \leq \sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle \leq B\langle f, f\rangle B^*.$$

□

Proposition 4.2. *Let $K \in Hom^*_{\mathcal{A}}(\mathcal{H})$ and $\{f_j\}_{j\in J}$ be a $*$-frame with the $*$-frame bounds A, B, then $\{Kf_j\}_{j\in J}$ is a $*$-K-frame with the $*$-K-frame bounds $A, B\|K\|$. The $*$-K-frame operator of $\{Kf_j\}_{j\in J}$ is $S' = KSK^*$, where S is the $*$-K-frame operator of $\{f_j\}_{j\in J}$.*

Proof. The first part is obvious by Proposition 4.1 ii), since every $*$-frame is a $*$-I-frame. But by definition of S, $SK^*f = \sum_{j\in J}\langle f, Kf_j\rangle f_j$. Thus

$$KSK^*f = K\sum_{j\in J}\langle f, Kf_j\rangle f_j = \sum_{j\in J}\langle f, Kf_j\rangle Kf_j. \tag{4.1}$$

Hence $S' = KSK^*$. □

Corollary 4.3. *Suppose that $K \in Hom^*_{\mathcal{A}}(\mathcal{H})$ and $\{f_j\}_{j\in J}$ is a $*$-frame, then $\{KS^{-1}f_j\}$ is a $*$-K-frame, when S is the $*$-frame operator of $\{f_j\}_{j\in J}$.*

Proof. If S is the $*$-frame operator of $\{f_j\}_{j\in J}$ then we know $\{S^{-1}f_j\}_{j\in J}$ is also a $*$-frame. Now applying Proposition 4.2, one may complete the proof. □

Proposition 4.4. *Let $\{f_j\}_{j\in J}$ be a $*$-frame with $*$-frame bounds A, B and the $*$-frame operator S. If $K, L \in Hom^*_{\mathcal{A}}(\mathcal{H})$ and $\overline{Rang(L^*)}$ is orthogonally complemented and $Rang(K) \subseteq Rang(L)$ then $\{Lf_j\}_{j\in J}$ is a $*$-K-frame with the $*$-K-frame operator $S' = L^*SL$.*

Proof. Since $Rang(K) \subseteq Rang(L)$ so by Theorem 1.1 there exists a positive real number $\lambda > 0$ such that $\lambda KK^* \leq LL^*$ therefore $\lambda\langle K^*f, K^*f\rangle \leq \langle L^*f, L^*f\rangle$ hence $(A\sqrt{\lambda})\langle K^*f, K^*f\rangle(A\sqrt{\lambda})^* \leq A\langle L^*f, L^*f\rangle A^*$, thus by the fact that $\{f_j\}_{j\in J}$ is a $*$-frame we have

$$\begin{aligned}(A\sqrt{\lambda})\langle K^*f, K^*f\rangle(A\sqrt{\lambda})^* \leq A\langle L^*f, L^*f\rangle A^* &\leq \sum_{j\in J}\langle f, Lf_j\rangle\langle Lf_j, f\rangle\\ &\leq B\langle L^*f, L^*f\rangle B^*\\ &\leq (B\|L\|)\langle f, f\rangle(B\|L\|)^*.\end{aligned}$$

So $\{Lf_j\}_{j\in J}$ is a $*$-K-frame with $*$-K-frame bounds $A\sqrt{\lambda}$ and $B\|L\|$. The proof of $S' = L^*SL$ is obvious. □

In [2], the authors have shown that a tensor product of $*$-frames in C^*-Hilbert modules is also a $*$frame. We study the subject for $*$-frames for operators in Hilbert C^*-modules. In the rest of this section, $\mathcal{H}$ and $\mathcal{K}$ stand for Hilbert C^*-modules over C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively.
The following theorem is a generalization of [2, Theorem 2.2] and [18, Lemma 3.1].

