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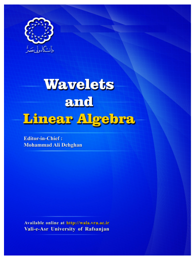

Some relations between ε-directional derivative and ε-generalized weak subdifferential

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Abstract

In this paper, we study ε-generalized weak subdifferential for vector valued functions defined on a real ordered topological vector space X. We give various characterizations of ε-generalized weak subdifferential for this class of functions. It is well known that if the function $f : X \to \mathbb{R}$ is subdifferentiable at $x_0 \in X$, then f has a global minimizer at x_0 if and only if $0 \in \partial f(x_0)$. We show that a similar result can be obtained for ε-generalized weak subdifferential. Finally, we investigate some relations between ε-directional derivative and ε-generalized weak subdifferential. In fact, in the classical subdifferential theory, it is well known that if the function $f : X \to \mathbb{R}$ is subdifferentiable at $x_0 \in X$ and it has directional derivative at x_0 in the direction $u \in X$, then the relation $f'(x_0, u) \geq \langle u, x^* \rangle, \forall\ x^* \in \partial f(x_0)$ is satisfied. We prove that a similar result can be obtained for ε-generalized weak subdifferential.

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1. Introduction

The ε-subgradient which was defined by Zalinescu [9] plays an important role in Optimization Theory. In the literature, Gasimov was the first to suggest an algorithm to solve non-convex optimization problems [4]. Subgradient was also defined by Y. Kücük, L. Atasever and M. Kücük for non-convex functions. Also, generalized weak subgradient and generalized weak subdifferential were defined for non-convex functions with values in an ordered vector space (see [7]). Azimov and Gasimov gave optimality conditions for a non-convex vector optimization problem by using weak subdifferentials that depend on supporting conic surfaces (see [1, 2]). So, weak subdifferentials and conic surfaces have important roles in non-convex optimization. Subgradient was also defined for convex functions with values in an ordered vector space (see [3, 8, 9, 10]). In this paper, we first define ε-generalized weak subdifferential for vector valued functions defined on a real topological vector space X. Next, we give various characterizations for ε-generalized weak subdifferential of this class of functions. Finally, we investigate some relations between ε-directional derivative and ε-generalized weak subdifferential. The paper is organized as follows: In Section 2, we recall some basic definitions. In Section 3, we give various characterizations for ε-generalized weak subdifferential of vector valued functions defined on a real ordered topological vector space X. Some properties of ε-generalized weak subdifferential are presented in Section 4. In Section 5, we examine some relations between ε-directional derivative and ε-generalized weak subdifferential.

2. Preliminaries

In this section, we give some basic definitions and results. Let Y be a real vector space and C_Y be a closed convex cone and pointed in Y (the later means that $C_Y \cap (-C_Y) = \{0\}$). The cone C_Y induces a relation $\leq_{C_Y}$ on Y which is defined by

$$x \leq_{C_Y} y \Leftrightarrow y - x \in C_Y, \quad (x, y \in Y).$$

It is clear that $\leq_{C_Y}$ is a partial order on Y, and so $(Y, \leq_{C_Y})$ is an ordered vector space. Moreover, if $intC_Y \neq \emptyset$, then we say that

$$x \ll y \Longleftrightarrow y - x \in intC_Y, (x, y \in Y).$$

Definition 2.1. ([5, 6]). Let $(Y, \leq_{C_Y})$ be a real ordered topological vector space with $intC_Y \neq \emptyset$.
(i) Let C be a subset of Y. A point $\bar{c} \in C$ is called a weakly maximal point of C if there is no $c \in C$ such that $\bar{c} \ll c$. The set of all weakly maximal points of C is called the weakly maximum of C and is denoted by $wmax\, C$.
(ii) Let C be a subset of Y. The supremum of C is defined as

$$SupC := wmax[cl(C - C_Y)],$$

where for a subset A of Y the $cl(A)$ is called the closure of A in Y.

Definition 2.2. ([5, 6]). Let $(Y, \leq_{C_Y})$ be a real ordered topological vector space and C be a non-empty subset of Y.
(i) An element $x \in Y$ such that $x \leq_{C_Y} c$ for all $c \in C$ is called a lower bound of C. An infimum of C, denoted by $\inf C$, is the greatest lower bound of C, that is, a lower bound x of C such that $z \leq_{C_Y} x$ for every other lower bound z of C.
(ii) An element $x \in Y$ such that $c \leq_{C_Y} x$ for all $c \in C$ is called an upper bound of C. A supremum of C, denoted by $\sup C$, is the least upper bound of C, that is, an upper bound x of C such that $x \leq_{C_Y} z$ for every other upper bound z of C.

Definition 2.3. ([7]). Let X be a real vector space and $(Y, \leq_{C_Y})$ be a real ordered vector space. A function $|||.||| : X \to C_Y$ is called a vectorial norm on X, if for all $x, z \in X$ and all $\lambda \in \mathbb{R}$ the following assertions are satisfied:
(i) $|||x||| = 0_Y \Leftrightarrow x = 0_X$.
(ii) $|||x||| = |\lambda|\ |||x|||$.
(iii) $|||x + z||| \leq_{C_Y} |||x||| + |||z|||$.

If $Y := \mathbb{R}$ and $C_Y := \mathbb{R}_+$, then $|||.|||$ is called a norm on X and denoted by $||.||$.

Let $(Y, \leq_{C_Y})$ be an ordered locally convex topological vector space. The topology that is induced by vectorial norm on X is the topology induced by the neighborhood base $\{X(a, U) : U \in B(0)\}$, where

$$X(a, U) := \{x \in X : |||x - a||| \in U\},$$

with $B(0)$ is a neighborhood base of the origin in Y and a running over X.

Definition 2.4. ([5, 6]). Let X be a real vector space and $(Y, \leq_{C_Y})$ be a real ordered vector space. Let S be a non-empty convex subset of X. A function $f : S \to Y$ is called C_Y-convex (or convex) if for all $x, y \in S$ and all $\lambda \in [0, 1]$

$$\lambda f(x) + (1 - \lambda) f(y) - f(\lambda x + (1 - \lambda) y) \in C_Y.$$

Definition 2.5. ([5, 6]). Let X be a real vector space and $(Y, \leq_{C_Y})$ be a real ordered vector space. Let S be a non-empty convex subset of X. Let the function $f : S \to Y$ be given. The set

$$epi(f) := \{(x, y) \in X \times Y : x \in S,\ f(x) \leq_{C_Y} y\}$$

is called the epigraph of f.
The set

$$hypo(f) := \{(x, y) \in X \times Y : x \in S,\ y \leq_{C_Y} f(x)\}$$

is called the hypograph of f.

