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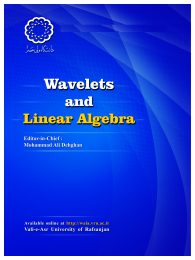


# Ultra Bessel sequences in direct sums of Hilbert spaces

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## Abstract

In this paper, we establish some new results in ultra Bessel sequences and ultra Bessel sequences of subspaces. Also, we investigate ultra Bessel sequences in direct sums of Hilbert spaces. Specially, we show that $\{(f_i, g_i)\}_{i=1}^{\infty}$ is a an ultra Bessel sequence for Hilbert space $\mathcal{H} \oplus \mathcal{K}$ if and only if $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ are ultra Bessel sequences for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively.

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## 1. Introduction

We say that a sequence $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is a frame for Hilbert space $\mathcal{H}$, if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leqslant \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leqslant B\|f\|^2, \quad f \in \mathcal{H}. \tag{1.1}$$

If $A = B$, we call $\{f_i\}_{i=1}^{\infty}$ a tight frame and if $A = B = 1$, it is called a Parseval frame. If the right-hand inequality of (1.1) holds for all $f \in \mathcal{H}$, we say $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence for $\mathcal{H}$. In 2008, the concept of ultra Bessel sequences in Hilbert spaces introduced and investigated by Faroughi and Najati [6].

**Definition 1.1.** Let $\mathcal{H}_0$ be an inner product space. Let $\{f_i\}_{i=1}^{\infty}$ be a sequence of members of $\mathcal{H}_0$. Then $\{f_i\}_{i=1}^{\infty}$ is called an ultra Bessel sequence in $\mathcal{H}_0$, if

$$\sup_{\|f\|=1} \sum_{i=n}^{\infty} |\langle f, f_i\rangle|^2 \to 0 \tag{1.2}$$

as $n \to \infty$, i.e., the series $\sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2$ converges uniformly in unit sphere of $\mathcal{H}_0$.

Frame of subspaces as a generalization of ordinary frame introduced by Casazza and Kutyniok in [4].

**Definition 1.2.** Let $\{v_i\}_{i=1}^{\infty}$ be a family of weights, i.e., for all $i \geq 1$, $v_i > 0$. A family of closed subspaces $\{W_i\}_{i=1}^{\infty}$ of a Hilbert space $\mathcal{H}$ is a frame of subspaces with respect to $\{v_i\}_{i=1}^{\infty}$ for $\mathcal{H}$, if there exist constants $0 < C \leq D < \infty$ such that

$$C\|f\|^2 \leq \sum_{i=1}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 \leq D\|f\|^2, \quad f \in \mathcal{H}. \tag{1.3}$$

If the right hand inequality of (1.3) holds for all $f \in \mathcal{H}$, we call $\{W_i\}_{i=1}^{\infty}$ a Bessel sequence of subspaces with respect to $\{v_i\}_{i=1}^{\infty}$ with Bessel bound $D$.

**Definition 1.3.** For each family of closed subspaces $\{W_i\}_{i=1}^{\infty}$ of $\mathcal{H}$, we define the set

$$\Big(\sum_{i=1}^{\infty} \oplus W_i\Big)_{\ell^2} = \{\{f_i\}_{i=1}^{\infty} | f_i \in W_i, \quad \sum_{i=1}^{\infty} \|f_i\|^2 < \infty\}.$$

It clear that $\Big(\sum_{i=1}^{\infty} \oplus W_i\Big)_{\ell^2}$ is a Hilbert space with the point wise operations and with the inner product given by

$$\langle \{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty} \rangle = \sum_{i=1}^{\infty} \langle f_i, g_i\rangle.$$

It is proved in [4], if $\{W_i\}_{i=1}^{\infty}$ is a Bessel sequence of subspaces with respect to $\{v_i\}_{i=1}^{\infty}$ for $\mathcal{H}$ then the operator

$$T_{W,v} : \Big(\sum_{i=1}^{\infty} \oplus W_i\Big)_{\ell^2} \to \mathcal{H}, \quad T_{W,v}(\{f_i\}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} v_i f_i$$

is bounded and its adjoint is

$$T^*_{W,v} : \mathcal{H} \to \Big(\sum_{i=1}^{\infty} \oplus W_i\Big)_{\ell^2}, \quad T^*_{W,v}(f) = \{v_i \pi_{W_i}(f)\}_{i=1}^{\infty}.$$

The operators $T_{W,v}$ and $T^*_{W,v}$ are called the synthesis and analysis operators for $\{W_i\}_{i=1}^{\infty}$ and $\{v_i\}_{i=1}^{\infty}$, respectively.

Also, it is proved in [4], if $\{W_i\}_{i=1}^{\infty}$ is a frame of subspaces with respect to $\{v_i\}_{i=1}^{\infty}$, the operator

$$S_{W,v} : \mathcal{H} \to \mathcal{H}, \quad S_{W,v}(f) = TT^*(f)$$

is a positive, self-adjoint and invertible operator on $\mathcal{H}$ and we have the reconstruction formula

$$f = \sum_{i=1}^{\infty} v_i^2 S^{-1}_{W,v} \pi_{W_i}(f), \quad f \in \mathcal{H}.$$

The operator $S_{W,v}$ is called the frame operator for $\{W_i\}_{i=1}^{\infty}$ and $\{v_i\}_{i=1}^{\infty}$.

Ultra Bessel sequence of subspaces was introduced in [2] (see also [1]).