Theorem 4.5. *Let $K \in Hom^*_{\mathcal{A}}(\mathcal{H})$ and $L \in Hom^*_{\mathcal{B}}(\mathcal{K})$. Let $\{f_j\}_{j\in J} \subseteq \mathcal{H}$ be a $*$-K-frame with $*$-K-frame bounds A and B and frame operator S_f and $\{h_j\}_{j\in J} \subseteq \mathcal{K}$ be a $*$-L-frame with $*$-K-frame bounds C and D and $*$-L-frame operator S_h. Then $\{f_j \otimes h_j\}_{j\in J}$ is a $*$-$K \otimes L$-frame for Hilbert $\mathcal{A}\otimes\mathcal{B}$-module $\mathcal{H}\otimes\mathcal{K}$ with $*$-$K\otimes L$-frame operator $S_f \otimes S_h$ and the lower and upper bounds $A\otimes C$ and $B \otimes D$, respectively.*

Proof. Since $\{f_j\}_{j\in J}$ and $\{h_j\}_{j\in J}$ are $*$-K-frame and $*$-L-frame, respectively, so for any $f \in \mathcal{H}$ and $h \in \mathcal{K}$ we have

$$\begin{aligned}
\sum_{j\in J}\sum_{i\in I}\langle f\otimes h, f_j\otimes h_i\rangle\langle f_j\otimes h_i, f\otimes h\rangle &= \sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle \otimes \sum_{i\in I}\langle h, h_i\rangle\langle h_i, h\rangle \\
&\geq A\langle K^*f, K^*f\rangle A^* \otimes C\langle L^*h, L^*h\rangle C^* \\
&= (A\otimes C)(\langle K^*f, K^*f\rangle \otimes \langle L^*h, L^*h\rangle)(A\otimes C)^* \\
&= (A\otimes C)\langle (K^*\otimes L^*)(f\otimes h), (K^*\otimes L^*)(f\otimes h)\rangle(A\otimes C)^* \\
&= (A\otimes C)\langle (K\otimes L)^*(f\otimes h), (K\otimes L)^*(f\otimes h)\rangle(A\otimes C)^*
\end{aligned}$$

The rest of the proof is similar to the proof of [2, Theorem 2.2]. □

5. Perturbations of $*$-K-frames

In this section, we will show that, under some conditions, some perturbation theorem for frames in Hilbert spaces remains valid for $*$-frames for operators on Hilbert C^*-modules.
The following theorem is a generalization of [17, Theorem 7.1].

Theorem 5.1. *Assume that $K, L \in Hom^*_{\mathcal{A}}(\mathcal{H})$ with $Rang(L) \subseteq Rang(K)$ and K has closed range. Let $\{f_j\}_{j\in J}$ be a $*$-K-frame with $*$-K-frame bounds A and B. If there exists a constant $M > 0$, such that for all f in $\mathcal{H}$*

$$\|\sum_{j\in J}\langle f, f_j - h_j\rangle\langle f_j - h_j, f\rangle\| \leq M\min\{\|\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle\|, \|\sum_{j\in J}\langle f, h_j\rangle\langle h_j, f\rangle\|\}. \tag{5.1}$$

Then $\{h_j\}_{j\in J}$ is a $*$-L-frame. If K is a co-isometry, $Rang(K) \subseteq Rang(L)$ and $Rang(L)$ is closed then the converse is valid.