Definition 2.6. ([6]). Let X and Y be real topological vector spaces. Let S be an open subset of X and $f : S \to Y$ be a given function. If for $\bar{x} \in S$ and $u \in X$ the limit

$$f'(\bar{x}, u) := \lim_{t \to 0^+} \frac{f(\bar{x} + tu) - f(\bar{x})}{t}$$

exists, then $f'(\bar{x}, u)$ is called the directional derivative of f at $\bar{x}$ in the direction u. If this limit exists for all $u \in X$, then, f is called directionally differentiable at $\bar{x}$.

Definition 2.7. Let X be a real topological vector space and $(Y, \leq_{C_Y})$ be a real ordered topological vector space with $intC_Y \neq \emptyset$. Let S be an open subset of X, $f : S \to Y$ be a given function and $\varepsilon \in \mathbb{R}_+$. If for $\bar{x} \in S$ and $u \in X$ the infimum

$$f'_\varepsilon(\bar{x}, u) := \inf_{t>0} \frac{f(\bar{x} + tu) - f(\bar{x}) + \varepsilon \mathbf{1}}{t}$$

exists in Y, then $f'_\varepsilon(\bar{x}, u)$ is called the ε-directional derivative of f at $\bar{x}$ in the direction u. If this infimum exists in Y for each $u \in X$, then, f is called ε-directionally differentiable at $\bar{x}$. Note that $\mathbf{1} \in intC_Y$. See Definition 2.2.

Definition 2.8. ([5, 6]). Let X and Y be real normed spaces and S be a non-empty open subset of X. Let $f : S \to Y$ be a given function and $\bar{x} \in X$. If there exists a continuous linear function $T : X \to Y$ such that

$$\lim_{\|h\| \to 0} \frac{\|f(\bar{x} + h) - f(\bar{x}) - T(h)\|}{\|h\|} = 0,$$

then T is called the Fr*é*chet derivative of f at $\bar{x}$ and denoted by $f'(\bar{x}) := T$. In this case, f is called Fr*é*chet differentiable at $\bar{x}$.

The proof of the following theorem can be found in [5, 6].

Theorem 2.9. *Let X be a real normed space, $(Y, \leq_{C_Y})$ be a real ordered normed space and S be a non-empty open subset of X. Let $f : S \to Y$ be Fréchet differentiable at every point $x \in S$. Then, f is C_Y-convex if and only if*

$$f'(y)(x - y) \leq_{C_Y} f(x) - f(y), \quad \forall x, y \in S.$$

3. ε-generalized weak subdifferential

In this section, we give various characterizations for ε-generalized weak subdifferential of a vector valued function f. Also, we present the definition of an ε-generalized lower locally Lipschitz function.

Definition 3.1. ([7]). Let X be a real topological vector space and $(Y, \leq_{C_Y})$ be a real ordered topological vector space. Assume that $f : X \to Y$ is a given function and $\bar{x} \in X$. Then a point $T \in B(X, Y)$ is called a subgradient of f at $\bar{x}$ if

$$f(x) - f(\bar{x}) - T(x - \bar{x}) \in C_Y, \quad \forall\ x \in X.$$

The set of all subgradients of f at $\bar{x}$ is called the subdifferential of f at $\bar{x}$ and denoted by

$$\partial f(\bar{x}) := \{T \in B(X, Y) : T \ is \ a \ subgradient \ of \ f \ at \ \bar{x}\},$$

where $B(X, Y)$ is the vector space of all continuous linear functions from X to Y.
Also, if $\partial f(\bar{x}) \neq \emptyset$, then, f is called subdifferentiable at $\bar{x}$.

Definition 3.2. Let X be a real topological vector space and $(Y, \leq_{C_Y})$ be a real ordered topological vector space. Assume that $intC_Y \neq \emptyset$, $f : X \to Y$ is a function, $\bar{x} \in X$ and $\varepsilon \in \mathbb{R}_+$. Then a point $T \in B(X, Y)$ is called an ε-subgradient of f at $\bar{x}$ if

$$f(x) - f(\bar{x}) - T(x - \bar{x}) + \varepsilon \mathbf{1} \in C_Y, \quad \forall\, x \in X.$$

The set of all ε-subgradients of f at $\bar{x}$ is called the ε-subdifferential of f at $\bar{x}$ and denoted by

$$\partial_\varepsilon f(\bar{x}) := \{T \in B(X, Y) : T\ is\ an\ \varepsilon - subgradient\ of\ f\ at\ \bar{x}\},$$

where $B(X, Y)$ is the vector space of all continuous linear functions from X to Y and $\mathbf{1} \in intC_Y$. Also, if $\partial_\varepsilon f(\bar{x}) \neq \emptyset$, then, f is called ε-subdifferentiable at $\bar{x}$.

Remark 3.3. Let X be a real topological vector space and $(Y, \leq_{C_Y})$ be a real ordered topological vector space. Assume that $intC_Y \neq \emptyset$, $f : X \to Y$ is a function and $\bar{x} \in X$. Suppose that $0 \leq \varepsilon_1 \leq \varepsilon_2$. Then, $\partial f(\bar{x}) = \partial_0 f(\bar{x}) \subseteq \partial_{\varepsilon_1} f(\bar{x}) \subseteq \partial_{\varepsilon_2} f(\bar{x})$, and for all $\varepsilon \in \mathbb{R}_+$ one has

$$\partial_\varepsilon f(\bar{x}) = \bigcap_{\delta > \varepsilon} \partial_\delta f(\bar{x}).$$

Definition 3.4. Let $(X, \|.\|)$ be a real normed space, $f : X \to \bar{\mathbb{R}}$ be a proper function, $\bar{x} \in X$ be such that $f(\bar{x}) \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}_+$. Then, $(x^*, c) \in X^* \times \mathbb{R}_+$ is called an ε-weak subgradient of f at $\bar{x}$ if

$$< x^*, x - \bar{x} > -c\|x - \bar{x}\| \leq f(x) - f(\bar{x}) + \varepsilon, \quad \forall\, x \in X.$$

The set of all ε-weak subgradients of f at $\bar{x}$ is called the ε-weak subdifferential of f at $\bar{x}$ and denoted by

$$\partial^w_\varepsilon f(\bar{x}) := \{(x^*, c) \in X^* \times \mathbb{R}_+ : (x^*, c)\ is\ an\ \varepsilon - weak\ subgradient\ of\ f\ at\ \bar{x}\}.$$

Also, if $\partial^w_\varepsilon f(\bar{x}) \neq \emptyset$, then, f is called ε-weak subdifferentiable at $\bar{x}$.

