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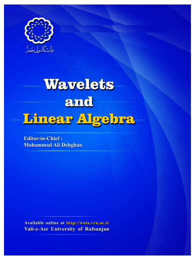

Max-Plus Algebra on tensors and its properties

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Abstract

In this paper we generalize the max plus algebra system of real matrices to the class of real tensors and derive its fundamental properties. Also we give some basic properties for the left (right) inverse, under the new system. The existence of order 2 left (right) inverses of tensors is characterized.

1. Introduction

In max plus algebra we work with the max plus semi-ring which is the set $\Re_{\max} = \Re \cup \{-\infty\}$ together with operations $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$. The additive and multiplicative identities are taken to be $\varepsilon = -\infty$ and $e = 0$ respectively. Max plus algebra is one of many idempotent semirings which have been considered in various fields of mathematics. It has many applications in many areas such as optimization, mathematical physics, algebraic geometry, control theory, machine scheduling, manufacturing systems, parallel processing systems and traffic control, see [6],

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[7], [8] and [10]. Many equations that are used to describe the behavior of these applications are nonlinear in conventional algebra but become linear in max-plus algebra. This is a primary reason for its utility in various areas [7].

A tensor can be regarded as a higher order generalization of a matrix, which takes the form

$$\mathbb{A} = (a_{i_1,\ldots,i_m}), \quad a_{i_1,\ldots,i_m} \in \Re, \quad 1 \leq \ i_1, \ldots, i_m \ \leq n,$$

where $\Re$ is a real field. Such a multi-array $\mathbb{A}$ is said to be an mth order n-dimensional square real tensor with n^m entries $a_{i_1,\ldots,i_m}$. In this regard, a vector is a first order tensor and a matrix is a second order tensor. Tensors of order more than two are called higher order tensors.

These are some basic knowledge of max plus algebra on matrices. Is it possible to extend them to higher orders tensors? In this paper we show the answer is affirmative.

The paper is organized as follows. In Section 2, the fundamental concepts of max plus algebra system and tensors are given briefly for readers. In Section 3, we generalize the max plus algebra system to tensors and we obtain some properties. In section 4, the inverse of tensor under the new system is defined. The left (right) inverse of diagonal tensors of any order are obtained and some basic properties for order 2 left (right) inverse of tensors are given.

We first add a comment on the notation that is used. Vectors are written as $(x, y, \ldots)$, matrices correspond to $(A, B, \ldots)$ and tensors are written as $(\mathbb{A}, \mathbb{B}, \ldots)$. The entry with row index i and column index j in a matrix A, i.e. $(A)_{ij}$ is symbolized by a_{ij} (also $(\mathbb{A})_{i_1 i_2 \ldots i_m} = a_{i_1 i_2 \ldots i_m}$). $\Re$ and C represents the real and complex field, respectively. For each nonnegative integer n, denote $[n] = \{1, 2, \ldots, n\}$. We denote $\Re_{\max} = \Re \cup \{-\infty\}$ and $\bar{\Re}_{\max} = \Re \cup \{\pm\infty\}$.

2. Preliminaries

2.1. Max plus algebra system

In this subsection we give the basic definition of the max plus algebra. For the proofs and more information about max plus algebra the reader is referred to [2, 3, 4, 9, 11]. If $a, b \in \Re_{max}$, then we set

$$a \oplus b = \max(a, b),$$

and

$$a \otimes b = a + b.$$

For example,

$$(-1) \oplus 2 = \max(-1, 2) = 2 = \max(2, -1) = 2 \oplus (-1),$$

$$7 \otimes 3 = 7 + 3 = 10 = 3 + 7 = 3 \otimes 7.$$

By max plus algebra we understand the analogue of linear algebra developed for the pair of operations $(\oplus, \otimes)$, extended to matrices and vectors as in conventional linear algebra. That is, for vectors $x = (x_i)$, $y = (y_i)$ in $\Re^n_{max}$ and $c \in \bar{\Re}_{\max}$ the vectors $x \oplus y = (\max\{x_i, y_i\})$ and $c \otimes x = (c \otimes x_i)$ are defined entrywise. The sum $A \oplus B$ of two matrices is defined analogously. If A, B, C are matrices of compatible sizes with entries from $\Re_{max}$, we write $C = A \oplus B$ if $c_{ij} = max(a_{ij}, b_{ij})$ for all $i, j \in [n]$.

If $A = (a_{ik}) \in M_n(\mathfrak{R}_{\max})$, then the map

$$x \in \mathfrak{R}_{max} \Rightarrow A \otimes x \in \mathfrak{R}_{max},$$

where $(A \otimes x)_i = \max_k (a_{ik} + x_k)$, $i \in [n]$, is linear in the sense given above, namely for all $x, y \in \mathfrak{R}^n_{max}$, $c \in \bar{\mathfrak{R}}_{\max}$

$$A \otimes (x \oplus y) = (A \otimes x) \oplus (A \otimes y), A \otimes (c \otimes x) = c \otimes (A \otimes x).$$

Also we write $C = (c_{il}) = A \otimes B$, if $c_{il} = \max_k (a_{ik} + b_{kl})$, for all $i, l \in [n]$ and $c \otimes A = A \otimes c = \left(c \otimes a_{ij}\right)$ for $c \in \bar{\mathfrak{R}}_{\max}$.