Proof. Suppose that $f \in \mathcal{H}$, so we have

$$\begin{aligned}
\|\sum_{j\in J}\langle f, h_j\rangle\langle h_j, f\rangle\|^{\frac{1}{2}} &= \|\{f, h_j\}\| \leq \|\{f, f_j - h_j\}\| + \|\{f, f_j\}\| \\
&= \|\sum_{j\in J}\langle f, f_j - h_j\rangle\langle f_j - h_j, f\rangle\|^{\frac{1}{2}} + \|\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle\|^{\frac{1}{2}} \\
&\leq \sqrt{M}\|\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle\|^{\frac{1}{2}} + \|\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle\|^{\frac{1}{2}} \\
&= (\sqrt{M}+1)\|\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle\|^{\frac{1}{2}} \leq \|B\|(1+\sqrt{M})\|f\|.
\end{aligned} \tag{5.2}$$

So by (5.2) and Lemma 1.3, $\{h_j\}_{j\in J}$ is a $*$-Bessel sequence with $*$-Bessel bound $1+\sqrt{M}\|B\|1_{\mathcal{A}}$.

On the other hand we have

$$\begin{aligned}
\|\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle\|^{\frac{1}{2}} &\leq \|\sum_{j\in J}\langle f, f_j - h_j\rangle\langle f_j - h_j, f\rangle\|^{\frac{1}{2}} + \|\sum_{j\in J}\langle f, h_j\rangle\langle h_j, f\rangle\|^{\frac{1}{2}} \\
&\leq \sqrt{M}\|\sum_{j\in J}\langle f, h_j\rangle\langle h_j, f\rangle\|^{\frac{1}{2}} + \|\sum_{j\in J}\langle f, h_j\rangle\langle h_j, f\rangle\|^{\frac{1}{2}} \\
&= (\sqrt{M} + 1)\|\sum_{j\in J}\langle f, h_j\rangle\langle h_j, f\rangle\|^{\frac{1}{2}}.
\end{aligned}$$

The operator $U : \mathcal{H} \to \ell^2(\mathcal{A})$ given by $Uf = \{\langle f, h_j\rangle\}_{j\in J}$ is well-defined since $\{h_j\}_{j\in J}$ is a $*$-Bessel sequence. Thus

$$\begin{aligned}
\|Uf\|^2 = \|\sum_{j\in J}\langle f, h_j\rangle\langle h_j, f\rangle\| &\geq \frac{1}{(\sqrt{M} + 1)^2}\|\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle\| \\
&\geq \frac{\|A\|^2}{(\sqrt{M} + 1)^2}\|K^* f\|^2.
\end{aligned}$$

This means that $\|Uf\|^2 \geq \frac{\|A\|^2}{(\sqrt{M}+1)^2}\|K^* f\|^2$, so by Theorem 1.2 there exists $\lambda > 0$ such that $\sum_{j\in J}\langle f, h_j\rangle\langle h_j, f\rangle \geq \sqrt{\lambda}1_{\mathcal{A}}\langle K^* f, K^* f\rangle\sqrt{\lambda}1_{\mathcal{A}}$. Thus by part ($iv$) of Proposition 4.1, $\{h_j\}_{j\in J}$ is a $*$-L-frame. For the converse suppose that $\{h_j\}_{j\in J}$ is a $*$-L-frame with the $*$-L-frame bounds C and D, respectively and K is a co-isometry operator on $\mathcal{H}$, i.e., $\|K^* f\| = \|f\|$, for any $f \in \mathcal{H}$. We obtain

$$\begin{aligned}
\|\sum_{j\in J}\langle f, f_j - h_j\rangle\langle f_j - h_j, f\rangle\|^{\frac{1}{2}} &\leq \|\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle\|^{\frac{1}{2}} + \|\sum_{j\in J}\langle f, h_j\rangle\langle h_j, f\rangle\|^{\frac{1}{2}} \\
&\leq \|\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle\|^{\frac{1}{2}} + \|D\|\|f\| \\
&= \|\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle\|^{\frac{1}{2}} + \|D\|\|K^* f\| \\
&\leq \|\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle\|^{\frac{1}{2}} + \frac{\|D\|}{\|A\|}\|\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle\|^{\frac{1}{2}} \\
&= \left(1 + \frac{\|D\|}{\|A\|}\right)\|\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle\|^{\frac{1}{2}}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\|\sum_{j\in J}\langle f, f_j - h_j\rangle\langle f_j - h_j, f\rangle\|^{\frac{1}{2}} &\leq \|\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle\|^{\frac{1}{2}} + \|\sum_{j\in J}\langle f, h_j\rangle\langle h_j, f\rangle\|^{\frac{1}{2}} \\
&\leq \|\sum_{j\in J}\langle f, h_j\rangle\langle h_j, f\rangle\|^{\frac{1}{2}} + \|B\|\|f\| \\
&= \|\sum_{j\in J}\langle f, h_j\rangle\langle h_j, f\rangle\|^{\frac{1}{2}} + \|B\|\|K^* f\|,
\end{aligned}$$