Definition 3.5. Let X be a real topological vector space and $(Y, \leq_{C_Y})$ be a real ordered topological vector space with $intC_Y \neq \emptyset$. Let $f : X \to Y$ be a function and $|||.||| : X \to C_Y$ be a vectorial norm on X and let $\bar{x} \in X$ and $\varepsilon \in \mathbb{R}_+$ be arbitrary. A point $(T, c) \in B(X, Y) \times \mathbb{R}_+$ is called an ε-generalized weak subgradient of f at $\bar{x}$ if

$$f(x) - f(\bar{x}) - T(x - \bar{x}) + c|||x - \bar{x}||| + \varepsilon \mathbf{1} \in C_Y, \quad \forall\, x \in X,$$

where $\mathbf{1} \in intC_Y$.

The set of all ε-generalized weak subgradients of f at $\bar{x}$ is called the ε-generalized weak subdifferential of f at $\bar{x}$ and denoted by

$$\partial^{gw}_\varepsilon f(\bar{x}) := \{(T, c) \in B(X, Y) \times \mathbb{R}_+ : (T, c)\ is\ an\ \varepsilon - generalized\ weak\ subgradient\ of\ f\ at\ \bar{x}\}.$$

Also, if $\partial^{gw}_\varepsilon f(\bar{x}) \neq \emptyset$, then, f is called ε-generalized weak subdifferentiable at $\bar{x}$.

Remark 3.6. In view of Definition 3.2 and Definition 3.4 it is easy to see that

$$(T, c) \in \partial_{\varepsilon}^{gw} f(\bar{x}) \Leftrightarrow T \in \partial_{\varepsilon}(f + c||| \cdot - \bar{x}|||)(\bar{x}),$$

where $f : X \to Y$ is a function, $|||.||| : X \to C_Y$ is a vectorial norm on X and $\bar{x} \in X$.

Lemma 3.7. *Let X be a real topological vector space and $(Y, \leq_{C_Y})$ be a real ordered topological vector space with $intC_Y \neq \emptyset$. Let $f : X \to Y$ be a function, $|||.||| : X \to C_Y$ be a vectorial norm on X and $\varepsilon \in \mathbb{R}_+$. If f is ε- subdifferentiable at $\bar{x} \in X$, then, f is ε-generalized weak subdifferentiable at $\bar{x}$.*

Proof. Since $\partial_{\varepsilon} f(\bar{x}) \neq \emptyset$, then there exists $T \in \partial_{\varepsilon} f(\bar{x})$ such that

$$T(x - \bar{x}) \leq_{C_Y} f(x) - f(\bar{x}) + \varepsilon \mathbf{1}, \quad \forall\, x \in X,$$

where $\mathbf{1} \in intC_Y$. So

$$f(x) - f(\bar{x}) - T(x - \bar{x}) + \varepsilon \mathbf{1} \in C_Y, \quad \forall\, x \in X.$$

Since $c|||x - \bar{x}||| \in C_Y$ for all $c \in \mathbb{R}_+$ and C_Y is a convex cone, it follows that

$$f(x) - f(\bar{x}) - T(x - \bar{x}) + \varepsilon \mathbf{1} + c|||x - \bar{x}||| \in C_Y, \quad \forall\, x \in X.$$

That is

$$T(x - \bar{x}) - c|||x - \bar{x}||| \leq_{C_Y} f(x) - f(\bar{x}) + \varepsilon \mathbf{1}, \quad \forall\, x \in X.$$

Hence, $(T, c) \in \partial_{\varepsilon}^{gw} f(\bar{x})$ for all $c \in \mathbb{R}_+$. Therefore, $\partial_{\varepsilon}^{gw} f(\bar{x}) \neq \emptyset$. □

The following example shows that the converse statement to Lemma 3.1 is not true.

Example 3.8. Let $X := \mathbb{R}$, $Y := \mathbb{R}^2$, $C_Y := \mathbb{R}^2_+$, $|||.||| : \mathbb{R} \to \mathbb{R}^2_+$ be defined by $|||x||| = (|x|, |x|)$ for all $x \in \mathbb{R}$, and let $f : \mathbb{R} \to \mathbb{R}^{\not\in}$ be defined by $f(x) = (-|x|, -|x|)$ for all $x \in \mathbb{R}$. Let $\varepsilon \in \mathbb{R}_+$. It is easy to see that f is not ε-subdifferentiable at $x = 0$, but f is ε-generalized weak subdifferentiable at $x = 0$.

Remark 3.9. Let X be a real topological vector space, $(Y, \leq_{C_Y})$ be a real ordered topological vector space with $intC_Y \neq \emptyset$ and $|||.||| : X \to C_Y$ be a vectorial norm on X. Let $\varepsilon \in \mathbb{R}_+$. Then

$$\partial_{\varepsilon}|||\bar{x}||| = \{T \in B(X, Y) : T(\bar{x}) = |||\bar{x}|||,\ T(x) \leq_{C_Y} |||x||| + \varepsilon \mathbf{1},\ \forall\, x \in X,\ \forall\, \varepsilon \in \mathbb{R}_+\},$$

where $\mathbf{1} \in intC_Y$.

Definition 3.10. Let X be a real topological vector space, $(Y, \leq_{C_Y})$ be a real ordered topological vector space with $intC_Y \neq \emptyset$ and $|||.||| : X \to C_Y$ be a vectorial norm on X. Let $\varepsilon \in \mathbb{R}_+$. A function $f : X \to Y$ is called ε-generalized lower locally Lipschitz at $\bar{x} \in X$ if there exists a non-negative real number L (Lipschitz constant) and a neighborhood $N(\bar{x})$ of $\bar{x}$ such that

$$-L|||x - \bar{x}||| \leq_{C_Y} f(x) - f(\bar{x}) + \varepsilon \mathbf{1}, \quad \forall\, x \in N(\bar{x}), \tag{3.1}$$

where $\mathbf{1} \in intC_Y$. If (3.1) holds for all $x \in X$, then, f is called ε-generalized lower Lipschitz at $\bar{x}$.

Theorem 3.11. *Let X be a real topological vector space, $(Y, \leq_{C_Y})$ be a real ordered topological vector space with $intC_Y \neq \emptyset$ and $|||.||| : X \to C_Y$ be a vectorial norm on X. Let $f : X \to Y$ be a function and let $\bar{x} \in X$ and $\varepsilon \in \mathbb{R}_+$. If f is ε-generalized lower Lipschitz at $\bar{x}$, then, f is ε-generalized weak subdifferentiable at $\bar{x}$.*

Proof. Suppose that f is ε-generalized lower Lipschitz at $\bar{x}$. Then there exists $L \geq 0$ such that

$$-L|||x - \bar{x}||| \leq_{C_Y} f(x) - f(\bar{x}) + \varepsilon\mathbf{1}, \quad \forall\, x \in X,$$

where $\mathbf{1} \in intC_Y$. So, we have

$$0(x - \bar{x}) - L|||x - \bar{x}||| \leq_{C_Y} f(x) - f(\bar{x}) + \varepsilon\mathbf{1}, \quad \forall\, x \in X.$$

Hence, $(0, L) \in \partial_\varepsilon^{gw} f(\bar{x})$, that is, f is ε-generalized weak subdifferentiable at $\bar{x}$. □

Theorem 3.12. *Under the hypotheses of Theorem 3.1 if f is ε- generalized lower Lipschitz at $\bar{x} \in X$, then there exists $p \geq 0$ and $q \in Y$ such that*

$$q - p|||x||| \leq_{C_Y} f(x) + \varepsilon\mathbf{1} \quad \forall\, x \in X,$$

where $\mathbf{1} \in intC_Y$.