2.2. Basic definition of tensor

In this subsection, we will cover some fundamental notions and properties on tensors. We denote the set of all mth order n-dimensional tensors by $\mathfrak{R}^{[m,n]}_{\max}$ such that all entries belong to $\mathfrak{R}_{\max}$. For a vector $x = (x_1, ..., x_n)^T \in \mathfrak{R}^n$, let $\mathbb{A}x^{m-1}$ be a vector in $\mathfrak{R}^n$ whose ith component is defined as the following [13]:

$$\left(\mathbb{A}x^{m-1}\right)_i = \sum_{i_2,...,i_m=1}^{n} a_{ii_2...i_m} x_{i_2} \dots x_{i_m}, \tag{2.1}$$

and let $x^{[m]} = \left(x_1^m, ..., x_n^m\right)^T$.

Definition 2.1. [14] Let $\mathbb{A}$ (and $\mathbb{B}$) be an order $m \geq 2$ (and order $k \geq 1$), dimension n tensor, respectively. The product $\mathbb{A}\mathbb{B}$ is defined to be the following tensor $\mathbb{C}$ of order $(m-1)(k-1)+1$ and dimension n:

$$c_{i\alpha_1...\alpha_{m-1}} = \sum_{i_2,...,i_m=1}^{n} a_{ii_2...i_m} b_{i_2\alpha_1} ... b_{i_m\alpha_{m-1}},$$

where $(i \in [n], \alpha_1, ..., \alpha_{m-1} \in [n]^{k-1})$.

It is easy to check from the definition that $I_n\mathbb{A} = \mathbb{A} = \mathbb{A}I_n$, where I_n is the identity matrix of order n. When $k = 1$ and $\mathbb{B} = x \in C^n$ is a vector of dimension n, then $(m-1)(k-1)+1 = 1$. Thus $\mathbb{A}\mathbb{B} = \mathbb{A}x$ is still a vector of dimension n, and we have

$$(\mathbb{A}x)_i = (\mathbb{A}\mathbb{B})_i = c_i = \sum_{i_2...i_m=1}^{n} a_{ii_2...i_m} x_{i_2} ... x_{i_m} = \left(\mathbb{A}x^{m-1}\right)_i,$$

Thus we have $\mathbb{A}x^{m-1} = \mathbb{A}x$. So the first application of the tensor product defined above is that now $\mathbb{A}x^{m-1}$ can be simply written as $\mathbb{A}x$.

Definition 2.2. [12] A tensor $\mathbb{A} \in \mathfrak{R}^{[m,n]}$ is reducible, if there exists a nonempty proper index subset $I \subset \{1, ..., n\}$ such that

$$a_{i_1,...,i_m} = 0, \quad \forall i_1 \in I, \quad \forall i_2, ..., i_m \notin I,$$

If $\mathbb{A}$ is not reducible, then we call $\mathbb{A}$ irreducible.

3. Max plus algebra on tensors: Basic definitions and properties

In this section we define the max plus algebra system on tensors and we investigate their basic properties.

Definition 3.1. The max plus algebraic addition ($\oplus$) and multiplication ($\otimes$) are defined as follows:
(i) Suppose that $\mathbb{A}, \mathbb{B}$ are mth order n dimensional tensors with entries from $\mathfrak{R}_{\max}$ then we have $\mathbb{A} \oplus \mathbb{B}$ is mth order n dimensional tensor and

$$(\mathbb{A} \oplus \mathbb{B})_{i_1 \ldots i_m} = a_{i_1 \ldots i_m} \oplus b_{i_1 \ldots i_m} = \max\left(a_{i_1 \ldots i_m}, b_{i_1 \ldots i_m}\right). \tag{3.1}$$

(ii) Suppose that $\mathbb{A} \in \mathfrak{R}_{max}^{[m,n]}$ and $\mathbb{B} \in \mathfrak{R}_{max}^{[k,n]}$ where $m \geq 2, k \geq 1$ then we have $\mathbb{A} \otimes \mathbb{B} \in \mathfrak{R}_{max}^{[(m-1)(k-1)+1,n]}$ and

$$\begin{aligned}(\mathbb{A} \otimes \mathbb{B})_{i\alpha_1 \ldots \alpha_{m-1}} &= \bigoplus_{i_2,\ldots,i_m=1}^{n} a_{ii_2 \ldots i_m} \otimes b_{i_2\alpha_1} \otimes \ldots \otimes b_{i_m \otimes \alpha_{m-1}} \\ &= \max_{1 \leq i_2,\ldots,i_m \leq n} \left\{a_{ii_2 \ldots i_m} + b_{i_2\alpha_1} + \ldots + b_{i_m\alpha_{m-1}}\right\},\end{aligned} \tag{3.2}$$

where $i \in \{1, \ldots, n\}$, $\alpha_1, \ldots, \alpha_{m-1} \in [n]^{k-1}$. In particular for $x \in \mathfrak{R}_{max}^{n}$ we have

$$(\mathbb{A} \otimes x)_i = \max_{1 \leq i_2 \ldots i_m \leq n} \left\{a_{ii_2 \ldots i_m} + x_{i_2} + \ldots + x_{i_m}\right\}.$$