By Theorem 1.2, there exists $\lambda > 0$ such that $\|K^* f\|^2 \leq \lambda \|L^* f\|^2$, $f \in \mathcal{H}$, since $Rang(K) \subseteq Rang(L)$ and $Rang(L)$ is closed. Hence

$$\begin{aligned}
\|\sum_{j\in J}\langle f, f_j - h_j\rangle\langle f_j - h_j, f\rangle\|^{\frac{1}{2}} &\leq \|\sum_{j\in J}\langle f, h_j\rangle\langle h_j, f\rangle\|^{\frac{1}{2}} + \|B\|\|K^* f\| \\
&\leq \|\sum_{j\in J}\langle f, h_j\rangle\langle h_j, f\rangle\|^{\frac{1}{2}} + \sqrt{\lambda}\|B\|\|L^* f\| \\
&\leq \|\sum_{j\in J}\langle f, h_j\rangle\langle h_j, f\rangle\|^{\frac{1}{2}} + \frac{\sqrt{\lambda}\|B\|}{\|C\|}\|\sum_{j\in J}\langle f, h_j\rangle\langle h_j, f\rangle\|^{\frac{1}{2}} \\
&= \left(1 + \frac{\sqrt{\lambda}\|B\|}{\|C\|}\right)\|\sum_{j\in J}\langle f, h_j\rangle\langle h_j, f\rangle\|^{\frac{1}{2}}.
\end{aligned}$$

So with $M = \min\{(1 + \frac{\|D\|}{\|A\|})^2, (1 + \frac{\sqrt{\lambda}\|B\|}{\|C\|})^2\}$, (5.3) holds. □

Corollary 5.2. *Assume that $K, L \in Hom^*_{\mathcal{A}}(\mathcal{H})$ with $Rang(L) \subseteq Rang(K)$ and K has closed range. Let $\{f_j\}_{j\in J}$ be a $*$-K-frame with $*$-K-frame bounds $A, B \in \mathcal{Z}(\mathcal{A})$. If there exists a constant $M > 0$, such that for any $f \in \mathcal{H}$,*

$$\|\sum_{j\in J}\langle f, f_j - h_j\rangle\langle f_j - h_j, f\rangle\| \leq M \min\{\|\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle\|, \|\sum_{j\in J}\langle f, h_j\rangle\langle h_j, f\rangle\|\}, \tag{5.3}$$

then $\{h_j\}_{j\in J}$ is a $*$-L-frame.

Proof. The proof is obvious by Theorem 5.1 and Lemma 3.3. □

Corollary 5.3. *Assume that $K \in Hom^*_{\mathcal{A}}(\mathcal{H})$ such that K has closed range. Let $\{f_j\}_{j\in J}$ be a $*$-K-frame with $*$-K-frame bounds A and B, respectively. If there exists a constant $M > 0$, such that*

$$\|\sum_{j\in J}\langle f, f_j - h_j\rangle\langle f_j - h_j, f\rangle\| \leq M \min\{\|\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle\|, \|\sum_{j\in J}\langle f, h_j\rangle\langle h_j, f\rangle\|\}, \quad (f \in \mathcal{H}),$$

then $\{h_j\}_{j\in J}$ is a $*$-K-frame. The converse is valid for any co-isometry operator K.