Proof. Let f be ε-generalized lower Lipschitz at $\bar{x}$. Then there exists $L \geq 0$ such that

$$-L|||x - \bar{x}||| \leq_{C_Y} f(x) - f(\bar{x}) + \varepsilon\mathbf{1}, \quad \forall\, x \in X.$$

So, one has

$$-L|||x||| - L|||\bar{x}||| \leq_{C_Y} -L|||x - \bar{x}||| \leq_{C_Y} f(x) - f(\bar{x}) + \varepsilon\mathbf{1}, \quad \forall\, x \in X. \tag{3.2}$$

Now, put $q := f(\bar{x}) - L|||x|||$ and $p := L$ in (3.2). Thus it is clear that $p \geq 0$, $q \in Y$ and

$$q - p|||x||| \leq_{C_Y} f(x) + \varepsilon\mathbf{1},$$

for all $x \in X$. □

4. Properties of ε-generalized weak subdifferential

In the classical subdifferential theory, it is well known that if the function $f : X \to \mathbb{R}$ is subdifferentiable at $x_0 \in X$, then f has a global minimizer at x_0 if and only if $0 \in \partial f(x_0)$. In this section, a similar result can be obtained for ε-generalized weak subdifferential (see Theorem 4.1, below).

Proposition 4.1. *Let X be a real topological vector space, $(Y, \leq_{C_Y})$ be a real ordered topological vector space with $intC_Y \neq \emptyset$ and $|||.||| : X \to C_Y$ be a vectorial norm on X. Let $f : X \to Y$ be a function, $\bar{x} \in X$ and $\varepsilon \in \mathbb{R}_+$. Then, $\partial_\varepsilon^{gw} f(\bar{x})$ is a convex set.*

Proof. Let $(T_1, c_1), (T_2, c_2) \in \partial_\varepsilon^{gw} f(\bar{x})$ be arbitrary and $0 \le \lambda \le 1$. Then we have

$$T_1(x-\bar{x}) - c_1|||x-\bar{x}||| \le_{C_Y} f(x) - f(\bar{x}) + \varepsilon\mathbf{1}, \quad \forall\, x \in X,$$

and

$$T_2(x-\bar{x}) - c_2|||x-\bar{x}||| \le_{C_Y} f(x) - f(\bar{x}) + \varepsilon\mathbf{1}, \quad \forall\, x \in X,$$

where $\mathbf{1} \in intC_Y$. So, since C_Y is a cone, we have

$$\lambda f(x) - \lambda f(\bar{x}) + \lambda\varepsilon\mathbf{1} - \lambda T_1(x-\bar{x}) + \lambda c_1|||x-\bar{x}||| \in C_Y, \quad \forall x \in X,$$

and

$$(1-\lambda)f(x) - (1-\lambda)f(\bar{x}) + (1-\lambda)\varepsilon\mathbf{1} - (1-\lambda)T_2(x-\bar{x}) + (1-\lambda)c_2|||x-\bar{x}||| \in C_Y,$$

for all $x \in X$. Since C_Y is a convex cone, it follows that

$$f(x) - f(\bar{x}) + \varepsilon\mathbf{1} - (\lambda T_1 + (1-\lambda)T_2)(x-\bar{x}) + (\lambda c_1 + (1-\lambda)c_2)|||x-\bar{x}||| \in C_Y,$$

for all $x \in X$. Hence

$$(\lambda T_1 + (1-\lambda)T_2)(x-\bar{x}) - (\lambda c_1 + (1-\lambda)c_2)|||x-\bar{x}||| \le_{C_Y} f(x) - f(\bar{x}) + \varepsilon\mathbf{1},$$

for all $x \in X$. So, one has

$$(\lambda T_1 + (1-\lambda)T_2, \lambda c_1 + (1-\lambda)c_2) \in \partial_\varepsilon^{gw} f(\bar{x}).$$

That is

$$\lambda(T_1, c_1) + (1-\lambda)(T_2, c_2) \in \partial_\varepsilon^{gw} f(\bar{x}).$$

□

Proposition 4.2. *Let X be a real normed space space, $(Y, \le_{C_Y})$ be a real ordered normed space with $intC_Y \neq \emptyset$ and $|||.||| : X \to C_Y$ be a vectorial norm on X. Let $f : X \to Y$ be a function, $\bar{x} \in X$ and $\varepsilon \in \mathbb{R}_+$. Then, $\partial_\varepsilon^{gw} f(\bar{x})$ is a closed set in $B(X, Y) \times \mathbb{R}_+$.*

Proof. If $\partial_\varepsilon^{gw} f(\bar{x}) = \emptyset$, then it is closed. Suppose that $\partial_\varepsilon^{gw} f(\bar{x}) \neq \emptyset$ and $(T, c) \in cl(\partial_\varepsilon^{gw} f(\bar{x}))$ is arbitrary. Then there exists a sequence $\{(T_n, c_n)\}_{n\ge1} \subset \partial_\varepsilon^{gw} f(\bar{x})$ such that $\|(T_n, c_n) - (T, c)\| \to 0$ as $n \to \infty$, where for an element $(S, c) \in B(X, Y) \times \mathbb{R}_+$ we define

$$\|(S, c)\| := \|S\| + |c|.$$

Thus we conclude that $\|T_n - T\| \to 0$ and $|c_n - c| \to 0$ as $n \to \infty$. This implies that

$$\|T_n(x) - T(x)\| \to 0 \; for\; each\; x \in X, \; and\; |c_n - c| \to 0, \; as\; n \to \infty. \tag{4.1}$$

Now, assume on the contrary that $(T, c) \notin \partial_\varepsilon^{gw} f(\bar{x})$. Then there exists $x_0 \in X$ such that

$$T(x_0 - \bar{x}) - c|||x_0 - \bar{x}||| \not\le_{C_Y} f(x_0) - f(\bar{x}) + \varepsilon\mathbf{1}. \tag{4.2}$$

Since $(T_n, c_n) \in \partial_\varepsilon^{gw} f(\bar{x})$ $(n = 1, 2, \cdots)$, in view of Definition 3.4 we have

$$T_n(x - \bar{x}) - c_n|||x - \bar{x}||| \leq_{C_Y} f(x) - f(\bar{x}) + \varepsilon\mathbf{1}, \quad \forall\, x \in X,\ \forall\, n \geq 1,$$

where $\mathbf{1} \in intC_Y$. That is

$$f(x) - f(\bar{x}) + \varepsilon\mathbf{1} - T_n(x - \bar{x}) + c_n|||x_0 - \bar{x}||| \in C_Y, \quad \forall\, x \in X,\ \forall\, n \geq 1. \tag{4.3}$$