Example 3.2. Let $\mathbb{A}$ and $\mathbb{B}$ be third-order two-dimensional tensors of the following form:

$$\begin{array}{llll} a_{111} = 1 & a_{121} = 2 & a_{112} = 1 & a_{122} = 2 \\ a_{211} = 2 & a_{221} = 1 & a_{212} = -2 & a_{222} = -8, \end{array}$$

$$\begin{array}{llll} b_{111} = 2 & b_{121} = 0 & b_{112} = 4 & b_{122} = -1 \\ b_{211} = 10 & b_{221} = -3 & b_{212} = 1 & b_{222} = 0, \end{array}$$

if $\mathbb{C} = \mathbb{A} \otimes \mathbb{B}$, then for example $c_{12112} = 5$.
If $x = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ then $(\mathbb{A} \otimes x) = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$.

The max plus algebraic addition ($\oplus$) and multiplication ($\otimes$) have the following properties:

Theorem 3.3. *Let* $\mathbb{A}, \mathbb{B}, \mathbb{C} \in \mathfrak{R}_{max}^{[m,n]}$, *then*
(i) $\mathbb{A} \oplus \mathbb{B} = \mathbb{B} \oplus \mathbb{A}$.
(ii) $\mathbb{A} \oplus (\mathbb{B} \oplus \mathbb{C}) = (\mathbb{A} \oplus \mathbb{B}) \oplus \mathbb{C}$.
(iii) $\mathbb{A} \oplus \mathbb{E} = \mathbb{A} = \mathbb{E} \oplus \mathbb{A}$ *where* $\mathbb{E}$ *is an mth order n dimensional tensor whose all entries are* ε.
(iv) $\mathbb{A} \oplus \mathbb{B} \geq \mathbb{A}$.
(v) $\mathbb{A} \oplus \mathbb{B} = \mathbb{A}$ *if and only if* $\mathbb{A} \geq \mathbb{B}$.

Theorem 3.4. *Let* $\mathbb{A}, \mathbb{B}, \mathbb{C} \in \mathfrak{R}_{max}^{[m,n]}$ *and* $\alpha \in \bar{\mathfrak{R}}_{\max}$, *then*
(i) $(\alpha \otimes \mathbb{A})_{i_1 i_2 \ldots i_m} = (\mathbb{A} \otimes \alpha)_{i_1 i_2 \ldots i_m} = \alpha + a_{i_1 i_2 \ldots i_m}$.
(ii) $\mathbb{A} \otimes E = \mathbb{E} = E \otimes \mathbb{A}$, *where* E *is an* $n \times n$ *matrix whose all entries are* ε.
(iii) $\alpha \otimes (\mathbb{B} \oplus \mathbb{C}) = (\alpha \otimes \mathbb{B}) \oplus (\alpha \otimes \mathbb{C})$.

(iv) $\mathbb{B} \otimes (\alpha \otimes \mathbb{C}) = ((m-1)\,\alpha) \otimes (\mathbb{B} \otimes \mathbb{C})$ *and* $(\alpha \otimes \mathbb{B}) \otimes \mathbb{C} = \alpha \otimes (\mathbb{B} \otimes \mathbb{C})$.
(v) *Let* $\mathbb{A}_1,\ \mathbb{A}_2 \in \mathfrak{R}_{max}^{[m,n]}$ *and* $\mathbb{B} \in \mathfrak{R}_{max}^{[k,n]}$ *then* $(\mathbb{A}_1 \oplus \mathbb{A}_2) \otimes \mathbb{B} = (\mathbb{A}_1 \otimes \mathbb{B}) \oplus (\mathbb{A}_2 \otimes \mathbb{B})$.
(vi) *Let A be an $n \times n$ matrix and* $\mathbb{B}_1, \mathbb{B}_2 \in \mathfrak{R}_{max}^{[k,n]}$ *then* $A \otimes (\mathbb{B}_1 \oplus \mathbb{B}_2) = (A \otimes \mathbb{B}_1) \oplus (A \otimes \mathbb{B}_2)$. *(Note that in general when A is not a matrix, then the right distributivity doesn't hold.)*
(vii) *Let T, S are both matrices. Then*

$$T \otimes (\mathbb{A} \oplus \mathbb{B}) \otimes S = (T \otimes \mathbb{A} \otimes S) \oplus (T \otimes \mathbb{B} \otimes S).$$