In the following theorem another perturbation of $*$-K-frames is given that is a generalization of [17, Theorem 7.3].

Theorem 5.4. *Assume that $K, L \in Hom^*_{\mathcal{A}}(\mathcal{H})$ with $Rang(L) \subseteq Rang(K)$ and K has closed range. Let $\{f_j\}_{j\in J}$ be a $*$-K-frame, with $*$-K-frame bounds A, B. If there exist $\alpha, \beta, \gamma \geq 0$ such that* $\max\{\alpha + \frac{\gamma}{\|A\|}, \beta\} < 1$ *and*

$$\|\sum_{j\in J}\langle f, f_j - h_j\rangle\langle f_j - h_j, f\rangle\|^{\frac{1}{2}} \leq \alpha\|\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle\|^{\frac{1}{2}} + \beta\|\sum_{j\in J}\langle f, h_j\rangle\langle h_j, f\rangle\|^{\frac{1}{2}} + \gamma\|K^* f\|, \tag{5.4}$$

then $\{h_j\}_{j\in J}$ is a $*$-L-frame.

Proof. For any $f \in \mathcal{H}$ we have

$$\begin{aligned}\|\sum_{j\in J}\langle f,h_j\rangle\langle h_j,f\rangle\|^{\frac{1}{2}} &= \|\{f,h_j\}\| \le \|\{f,f_j-h_j\}\| + \|\{f,f_j\}\| \\ &= \|\sum_{j\in J}\langle f,f_j-h_j\rangle\langle f_j-h_j,f\rangle\|^{\frac{1}{2}} + \|\sum_{j\in J}\langle f,f_j\rangle\langle f_j,f\rangle\|^{\frac{1}{2}} \\ &\le (1+\alpha)\|\sum_{j\in J}\langle f,f_j\rangle\langle f_j,f\rangle\|^{\frac{1}{2}} + \beta\|\sum_{j\in J}\langle f,h_j\rangle\langle h_j,f\rangle\|^{\frac{1}{2}} + \gamma\|K^*f\|.\end{aligned}$$

So

$$\begin{aligned}(1-\beta)\|\sum_{j\in J}\langle f,h_j\rangle\langle h_j,f\rangle\|^{\frac{1}{2}} &\le (1+\alpha)\|\sum_{j\in J}\langle f,f_j\rangle\langle f_j,f\rangle\|^{\frac{1}{2}} + \gamma\|K^*f\| \\ &\le ((1+\alpha)+\frac{\gamma}{\|A\|})\|\sum_{j\in J}\langle f,f_j\rangle\langle f_j,f\rangle\|^{\frac{1}{2}}.\end{aligned}$$

Hence

$$\|\sum_{j\in J}\langle f,h_j\rangle\langle h_j,f\rangle\|^{\frac{1}{2}} \le \|B\|(1+\frac{\alpha+\beta+\frac{\gamma}{\|A\|}}{1-\beta})\|f\|. \tag{5.5}$$

Similarly

$$\|\sum_{j\in J}\langle f,h_j\rangle\langle h_j,f\rangle\|^{\frac{1}{2}} \ge (1-\alpha-\frac{\gamma}{\|A\|})\|\sum_{j\in J}\langle f,f_j\rangle\langle f_j,f\rangle\|^{\frac{1}{2}} - \beta\|\sum_{j\in J}\langle f,h_j\rangle\langle h_j,f\rangle\|^{\frac{1}{2}}.$$