Put $x := x_0$ in (4.3), therefore one has

$$f(x_0) - f(\bar{x}) + \varepsilon\mathbf{1} - T_n(x_0 - \bar{x}) + c_n|||x_0 - \bar{x}||| \in C_Y, \quad \forall\, n \geq 1. \tag{4.4}$$

But we have

$$\begin{aligned}
&||[f(x_0) - f(\bar{x}) + \varepsilon\mathbf{1} - T_n(x_0 - \bar{x}) + c_n|||x_0 - \bar{x}|||] \\
&-[f(x_0) - f(\bar{x}) + \varepsilon\mathbf{1} - T(x_0 - \bar{x}) + c|||x_0 - \bar{x}|||]|| \\
&= ||[T_n(x_0 - \bar{x}) - T(x_0 - \bar{x})] + (c_n - c)|||x_0 - \bar{x}|||\,|| \\
&\leq ||T_n(x_0 - \bar{x}) - T(x_0 - \bar{x})|| + |c_n - c|\,||\,|||x_0 - \bar{x}|||\,||, \quad \forall\, n \geq 1.
\end{aligned} \tag{4.5}$$

In view of (4.1) it follows from (4.5) that

$$||[f(x_0) - f(\bar{x}) + \varepsilon\mathbf{1} - T_n(x_0 - \bar{x}) + c_n|||x_0 - \bar{x}|||] - [f(x_0) - f(\bar{x}) + \varepsilon\mathbf{1} - T(x_0 - \bar{x}) + c|||x_0 - \bar{x}|||]|| \to 0, \tag{4.6}$$

whenever $n \to \infty$. Since C_Y is closed, so we conclude from (4.4) and (4.6) that

$$f(x_0) - f(\bar{x}) + \varepsilon\mathbf{1} - T(x_0 - \bar{x}) + c|||x_0 - \bar{x}||| \in C_Y.$$

That is

$$T(x_0 - \bar{x}) - c|||x_0 - \bar{x}||| \leq_{C_Y} f(x_0) - f(\bar{x}) + \varepsilon\mathbf{1},$$

which contradicts (4.2). Hence $(T, c) \in \partial_\varepsilon^{gw} f(\bar{x})$, and the proof is complete. □

Proposition 4.3. *Let X be a real topological vector space, $(Y, \leq_{C_Y})$ be a real ordered topological vector space with $intC_Y \neq \emptyset$ and $|||.||| : X \to C_Y$ be a vectorial norm on X. Let $f : X \to Y$ be a function, $\bar{x} \in X$ and $\varepsilon \in \mathbb{R}_+$ be arbitrary. If f is $\frac{\varepsilon}{\lambda}$-generalized weak subdifferentiable at $\bar{x} \in X$, then, $\partial_\varepsilon^{gw}(\lambda f)(\bar{x}) = \lambda\partial_{\frac{\varepsilon}{\lambda}}^{gw} f(\bar{x})$ for each $\lambda > 0$.*

Proof. Let $\lambda > 0$ be arbitrary. Since C_Y is a cone, it follows that $(T, c) \in \partial_\varepsilon^{gw}(\lambda f)(\bar{x})$ if and only if

$$\lambda f(x) - \lambda f(\bar{x}) + \varepsilon\mathbf{1} - T(x - \bar{x}) + |||x - \bar{x}||| \in C_Y, \quad \forall\, x \in X$$

if only and if

$$f(x) - f(\bar{x}) + \frac{\varepsilon}{\lambda}\mathbf{1} - \frac{T}{\lambda}(x - \bar{x}) + \frac{c}{\lambda}|||x - \bar{x}||| \in C_Y, \quad \forall\, x \in X$$

if and only if $(\frac{T}{\lambda}, \frac{c}{\lambda}) \in \partial_{\frac{\varepsilon}{\lambda}}^{gw} f(\bar{x})$ if and only if $(T, c) \in \lambda\partial_{\frac{\varepsilon}{\lambda}}^{gw} f(\bar{x})$. □

Theorem 4.4. *Let X be a real topological vector space, $(Y, \leq_{C_Y})$ be a real ordered normed space with $intC_Y \neq \emptyset$ and $|||.||| : X \to C_Y$ be a vectorial norm on X. Let $\varepsilon \in \mathbb{R}_+$ and $f : X \to Y$ be a function such that f is ε-generalized weak subdifferentiable at $\bar{x} \in X$. Then, f has a global minimizer at $\bar{x}$ if and only if $(0,0) \in \partial_\varepsilon^{gw} f(\bar{x})$ for all $\varepsilon \in \mathbb{R}_+$.*

Proof. Suppose that f has a global minimizer at $\bar{x}$. Then one has $f(\bar{x}) \leq_{C_Y} f(x)$ for all $x \in X$, and also we have $0_Y \leq_{C_Y} \varepsilon\mathbf{1}$. So $f(\bar{x}) \leq_{C_Y} f(x) + \varepsilon\mathbf{1}$ for all $x \in X$. Therefore

$$0(x - \bar{x}) - 0|||x - \bar{x}||| \leq_{C_Y} f(x) - f(\bar{x}) + \varepsilon\mathbf{1}, \quad \forall\, x \in X.$$

Hence $(0,0) \in \partial_\varepsilon^{gw} f(\bar{x})$ for all $\varepsilon \in \mathbb{R}_+$.
Conversely, assume that $(0,0) \in \partial_\varepsilon^{gw} f(\bar{x})$ for all $\varepsilon \in \mathbb{R}_+$. Thus

$$0 \leq_{C_Y} f(x) - f(\bar{x}) + \varepsilon\mathbf{1}, \quad \forall\, x \in X,\ \forall\, \varepsilon \in \mathbb{R}_+.$$

That is

$$f(x) - f(\bar{x}) + \varepsilon\mathbf{1} \in C_Y, \quad \forall\, x \in X,\ \forall\, \varepsilon \in \mathbb{R}_+. \tag{4.7}$$