Proof. The proof of **(i)**, **(ii)**, **(iii)** and **(iv)** is trivial. Also we have

$$\begin{aligned}
&((\mathbb{A}_1 \oplus \mathbb{A}_2) \otimes \mathbb{B})_{i\alpha_1\ldots\alpha_{m-1}} = \max_{1\le i_2,\ldots,i_m\le n}\left((\mathbb{A}_1 \oplus \mathbb{A}_2)_{ii_2\ldots i_m} + b_{i_2\alpha_1} + \ldots + b_{i_m\alpha_{m-1}}\right)\\
&= \max_{1\le i_2,\ldots,i_m\le n}\left(\max\left((\mathbb{A}_1)_{ii_2\ldots i_m}, (\mathbb{A}_2)_{ii_2\ldots i_m}\right) + b_{i_2\alpha_1} + \ldots + b_{i_m\alpha_{m-1}}\right)\\
&= \max_{1\le i_2,\ldots,i_m\le n}\left(\max\left((\mathbb{A}_1)_{ii_2\ldots i_m} + b_{i_2\alpha_1} + \ldots + b_{i_m\alpha_{m-1}}, (\mathbb{A}_2)_{ii_2\ldots i_m} + b_{i_2\alpha_1} + \ldots + b_{i_m\alpha_{m-1}}\right)\right)\\
&= \max\left(\max_{1\le i_2,\ldots,i_m\le n}\left((\mathbb{A}_1)_{ii_2\ldots i_m} + b_{i_2\alpha_1} + \ldots + b_{i_m\alpha_{m-1}}, (\mathbb{A}_2)_{ii_2\ldots i_m} + b_{i_2\alpha_1} + \ldots + b_{i_m\alpha_{m-1}}\right)\right)\\
&= \max\left(\max_{1\le i_2,\ldots,i_m\le n}(\mathbb{A}_1)_{ii_2\ldots i_m} + b_{i_2\alpha_1} + \ldots + b_{i_m\alpha_{m-1}}, \max_{1\le i_2,\ldots,i_m\le n}(\mathbb{A}_2)_{ii_2\ldots i_m} + b_{i_2\alpha_1} + \ldots + b_{i_m\alpha_{m-1}}\right)\\
&= ((\mathbb{A}_1 \otimes \mathbb{B}) \oplus (\mathbb{A}_2 \otimes \mathbb{B}))_{i\alpha_1\ldots\alpha_{m-1}}.
\end{aligned}$$

Thus the proof of **(v)** is complete. The proof of **(vi)** is similar. By the left distributive law and right distributive, the proof of part **(vii)** is complete. □

Now we use a method similar with the proof of Theorem 3.4 in [1] to show the associative law.

Theorem 3.5. *Let $\mathbb{A}$ (and $\mathbb{B}$, $\mathbb{C}$) be an order $m+1$ (and order $k+1$, order $r+1$), dimension n tensor, respectively. Then we have*

$$\mathbb{A} \otimes (\mathbb{B} \otimes \mathbb{C}) = (\mathbb{A} \otimes \mathbb{B}) \otimes \mathbb{C}.$$

Proof. For $\beta_1, \ldots, \beta_m \in ([n]^r)^k$, we write:

$$\beta_1 = \theta_{11}\ldots\theta_{1k},\ \ldots,\ \beta_m = \theta_{m1}\ldots\theta_{mk} \quad \left(\theta_{ij} \in [n]^r,\ i = 1, \ldots, m;\ j = 1, \ldots, k\right).$$

Then we have:

$$\begin{aligned}
(\mathbb{A} \otimes (\mathbb{B} \otimes \mathbb{C}))_{i\beta_1\ldots\beta_m} &= \max_{1\le i_1,\ldots,i_m\le n}\left\{a_{ii_1\ldots i_m} + \left(\sum_{j=1}^{m}(\mathbb{B} \otimes \mathbb{C})_{i_j\beta_j}\right)\right\}\\
&= \max_{1\le i_1,\ldots,i_m\le n}\left\{a_{ii_1\ldots i_m} + \left(\sum_{j=1}^{m}(\mathbb{B} \otimes \mathbb{C})_{i_j\theta_{j1}\ldots\theta_{jk}}\right)\right\}\\
&= \max_{1\le i_1,\ldots,i_m\le n}\left\{a_{ii_1\ldots i_m} + \left(\sum_{j=1}^{m}\max_{1\le t_{j1},\ldots,t_{jk}\le n} b_{i_jt_{j1}\ldots t_{jk}} + \left(c_{t_{j1}\theta_{j1}} + \ldots + c_{t_{jk}\theta_{jk}}\right)\right)\right\}\\
&= \max_{1\le i_1,\ldots,i_m\le n}\left\{a_{ii_1\ldots i_m} + \max_{1\le t_{jh}\le n(1\le j\le m;1\le h\le k)}\left(\sum_{j=1}^{m} b_{i_jt_{j1}\ldots t_{jk}} + \left(c_{t_{j1}\theta_{j1}} + \ldots + c_{t_{jk}\theta_{jk}}\right)\right)\right\}.
\end{aligned}$$

On the other hand, for $\alpha_1, ..., \alpha_m \in [n]^k$, we write:

$$\alpha_1 = t_{11}...t_{1k}, ..., \alpha_m = t_{m1}...t_{mk} \ \left(t_{ij} \in [n]\,,\ i = 1, ..., m;\ j = 1..., k\right).$$