Also $\{h_j\}_{j\in J}$ is a $*$-Bessel sequence which implies that the operator $T : \mathcal{H} \to \ell^2(\mathcal{A})$ defined by $Tf = \{\langle f,h_j\rangle\}_{j\in J}$ is well-defined. Thus $Rang(L) \subseteq Rang(K)$ and Theorem 1.2 imply that there exists $\mu > 0$ such that $\mu\|K^*f\|^2 \ge \|L^*f\|^2$, for every $f \in \mathcal{H}$. So for any $f \in \mathcal{H}$

$$\begin{aligned}\|Tf\| = \|\sum_{j\in J}\langle f,h_j\rangle\langle h_j,f\rangle\|^{\frac{1}{2}} &\ge (1-\frac{\alpha+\beta+\frac{\gamma}{\|A\|}}{1+\beta})\|\sum_{j\in J}\langle f,f_j\rangle\langle f_j,f\rangle\|^{\frac{1}{2}} \\ &\ge \|A\|(1-\frac{\alpha+\beta+\frac{\gamma}{\|A\|}}{1+\beta})\|K^*f\|, \quad f\in\mathcal{H}.\end{aligned} \tag{5.6}$$

So by (5.5), (5.6), Theorem 1.2, Lemma 1.3 and part (*iv*) of Proposition 4.1, $\{h_j\}_{j\in J}$ is a $*$-L-frame. □

Corollary 5.5. *Assume that $K, L \in Hom^*_{\mathcal{A}}(\mathcal{H})$ with $Rang(L) \subseteq Rang(K)$ and K has closed range. Let $\{f_j\}_{j\in J}$ be a $*$-K-frame with $*$-K-frame bounds $A, B \in \mathcal{Z}(\mathcal{A})$. If there exist $\alpha,\beta,\gamma \ge 0$ such that* $\max\{\alpha+\frac{\gamma}{\|A\|},\beta\} < 1$ *and*

$$\|\sum_{j\in J}\langle f,f_j-h_j\rangle\langle f_j-h_j,f\rangle\|^{\frac{1}{2}} \le \alpha\|\sum_{j\in J}\langle f,f_j\rangle\langle f_j,f\rangle\|^{\frac{1}{2}} + \beta\|\sum_{j\in J}\langle f,h_j\rangle\langle h_j,f\rangle\|^{\frac{1}{2}} + \gamma\|K^*f\|, \tag{5.7}$$

then $\{h_j\}_{j\in J}$ is a $*$-L-frame.

Proof. It can be proved by using Theorem 5.4 and Lemma 3.3. □

Corollary 5.6. *Assume that $K \in Hom^*_{\mathcal{A}}(\mathcal{H})$ has closed range. Let $\{f_j\}_{j\in J}$ be a $*$-K-frame, with $*$-K-frame bounds A, B. If there exist $\alpha,\beta,\gamma \geq 0$ such that* $\max\{\alpha + \frac{\gamma}{\|A\|},\beta\} < 1$ *and*

$$\|\sum_{j\in J}\langle f, f_j - h_j\rangle\langle f_j - h_j, f\rangle\|^{\frac{1}{2}} \leq \alpha\|\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle\|^{\frac{1}{2}} + \|\sum_{j\in J}\langle f, h_j\rangle\langle h_j, f\rangle\|^{\frac{1}{2}} + \gamma\|K^* f\|, \quad (5.8)$$

then $\{h_j\}_{j\in J}$ is a $*$-K-frame.

Corollary 5.7. *Let $\{f_j\}_{j\in J}$ be a $*$-frame, with $*$-frame bounds A, B. If there exists $\alpha,\beta,\gamma \geq 0$ such that* $\max\{\alpha + \frac{\gamma}{\|A\|},\beta\} < 1$ *and*

$$\|\sum_{j\in J}\langle f, f_j - h_j\rangle\langle f_j - h_j, f\rangle\|^{\frac{1}{2}} \leq \alpha\|\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle\|^{\frac{1}{2}} + \|\sum_{j\in J}\langle f, h_j\rangle\langle h_j, f\rangle\|^{\frac{1}{2}} + \gamma\|K^* f\|, \quad (5.9)$$

then $\{h_j\}_{j\in J}$ is a $*$-frame.

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