Therefore

$$\lim_{\varepsilon \to 0^+} ||[f(x) - f(\bar{x}) + \varepsilon\mathbf{1}] - [f(x) - f(\bar{x})]|| = \lim_{\varepsilon \to 0^+} ||\varepsilon\mathbf{1}|| = \lim_{\varepsilon \to 0^+} \varepsilon = 0, \tag{4.8}$$

for each $x \in X$. Note that $||\mathbf{1}|| = 1$. Since C_Y is a closed set in Y, it follows from (4.7) and (4.8) that

$$f(x) - f(\bar{x}) \in C_Y, \quad \forall\, x \in X.$$

So

$$0 \leq_{C_Y} f(x) - f(\bar{x}), \quad \forall\, x \in X.$$

That is $\bar{x}$ is a global minimizer of f at $\bar{x}$. □

Theorem 4.5. *Let X be a real normed space and $f : X \to \bar{\mathbb{R}}$ be a proper function. Let $\bar{x} \in domf$ and $\varepsilon \in \mathbb{R}_+$ be given. Then*

$$(x^*, c) \in \partial_\varepsilon^{w} f(\bar{x}) \Leftrightarrow f(\bar{x}) + (f + c||. - \bar{x}||)^*(x^*) \leq \langle x^*, \bar{x}\rangle + \varepsilon.$$

Proof. We have

$$(x^*, c) \in \partial_\varepsilon^{w} f(\bar{x}) \Leftrightarrow \langle x^*, x - \bar{x}\rangle - c||x - \bar{x}|| \leq f(x) - f(\bar{x}) + \varepsilon, \quad \forall\, x \in X.$$

$$\Leftrightarrow \langle x^*, x\rangle - f(x) - c||x - \bar{x}|| \leq \langle x^*, \bar{x}\rangle - f(\bar{x}) + \varepsilon, \quad \forall\, x \in X.$$

$$\Leftrightarrow \sup_{x \in X}\{\langle x^*, x\rangle - (f + c||\cdot - \bar{x}||)(x)\} \leq \langle x^*, \bar{x}\rangle - f(\bar{x}) + \varepsilon.$$

$$\Leftrightarrow (f + c||\cdot - \bar{x}||)^*(x^*) \leq \langle x^*, \bar{x}\rangle - f(\bar{x}) + \varepsilon.$$

$$\Leftrightarrow f(\bar{x}) + (f + c||\cdot - \bar{x}||)^*(x^*) \leq \langle x^*, \bar{x}\rangle + \varepsilon.$$

□

Proposition 4.6. *Let X be a real topological vector space, $(Y, \leq_{C_Y})$ be a real ordered topological vector space with $intC_Y \neq \emptyset$ and $|||.||| : X \to C_Y$ be a vectorial norm on X. Let $\varepsilon_1, \varepsilon_2 \in \mathbb{R}_+$ and $\bar{x} \in X$. Suppose that $f, g : X \to Y$ are functions such that f is ε_1-generalized weak subdifferentiable at $\bar{x}$ and g is ε_2-generalized weak subdifferentiable at $\bar{x}$. Then*

$$\partial^{gw}_{\varepsilon_1} f(\bar{x}) + \partial^{gw}_{\varepsilon_2} g(\bar{x}) \subseteq \partial^{gw}_{\varepsilon_1+\varepsilon_2}(f+g)(\bar{x}).$$

Proof. Let $(T_1, c_1) + (T_2, c_2) \in \partial^{gw}_{\varepsilon_1} f(\bar{x}) + \partial^{gw}_{\varepsilon_2} g(\bar{x})$, where $(T_1, c_1) \in \partial^{gw}_{\varepsilon_1} f(\bar{x})$ and $(T_2, c_2) \in \partial^{gw}_{\varepsilon_2} g(\bar{x})$. Then we have

$$f(x) - f(\bar{x}) + \varepsilon_1 \mathbf{1} - T_1(x - \bar{x}) + c_1|||x - \bar{x}||| \in C_Y, \quad \forall x \in X,$$

and

$$g(x) - g(\bar{x}) + \varepsilon_2 \mathbf{1} - T_2(x - \bar{x}) + c_2|||x - \bar{x}||| \in C_Y, \quad \forall\, x \in X.$$

Since C_Y is a convex cone, it follows that

$$(f+g)(x) - (f+g)(\bar{x}) + (\varepsilon_1 + \varepsilon_2)\mathbf{1} - (T_1 + T_2)(x - \bar{x}) + (c_1 + c_2)|||x - \bar{x}||| \in C_Y,$$

for all $x \in X$. That is

$$(T_1 + T_2)(x - \bar{x}) - (c_1 + c_2)|||x - \bar{x}||| \leq_{C_Y} (f+g)(x) - (f+g)(\bar{x}) + (\varepsilon_1 + \varepsilon_2)\mathbf{1},$$

for all $x \in X$. So, one has $(T_1 + T_2, c_1 + c_2) \in \partial^{gw}_{\varepsilon_1+\varepsilon_2}(f+g)(\bar{x})$. Hence $(T_1, c_1) + (T_2, c_2) \in \partial^{gw}_{\varepsilon_1+\varepsilon_2}(f+g)(\bar{x})$. □

5. Some relations between ε-directional derivative and ε-generalized weak subdifferential

In the classical subdifferential theory, it is well known that if the function $f : X \to \mathbb{R}$ is subdifferentiable at $x_0 \in X$ and it has directional derivative at x_0 in the direction $u \in X$, then the relation

$$f'(x_0, u) \geq \langle u, x^* \rangle, \quad \forall\, x^* \in \partial f(x_0)$$

is satisfied. In this section, a similar result can be obtained for ε-generalized weak subdifferential (see Theorem 5.2, below).

In the sequel, we give the following Definition (see [7]).

Definition 5.1. Let X be a real topological vector space and $(Y, \leq_{C_Y})$ be a real ordered topological vector space. Let $f : X \to Y$ be a function and $|||.||| : X \to C_Y$ be a vectorial norm on X and let $\bar{x} \in X$ be arbitrary. A point $(T, c) \in B(X, Y) \times \mathbb{R}_+$ is called a generalized weak subgradient of f at $\bar{x}$ if

$$T(x - \bar{x}) - c|||x - \bar{x}||| \leq_{C_Y} f(x) - f(\bar{x}), \quad \forall\, x \in X.$$

The set of all generalized weak subgradients of f at $\bar{x}$ is called the generalized weak subdifferential of f at $\bar{x}$ and denoted by

$$\partial^{gw} f(\bar{x}) := \{(T, c) \in B(X, Y) \times \mathbb{R}_+ : (T, c) \textit{ is a generalized weak subgradient of } f \textit{ at } \bar{x}\}.$$

Also, if $\partial^{gw} f(\bar{x}) \neq \emptyset$, then, f is called generalized weak subdifferentiable at $\bar{x}$.