Then we also have:

$$\begin{aligned}((\mathbb{A} \otimes \mathbb{B}) \otimes \mathbb{C})_{i\beta_1...\beta_m} &= \max_{\alpha_1,...,\alpha_m \in [n]^k} \left\{ (\mathbb{A} \otimes \mathbb{B})_{i\alpha_1...\alpha_m} + \left(\sum_{j=1}^{n} \left(c_{t_{j1}\theta_{j1}} + ... + c_{t_{jk}\theta_{jk}} \right) \right) \right\} \\ &= \max_{1 \le t_{jh} \le n (1 \le j \le m; 1 \le h \le k)} \left\{ \max_{1 \le i_1,...,i_m \le n} \left\{ a_{ii_1...i_m} + \left(\sum_{j=1}^{m} b_{i_j\alpha_j} \right) + \left(\sum_{j=1}^{m} c_{t_{j1}\theta_{j1}} + ... + c_{t_{jk}\theta_{jk}} \right) \right\} \right\} \\ &= \max_{1 \le i_1,...,i_m \le n} \left\{ a_{ii_1...i_m} + \max_{1 \le t_{jh} \le n (1 \le j \le m; 1 \le h \le k)} \left(\sum_{j=1}^{m} b_{i_j t_{j1}...t_{jk}} + \left(c_{t_{j1}\theta_{j1}} + ... + c_{t_{jk}\theta_{jk}} \right) \right) \right\}.\end{aligned}$$

Thus the proof is complete. □

Theorem 3.6. *Let $\mathbb{A}, \mathbb{B}$ and $\mathbb{C}$ be tensors over $\Re_{max}$ of compatible sizes and $\alpha \in \bar{\Re}_{\max}$, then*

(i) $\mathbb{A} \ge \mathbb{B} \ \Rightarrow (\mathbb{A} \oplus \mathbb{C}) \ge (\mathbb{B} \oplus \mathbb{C})$.
(ii) $\mathbb{A} \ge \mathbb{B} \ \Rightarrow (\mathbb{A} \otimes \mathbb{C}) \ge (\mathbb{B} \otimes \mathbb{C})$.
(iii) $\mathbb{A} \ge \mathbb{B} \ \Rightarrow (\mathbb{C} \otimes \mathbb{A}) \ge (\mathbb{C} \otimes \mathbb{B})$.
(iv) $\mathbb{A} \ge \mathbb{B} \ \Rightarrow (\alpha \otimes \mathbb{A}) \ge (\alpha \otimes \mathbb{B})$.

Proof. The proof is clear. □

Definition 3.7. A square matrix is called diagonal if all its diagonal entries are real numbers and off-diagonal entries are ε. A diagonal with all diagonal entries equal to 0 is called the unit matrix and denoted I.

Obviously, $A \otimes I = I \otimes A = A$ whenever A and I are of compatible sizes.

Theorem 3.8. *Let $\mathbb{A} \in \Re_{max}^{[m,n]}$.Then*

$$\mathbb{A} \otimes I = \mathbb{A} = I \otimes \mathbb{A},$$

whenever I is of a suitable dimension.

Definition 3.9. A permutation matrix is a matrix in which each row and each column contains exactly one entry equal to 0 and all other entries are equal to ε. If $\sigma : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ is a permutation we define the max plus permutation matrix $P_\sigma = \left(p_{ij}\right)$ where

$$p_{ij} = \begin{cases} 0 & i = \sigma(j) \\ \varepsilon & i \ne \sigma(j) \end{cases}$$

So that the j^{th} column of P_σ has 0 in the $\sigma(j)^{th}$ row.

Notice that a permutation matrix obtain from a unit matrix by permuting the rows or columns. Also we have $P^{-1} = P^T$.

Definition 3.10. Two mth order n dimensional tensor $\mathbb{A} = (a_{i_1,\ldots,i_m})$ and $\mathbb{B} = (b_{i_1,\ldots,i_m})$, are said to have the same ε-pattern if $a_{i_1,\ldots,i_m} = \varepsilon$ whenever $b_{i_1,\ldots,i_m} = \varepsilon$, and vice versa.

Definition 3.11. A matrix that has the same ε-pattern as a permutation matrix is called a generalized permutation matrix.

Definition 3.12. An mth order n dimensional tensor $\mathbb{A}$ is called diagonal if all its diagonal entries are real numbers and off-diagonal entries are ε. A diagonal tensor with all diagonal entries equal to 0 is called the unit tensor and denoted by $\mathbb{I}$.

Theorem 3.13. *Let $\mathbb{A}, \mathbb{I} \in \mathfrak{R}_{max}^{[m,n]}$, and $\mathbb{E}$ be a tensor whose all entries are ε.*
(i) *If $\mathbb{A} \otimes I = \mathbb{E}$, then $\mathbb{A} = \mathbb{E}$.*
(ii) *If $I \otimes \mathbb{A} = \mathbb{E}$, then $\mathbb{A} = \mathbb{E}$.*
(iii) *If $\mathbb{A} \otimes \mathbb{I} = 0$, then $\mathbb{A} = 0$.*
(iv) *If $\mathbb{I} \otimes \mathbb{A} = 0$, then $\mathbb{A} = 0$.*

Proof. **(i):**
By Definition 3.1, we have

$$(\mathbb{A} \otimes I)_{i\alpha_1\ldots\alpha_{m-1}} = \begin{cases} a_{ii_2\ldots i_m} & \alpha_j = i_{j+1},\ j = 1, 2, \ldots, m-1 \\ \varepsilon & otherwise \end{cases}$$