Theorem 5.2. *Let X be a real topological vector space, $(Y, \leq_{C_Y})$ be a real ordered normed space with $intC_Y \neq \emptyset$, $|||.||| : X \to C_Y$ be a vectorial norm on X, $\bar{x} \in X$ and $\varepsilon \in \mathbb{R}_+$ be arbitrary. Let $f : X \to Y$ be a function such that f is generalized weak subdifferentiable and ε-directionally differentiable at $\bar{x}$. Then*

$$\partial_{\varepsilon}^{gw} f(\bar{x}) = \partial^{gw} f'_{\varepsilon}(\bar{x}, .)(0).$$

Proof. Let $(T, c) \in \partial_{\varepsilon}^{gw} f(\bar{x})$ be arbitrary. Then in view of Definition 3.4 one has

$$T(x - \bar{x}) - c|||x - \bar{x}||| \leq_{C_Y} f(x) - f(\bar{x}) + \varepsilon\mathbf{1}, \quad \forall\, x \in X, \tag{5.1}$$

where $\mathbf{1} \in intC_Y$. Let $u \in X$ and $t > 0$ be arbitrary. Put $x := \bar{x} + tu$ in (5.1), thus we have

$$tT(u) - tc|||u||| \leq_{C_Y} f(\bar{x} + tu) - f(\bar{x}) + \varepsilon\mathbf{1}.$$

So

$$T(u) - c|||u||| \leq_{C_Y} \frac{f(\bar{x} + tu) - f(\bar{x}) + \varepsilon\mathbf{1}}{t}, \quad \forall\, u \in X,\ \forall\, t > 0. \tag{5.2}$$

Therefore, by Definition 2.7 and (5.2) we obtain

$$T(u) - c|||u||| \leq_{C_Y} \inf_{t>0} \frac{f(\bar{x} + tu) - f(\bar{x}) + \varepsilon\mathbf{1}}{t} = f'_{\varepsilon}(\bar{x}, u), \tag{5.3}$$

for all $u \in X$. Since $f'_{\varepsilon}(\bar{x}, 0) = 0$, it follows that

$$T(u) - c|||u||| \leq_{C_Y} f'_{\varepsilon}(\bar{x}, u) - f'_{\varepsilon}(\bar{x}, 0), \quad \forall u \in X.$$

That is, $(T, c) \in \partial^{gw} f'_{\varepsilon}(\bar{x}, \cdot)(0)$. Conversely, let $(T, c) \in \partial^{gw} f'_{\varepsilon}(\bar{x}, \cdot)(0)$ be arbitrary. Then by Definition 5.1 and Definition 2.7 we have

$$\begin{aligned} T(u - 0) - c|||u - 0||| &\leq_{C_Y} f'_{\varepsilon}(\bar{x}, u) - f'_{\varepsilon}(\bar{x}, 0) \\ &= f'_{\varepsilon}(\bar{x}, u) = \inf_{t>0} \tfrac{f(\bar{x}+tu) - f(\bar{x}) + \varepsilon\mathbf{1}}{t} \\ &\leq_{C_Y} \tfrac{f(\bar{x}+tu) - f(\bar{x}) + \varepsilon\mathbf{1}}{t}, \quad \forall\, u \in X,\ \forall\, t > 0. \end{aligned}$$

So one has

$$T(tu) - c|||tu||| \leq_{C_Y} f(\bar{x} + tu) - f(\bar{x}) + \varepsilon\mathbf{1}, \quad \forall\, u \in X,\ \forall\, t > 0. \tag{5.4}$$

Let $x \in X$ be arbitrary. Then by putting $u := \frac{x - \bar{x}}{t}$ in (5.4), we conclude that

$$T(x - \bar{x}) - c|||x - \bar{x}||| \leq_{C_Y} f(x) - f(\bar{x}) + \varepsilon\mathbf{1}, \quad \forall\, x \in X.$$

Hence $(T, c) \in \partial_{\varepsilon}^{gw} f(\bar{x})$, and the proof is complete. □

Theorem 5.3. *Let X be a real topological vector space, $(Y, \leq_{C_Y})$ be a real ordered normed space with $intC_Y \neq \emptyset$ and $|||.||| : X \to C_Y$ be a vectorial norm on X. Let $\varepsilon \in \mathbb{R}_+$ and $f : X \to Y$ be a function such that f is ε-generalized weak subdifferentiable at $\bar{x} \in X$ and ε-directionally differentiable at $\bar{x}$ in the direction $u \in X$. Then*

$$v \leq_{C_Y} f'_\varepsilon(\bar{x}, u), \quad \forall\, v \in D,$$

where

$$D := Sup\{T(u) - c|||u||| : (T, c) \in \partial^{gw}_\varepsilon f(\bar{x})\}.$$

Also, see Definition 2.1.

Proof. We claim that

$$T(u) - c|||u||| \leq_{C_Y} f'_\varepsilon(\bar{x}, u), \quad \forall\, (T, c) \in \partial^{gw}_\varepsilon f(\bar{x}). \tag{5.5}$$

Since by the hypothesis $f'_\varepsilon(\bar{x}, u)$ exists in Y, then in view of Definition 2.7 there exists a sequence $\{t_n\}_{n\geq 1}$ of positive real numbers such that

$$\lim_{n\to\infty} \|\frac{f(\bar{x} + t_n u) - f(\bar{x}) + \varepsilon\mathbf{1}}{t_n} - f'_\varepsilon(\bar{x}, u)\| = 0. \tag{5.6}$$

Now, let $(T, c) \in \partial^{gw}_\varepsilon f(\bar{x})$ be arbitrary. Then one has

$$T(x - \bar{x}) - c|||x - \bar{x}||| \leq_{C_Y} f(x) - f(\bar{x}) + \varepsilon\mathbf{1}, \quad \forall\, x \in X. \tag{5.7}$$

Put $x := \bar{x} + t_n u$ $(n = 1, 2, \cdots)$ in (5.7), it follows that

$$t_n T(u) - ct_n|||u||| \leq_{C_Y} f(\bar{x} + t_n u) - f(\bar{x}) + \varepsilon\mathbf{1}, \quad n = 1, 2, \cdots. \tag{5.8}$$

Since C_Y is a cone, we conclude from (5.8) that

$$\frac{f(\bar{x} + t_n u) - f(\bar{x}) + \varepsilon\mathbf{1}}{t_n} - (T(u) - c|||u|||) \in C_Y, \quad n = 1, 2, \cdots. \tag{5.9}$$

But it follows from (5.6) that

$$\begin{aligned} &\lim_{n\to\infty} \|[\tfrac{f(\bar{x}+t_n u)-f(\bar{x})+\varepsilon\mathbf{1}}{t_n} - (T(u) - c|||u|||)] - [f'_\varepsilon(\bar{x}, u) - (T(u) - c|||u|||)]\| \\ &= \lim_{n\to\infty} \|\tfrac{f(\bar{x}+t_n u)-f(\bar{x})+\varepsilon\mathbf{1}}{t_n} - f'_\varepsilon(\bar{x}, u)\| \\ &= 0. \end{aligned}$$

Because of C_Y is closed, in view of (5.9) and (5.10) one has

$$f'_\varepsilon(\bar{x}, u) - (T(u) - c|||u|||) \in C_Y.$$

Hence (5.5) holds. Now, we show that

$$v \leq_{C_Y} f'_\varepsilon(\bar{x}, u), \quad \forall\, v \in cl(D_0 - C_Y),$$

where $D_0 := \{T(u) - c|||u||| : (T, c) \in \partial^{gw}_\varepsilon f(\bar{x})\}$.

For this end, let $v \in cl(D_0 - C_Y)$ be arbitrary. Then there exist sequences $\{v_n\}_{n\geq 1} \subset D_0$ and $\{d_n\}_{n\geq 1} \subset C_Y$ such that

$$\lim_{n\to\infty} \|(v_n - d_n) - v\| = 0. \tag{5.10}$$

Since $v_n \in D_0$ $(n = 1, 2, \cdots)$, it follows that there exists a sequence $\{(T_n, c_n)\}_{n\geq 1} \subset \partial^{gw}_\varepsilon f(\bar{x})$ such that $v_n = T_n(u) - c_n|||u|||$, $n = 1, 2, \cdots$. Let $w_n := v_n - d_n = T_n(u) - c_n|||u||| - d_n$, $n = 1, 2, \cdots$. This implies that $T_n(u) - c_n|||u||| - w_n = d_n \in C_Y$ for all $n = 1, 2, \cdots$. Thus we deduce that

$$w_n \leq_{C_Y} T_n(u) - c_n|||u|||, \quad \forall\, n \geq 1. \tag{5.11}$$

Since $(T_n, c_n) \in \partial^{gw}_\varepsilon f(\bar{x})$, $n = 1, 2, \cdots$, it follows from (5.5) and (5.12) that

$$w_n \leq_{C_Y} f'_\varepsilon(\bar{x}, u), \quad \forall\, n \geq 1.$$

That is

$$f'_\varepsilon(\bar{x}, u) - w_n \in C_Y, \quad \forall\, n \geq 1. \tag{5.12}$$

But by (5.11) we have

$$\lim_{n\to\infty} |||[f'_\varepsilon(\bar{x}, u) - w_n] - [f'_\varepsilon(\bar{x}, u) - v]||| = \lim_{n\to\infty} \|w_n - v\| = 0. \tag{5.13}$$

Because of C_Y is closed, we conclude from (5.13) and (5.14) that $f'_\varepsilon(\bar{x}, u) - v \in C_Y$. That is

$$v \leq_{C_Y} f'_\varepsilon(\bar{x}, u), \quad \forall\, v \in cl(D_0 - C_Y). \tag{5.14}$$

But in view of Definition 2.1 one has

$$D = Sup\{T(u) - c|||u||| : (T, c) \in \partial^{gw}_\varepsilon f(\bar{x})\}$$

$$= wmax[cl(\{T(u) - c|||u||| : (T, c) \in \partial^{gw}_\varepsilon f(\bar{x})\} - C_Y)]$$

$$\subseteq cl(D_0 - C_Y).$$

Therefore in view of (5.15) and (5.16) we obtain

$$v \leq_{C_Y} f'_\varepsilon(\bar{x}, u), \quad \forall\, v \in D,$$

which completes the proof. □

The following theorem gives a convexity characterization of a vector valued function which is Fr*é*chet differentiable on its domain by using ε-generalized weak subdifferential.

Theorem 5.4. *Let X be a real normed space and* $(Y, \leq_{C_Y})$ *be a real ordered normed space with* $intC_Y \neq \emptyset$. *Let* $|||.||| : X \to C_Y$ *be a vectorial norm on X and* $f : X \to Y$ *be Fréchet differentiable and* ε*-generalized weak subdifferentiable at every point* $x \in X$. *Then, f is* C_Y*-convex if and only if* $(f'(\bar{x}), 0) \in \partial_{\varepsilon}^{gw} f(\bar{x})$ *for all* $\bar{x} \in X$ *and all* $\varepsilon \in \mathbb{R}_+$.

Proof. Suppose that f is C_Y-convex. Let $\bar{x} \in X$ and $\varepsilon \in \mathbb{R}_+$ be arbitrary. Then in view of Theorem 2.1 we have

$$f'(\bar{x})(x - \bar{x}) \leq_{C_Y} f(x) - f(\bar{x}), \quad \forall\, x \in X.$$

This implies that

$$f(x) - f(\bar{x}) - f'(\bar{x})(x - \bar{x}) \in C_Y, \quad \forall\, x \in X. \tag{5.15}$$

Since $\varepsilon\mathbf{1} \in C_Y$ for all $\varepsilon \in \mathbb{R}$ and C_Y is a convex cone, it follows from (5.17) that

$$f'(\bar{x})(x - \bar{x}) - 0|||x - \bar{x}||| \leq_{C_Y} f(x) - f(\bar{x}) + \varepsilon\mathbf{1}, \quad \forall\, x \in X. \tag{5.16}$$

Since f is Fr$\acute{e}$chet differentiable on X, in view of Definitioc 2.8 one has $f'(\bar{x}) \in B(X, Y)$. So we conclude from (5.18) that $(f'(\bar{x}), 0) \in \partial_{\varepsilon}^{gw} f(\bar{x})$.
Conversely, assume that $(f'(\bar{x}), 0) \in \partial_{\varepsilon}^{gw} f(\bar{x})$ for all $\bar{x} \in X$ and all $\varepsilon \in \mathbb{R}_+$. Thus

$$f'(\bar{x})(x - \bar{x}) - 0|||x - \bar{x}||| \leq_{C_Y} f(x) - f(\bar{x}) + \varepsilon\mathbf{1}, \quad \forall\, x \in X,\ \forall\, \varepsilon \in \mathbb{R}_+,$$

where $\mathbf{1} \in intC_Y$. That is

$$f(x) - f(\bar{x}) + \varepsilon\mathbf{1} - f'(\bar{x})(x - \bar{x}) \in C_Y, \quad \forall\, x \in X,\ \forall\, \varepsilon \in \mathbb{R}_+. \tag{5.17}$$

Since $||\mathbf{1}|| = 1$, it follows that

$$\lim_{\varepsilon\to 0^+} ||[f(x) - f(\bar{x}) + \varepsilon\mathbf{1} - f'(\bar{x})(x - \bar{x})] - [f(x) - f(\bar{x}) - f'(\bar{x})(x - \bar{x})]||$$

$$= \lim_{\varepsilon\to 0^+} ||\varepsilon\mathbf{1}||$$

$$= \lim_{\varepsilon\to 0^+} \varepsilon = 0, \quad \forall\, x \in X.$$

Since C_Y is closed, it follows from (5.19) and (5.20) that

$$f(x) - f(\bar{x}) - f'(\bar{x})(x - \bar{x}) \in C_Y, \quad \forall\, x \in X.$$

That is

$$f'(\bar{x})(x - \bar{x}) \leq_{C_Y} f(x) - f(\bar{x}), \quad \forall\, x \in X.$$

Therefore in view of Theorem 2.1 one has f is C_Y-convex. □

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