Hence $\mathbb{A} \otimes I = \mathbb{E}$ implies that $\mathbb{A} = \mathbb{E}$.
(ii):
Suppose that $\mathbb{A}$ has a non-ε entry $a_{i_1 i_2\ldots i_m}$, By Definition 3.1, we have $I \otimes \mathbb{A}$ has a non-ε entry $(I \otimes \mathbb{A})_{i_1\alpha} = a_{i_1\alpha}$, for $\alpha = i_2\ldots i_m$, which is a contradiction.
The proof of **(iii)** and **(iv)** are similar. □

Definition 3.14. A tensor $\mathbb{A} \in \mathfrak{R}_{max}^{[m,n]}$ is called reducible, if there exists a nonempty proper index subset $I \subset \{1, \ldots, n\}$ such that

$$a_{i_1,\ldots,i_m} = \varepsilon, \quad \forall i_1 \in I, \quad \forall i_2, \ldots, i_m \notin I,$$

If $\mathbb{A}$ is not reducible, then we call $\mathbb{A}$ irreducible.

Theorem 3.15. *If $\mathbb{A} \in \mathfrak{R}_{max}^{[m,n]}$ is irreducible, then $\mathbb{A}$ has no ε-face, that is*

$$\max_{1 \le i_2,\ldots,i_m \le n} \{a_{ii_2\ldots i_m}\} > \varepsilon, \quad \forall\, 1 \le i \le n.$$

Proof. Suppose not, then there exists i_0 so that $\max_{1 \le i_2,\ldots,i_m \le n} \{a_{i_0 i_2\ldots i_m}\} = \varepsilon$. Thus $a_{i_0 i_2\ldots i_m} = \varepsilon$, for all $i_2, \ldots, i_m$. In particular, if we let $I = \{i_0\}$, then $a_{i_1 i_2\ldots i_m} = \varepsilon$ for all $i_1 \in I$ and $i_2, \ldots, i_m \notin I$. this contradicts irreducibility. □

Lemma 3.16. *Let $\mathbb{A} \in \mathfrak{R}_{max}^{[m,n]}$ and P, Q be two matrices. then*

$$(P \otimes \mathbb{A} \otimes Q)_{i_1 \dots i_m} = \max_{1 \le j_1, \dots, j_m \le n} \left\{ a_{j_1 \dots j_m} + p_{i_1 j_1} + q_{j_2 i_2} + \dots + q_{j_m i_m} \right\}.$$

Proof. By Definition 3.1 we have

$$(P \otimes \mathbb{A} \otimes Q)_{i_1 \dots i_m} = \max_{1 \le j_2, \dots, j_m \le n} \left\{ \max_{1 \le j_1 \le n} \left(a_{j_1 \dots j_m} + p_{i_1 j_1} \right) + q_{j_2 i_2} + \dots + q_{j_m i_m} \right\}$$

$$= \max_{1 \le j_1, \dots, j_m \le n} \left\{ a_{j_1 \dots j_m} + p_{i_1 j_1} + q_{j_2 i_2} + \dots + q_{j_m i_m} \right\}.$$

□

Theorem 3.17. *Let $\sigma \in S_n$ be a permutation on the set $\{1, \dots, n\}$, $P = P_\sigma = \left(p_{ij}\right)$ be the corresponding permutation matrix of σ (where $p_{ij} = 0 \Leftrightarrow j = \sigma(i)$). Let $\mathbb{A}, \mathbb{B} \in \mathfrak{R}_{max}^{[m,n]}$ such that $\mathbb{B} = P \otimes \mathbb{A} \otimes P^T$, Then we have:*
(i) $b_{i_1 \dots i_m} = a_{\sigma(i_1) \dots \sigma(i_m)}$.
(ii) $P \otimes \mathbb{I} \otimes P^T = \mathbb{I}$.
(iii) *Let $D = diag(d_{11}, \dots, d_{nn})$ be an invertible diagonal matrix. Then*

$$D^{-(m-1)} \otimes \mathbb{I} \otimes D = \mathbb{I}.$$

Proof. By using Lemma 3.16 we have

$$b_{i_1 \dots i_m} = \left(P \otimes \mathbb{A} \otimes P^T\right)_{i_1 \dots i_m} = \max_{1 \le j_1, \dots, j_m \le n} \left\{ a_{j_1 \dots j_m} + p_{i_1 j_1} + \left(P^T\right)_{j_2 i_2} + \dots + \left(P^T\right)_{j_m i_m} \right\}$$

$$= \max_{1 \le j_1, \dots, j_m \le n} \left\{ a_{j_1 \dots j_m} + p_{i_1 j_1} + p_{i_2 j_2} + \dots + p_{i_m j_m} \right\} = a_{\sigma(i_1) \dots \sigma(i_m)}.$$

The proofs of **(ii)** and **(iii)** are trivial by using part **(i)**. □

Theorem 3.18. *Let $\mathbb{A} \in \mathfrak{R}_{max}^{[m,n]}$ be a reducible tensor and P be an $n \times n$ permutation matrix. Then $P \otimes \mathbb{A} \otimes P^T$ is a reducible tensor.*

Proof. By using the previous theorem and Definition 3.14 the assertion is clear. □

4. Inverse tensor under the new system

Since the operation $\oplus$ in max plus algebra is not invertible, inverse matrices are almost non-existent and thus some tools used in linear algebra are unavailable. It is known that in max plus algebra, generalized permutation matrices are the only type of invertible matrices [6, 4]:

Theorem 4.1. *Let $A \in M_n(\mathfrak{R}_{\max})$. Then a matrix $B = \left(b_{ij}\right)$ such that*

$$A \otimes B = I = B \otimes A,$$

exists if and only if A is a generalized permutation matrix.

Proof. See [4]. □

Recently in [5] the left and right inverse of tensors under the general product, are defined. In conventional multilinear algebra we know that, not all tensors have inverses. We will see that in max plus algebra the invertible tensors are even more limited.

Definition 4.2. Let $\mathbb{A}$ be a tensor of order m and dimension n, and let $\mathbb{B}$ be a tensor of order k and dimension n. If $\mathbb{A} \otimes \mathbb{B} = \mathbb{I}$, then $\mathbb{A}$ is called an order m left inverse of $\mathbb{B}$ in the max-plus sense, and $\mathbb{B}$ is called an order k right inverse of $\mathbb{A}$ in the max-plus sense.

Theorem 4.3. *Let $\mathbb{A} \in \mathfrak{R}_{max}^{[m,n]}$ be a diagonal tensor. Then the following statements hold:*
(i). *$\mathbb{A}$ has an order k left inverse if and only if $a_{ii...i} \neq \varepsilon$, $i = 1, 2, ..., n$.*
Moreover, an order k diagonal tensor $\mathbb{L}$ with diagonal entry $l_{ii...i} = -(k-1)\,a_{ii...i}$ is the unique order k left inverse of $\mathbb{A}$.
(ii). *$\mathbb{A}$ has an order k right inverse if and only if $a_{ii...i} \neq \varepsilon$, $i = 1, 2, ..., n$. In this case, an order k diagonal tensor $\mathbb{R}$ with diagonal entry $r_{ii...i} = \frac{-a_{ii...i}}{(m-1)}$ is the unique order k right inverse of $\mathbb{A}$.*

Proof. Let $\mathbb{A}$ has an order k left inverse, then there exists an order k dimension n tensor $\mathbb{L}$ such that $\mathbb{L} \otimes \mathbb{A} = \mathbb{I}$. Since $\mathbb{A}$ is diagonal, by Definition 3.1, $\mathbb{A}$ has an order k left inverse if and only if $a_{ii...i} \neq \varepsilon$, $i = 1, 2, ..., n$. By $\mathbb{L} \otimes \mathbb{A} = \mathbb{I}$ and Definition 3.1 we have $l_{ii...i} = -(k-1)\,a_{ii...i}$. Hence part **(i)** holds.
If $\mathbb{A}$ has an order k right inverse, then there exists an order k dimension n tensor $\mathbb{R}$ such that $\mathbb{A} \otimes \mathbb{R} = \mathbb{I}$. Since $\mathbb{A}$ is diagonal, by Definition 3.1, $\mathbb{A}$ has an order k right inverse if and only if $a_{ii...i} \neq \varepsilon$, $i = 1, 2, ..., n$. Also we have $(m-1)\,r_{ii...i} = -a_{ii...i}$. Hence part **(ii)** holds. □

Now we characterize all left (right) inverse of order 2 of an arbitrary tensor. In addition, we characterize those tensors which have left (right) inverse of order 2. For this purpose we need the following two lemmas which is given without proofs.

Lemma 4.4. *If $\mathbb{A} \in \mathfrak{R}_{max}^{[m,n]}$ and $\mathbb{A}$ has an order 2 left inverse L, then*
(i) *L does not have a real row and column.*
(ii) *L does not have an ε-row and ε-column.*
(iii) *$\mathbb{A}$ does not have an ε-face and real-face.*

Lemma 4.5. *If $\mathbb{A} \in \mathfrak{R}_{max}^{[m,n]}$ and $\mathbb{A}$ has an order 2 right inverse R, then*
(i) *R does not have a real row and column.*
(ii) *R does not have an ε-row and ε-column.*
(iii) *$\mathbb{A}$ does not have an ε-face and real-face.*

Theorem 4.6. *If $\mathbb{A} \in \mathfrak{R}_{max}^{[m,n]}$, and $\mathbb{A}$ has an order 2 left (right) inverse G, then G must be a generalised permutation matrix.*

Proof. Let $\mathbb{A}$ has an order 2 left (right) inverse G. We know that G does not have an ε-row (ε-column), so there will be at least one real entry in each row (column). Notice that if there exists one column of G such that has two real entries then $\mathbb{A}$ has an ε-face. Therefore there exists exactly one real entry in each column and row. □

Theorem 4.7. *If $\mathbb{A} \in \mathfrak{R}_{max}^{[m,n]}$, then $\mathbb{A}$ has an order 2 left inverse if and only if there exists a generalised permutation matrix G such that $\mathbb{A} = G \otimes \mathbb{I}$. Moreover, G^{-1} is the unique order 2 left inverse of $\mathbb{A}$.*

Proof. If $\mathbb{A} = G \otimes \mathbb{I}$, for a generalised permutation matrix G, then $\mathbb{A}$ has an order 2 left inverse G^{-1}. Assume C is an order 2 left inverse of $\mathbb{A}$, then $C \otimes \mathbb{A} = \mathbb{I}$, this equation conclude that C must be a generalised permutation matrix (by Theorem 4.6), thus $\mathbb{A} = C^{-1} \otimes \mathbb{I}$. Suppose that B is also an order 2 left inverse of $\mathbb{A}$, we can also get $\mathbb{A} = B^{-1} \otimes \mathbb{I}$. Hence $\left(C^{-1} - B^{-1}\right) \otimes \mathbb{I} = 0$, By Theorem 3.13, we have $C^{-1} = B^{-1}$. By the fact that a nonsingular matrix has a unique inverse matrix, it follows that $B = C$ and the desired results hold. □

Theorem 4.8. *If $\mathbb{A} \in \mathfrak{R}_{max}^{[m,n]}$, then $\mathbb{A}$ has an order 2 right inverse if and only if there exists a generalised permutation matrix Q such that $\mathbb{A} = \mathbb{I} \otimes Q$. In this case, Q^{-1} is the unique order 2 right inverse of $\mathbb{A}$.*

Proof. If $\mathbb{A} = \mathbb{I} \otimes Q$ for a generalised permutation matrix Q, then $\mathbb{A}$ has an order 2 left inverse Q^{-1}. If T is an order 2 right inverse of $\mathbb{A}$, then $\mathbb{A} \otimes T = \mathbb{I}$, imply that T is a generalised permutation matrix (by Theorem 4.6). So $\mathbb{A} = \mathbb{I} \otimes T^{-1}$. Hence if $\mathbb{A}$ has an order 2 right inverse, then there exists a generalised permutation matrix T such that $\mathbb{A} = \mathbb{I} \otimes T$.
If $\mathbb{R}$ is any order 2 right inverse of $\mathbb{A}$, then $\mathbb{A} \otimes R = \mathbb{I} \otimes Q \otimes R = \mathbb{I}$. Set $D = Q \otimes R$, then $\mathbb{I} = \mathbb{I} \otimes D$. By Definition 3.1, D must be the identity matrix of dimension n. Hence the proof is complete. □

Notice that for $m = 2$ (when $\mathbb{A}$ is a matrix), we have the right inverse is equal to left inverse, (refer to max algebra theory).

Theorem 4.9. *Let $\mathbb{A}$ and $\mathbb{B}$ be tensors such that $\mathbb{A} \otimes \mathbb{B} = 0$. Then the following hold:*
(i) *If the order 2 left inverse of a tensor $\mathbb{A}$ (resp. $\mathbb{B}$) exists, then $\mathbb{B} = 0$ (resp. $\mathbb{A} = 0$).*
(ii) *If the order 2 right inverse of a tensor $\mathbb{A}$ (resp. $\mathbb{B}$) exists, then $\mathbb{B} = 0$ (resp. $\mathbb{A} = 0$).*

Proof. If the order 2 left inverse of a tensor $\mathbb{A}$ exists, then by Theorem 4.7, there exists a generalised permutation matrix G such that $G \otimes \mathbb{I} \otimes \mathbb{B} = 0$. Thus $\mathbb{I} \otimes \mathbb{B} = 0$. By Theorem 3.13 we get $\mathbb{B} = 0$. Similarly if the order 2 left inverse of a tensor $\mathbb{B}$ exists, then Theorems 4.7 and 3.13 imply that $\mathbb{A} = 0$. Hence part **(i)** holds. The proof of **(ii)** follows in a manner similar to the proof of **(i)**, using the Theorems 4.8 and 3.13. □

Definition 4.10. We define a new class for tensors as follows:

$$\Gamma = \left\{ \mathbb{A} \in \mathfrak{R}_{\max}^{[m,n]} : \mathbb{A} = G \otimes \mathbb{I} = \mathbb{I} \otimes G, \ \textit{where } G \textit{ is a generalized permutation matrix} \right\}.$$

For example the unit tensor is belong to this class.

The following theorem is an interesting and fundamental extension of Theorem 4.1 for tensors, in which we charactrize the invertible tensors completely.

Theorem 4.11. *Let $\mathbb{A} \in \mathfrak{R}_{max}^{[m,n]}$.Then a matrix B such that*

$$\mathbb{A} \otimes B = \mathbb{I} = B \otimes \mathbb{A},$$

exists if and only if $\mathbb{A}$ is belong to Γ.

Proof. Let $\mathbb{A} \in \Gamma$, thus there exists a generalised permutation matrix G such that $\mathbb{A} = G \otimes \mathbb{I} = \mathbb{I} \otimes G$. By puting $B = G^{-1}$, we will have $\mathbb{A} \otimes B = \mathbb{I} = B \otimes \mathbb{A}$. On the other hand, if $\mathbb{A} \otimes B = \mathbb{I} = B \otimes \mathbb{A}$, Theorems 4.7 and 4.8 conclude that $\mathbb{A}$ is belong to Γ. □

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