**Definition 1.4.** Let $\mathcal{H}_0$ be an inner product space and $\{v_i\}_{i=1}^{\infty}$ be a family of weights. Let $\{W_i\}_{i=1}^{\infty}$ be a family of closed subspaces of $\mathcal{H}_0$. Then $\{W_i\}_{i=1}^{\infty}$ is called an ultra Bessel sequence of subspaces in $\mathcal{H}_0$, if

$$\sup_{\|f\|=1} \sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 \to 0, \tag{1.4}$$

as $n \to \infty$, i.e., the series $\sum_{i=1}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2$ converges uniformly in the unit sphere of $\mathcal{H}_0$.

## 2. New results in ultra Bessel sequences

In this section, we provide some new results about ultra Bessel sequences in Hilbert spaces and direct sums of Hilbert spaces.

**Proposition 2.1.** *Let $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ be ultra Bessel sequences for Hilbert space $\mathcal{H}$ and let $S, T \in B(\mathcal{H})$ such that $\|T\| + \|S\| \neq 0$. Then $\{Tf_i + Sg_i\}_{i=1}^{\infty}$ is an ultra Bessel sequences for $\mathcal{H}$.*

*Proof.* Let $\|f\| = 1$. Then

$$\begin{aligned}
&\frac{1}{\|T\|^2 + \|S\|^2} \sum_{i=n}^{\infty} |\langle f, Tf_i + Sg_i\rangle|^2 \\
&\qquad \leq \frac{2}{\|T\|^2} \sum_{i=n}^{\infty} |\langle f, Tf_i\rangle|^2 + \frac{2}{\|S\|^2} \sum_{i=n}^{\infty} |\langle f, Sg_i\rangle|^2 \\
&\qquad \leq 2 \sum_{i=n}^{\infty} |\langle \frac{T^*f}{\|T^*\|}, f_i\rangle|^2 + 2 \sum_{i=n}^{\infty} |\langle \frac{S^*f}{\|S^*\|}, g_i\rangle|^2 \\
&\qquad \leq 2 \sup_{\|h\|\leq 1} \sum_{i=n}^{\infty} |\langle h, f_i\rangle|^2 + 2 \sup_{\|g\|\leq 1} \sum_{i=n}^{\infty} |\langle g, g_i\rangle|^2 \\
&\qquad \leq 2 \sup_{\|h\|=1} \sum_{i=n}^{\infty} |\langle h, f_i\rangle|^2 + 2 \sup_{\|g\|=1} \sum_{i=n}^{\infty} |\langle g, g_i\rangle|^2.
\end{aligned}$$

□

**Corollary 2.2.** *Let $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ be ultra Bessel sequences for Hilbert space $\mathcal{H}$ and let $\alpha, \beta$ be scalars such that $|\alpha| + |\beta| \neq 0$. Then $\{\alpha f_i + \beta g_i\}_{i=1}^{\infty}$ is an ultra Bessel sequences.*

*Proof.* It is sufficient to take $T = \alpha I$ and $S = \beta I$ in Proposition 2.1. □

**Example 2.3.** Let $\mathcal{H}$ be a separable Hilbert space and $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Let $f_i = e_i + e_{i+1}$, for all $i \in \mathbb{N}$. Then $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence for $\mathcal{H}$, but it is not an ultra Bessel sequence for $\mathcal{H}$.
In fact,

$$\begin{aligned}
\sum_{i=1}^{\infty} |\langle f, e_i + e_{i+1}\rangle|^2 &= \sum_{i=1}^{\infty} |\langle f, e_i\rangle + \langle f, e_{i+1}\rangle|^2 \\
&\leq 2 \sum_{i=1}^{\infty} |\langle f, e_i\rangle|^2 + 2 \sum_{i=1}^{\infty} |\langle f, e_{i+1}\rangle|^2 \\
&\leq 4\|f\|,
\end{aligned}$$

and

$$\sup_{\|f\|=1} \sum_{i=n}^{\infty} |\langle f, e_i + e_{i+1}\rangle|^2 \geq |\langle e_n, e_n + e_{n+1}\rangle|^2 = 1.$$

We recall that if $\mathcal{H}_j$ is a Hilbert spaces for $j = 1, 2, ..., M$, then the space

$$\bigoplus_{j=1}^{M} \mathcal{H}_j = \{\{g_j\}_{j=1}^{M} : g_j \in \mathcal{H}_j, j = 1, 2, ..., M\}$$

is a Hilbert space with the point wise operations and with the inner product given by

$$\left\langle \{g_j\}_{j=1}^M, \{h_j\}_{j=1}^M \right\rangle = \sum_{j=1}^M \langle g_j, h_j \rangle.$$

We intend to study ultra Bessel sequences in Hilbert space $\bigoplus_{j=1}^M \mathcal{H}_j$. We first prove the following lemma.

**Lemma 2.4.** *Let $\{f_i\}_{i=1}^\infty$ be a Bessel sequence for Hilbert space $\mathcal{H}$. Then*

$$\sup_{\|f\|<1} \sum_{i=n}^\infty |\langle f, f_i\rangle|^2 \leq \sup_{\|f\|=1} \sum_{i=n}^\infty |\langle f, f_i\rangle|^2, \quad n \in \mathbb{N}.$$

*Proof.* Let us consider $\alpha = \sup_{\|f\|=1} \sum_{i=n}^\infty |\langle f, f_i\rangle|^2$. If $f = 0$, it is clear that $\sum_{i=n}^\infty |\langle f, f_i\rangle|^2 \leq \alpha$. Let $0 < \|f\| = k < 1$. Then

$$\frac{1}{k^2} \sum_{i=n}^\infty |\langle f, f_i\rangle|^2 = \sum_{i=n}^\infty |\langle \frac{f}{k}, f_i\rangle|^2 \leq \alpha.$$

So $\sum_{i=n}^\infty |\langle f, f_i\rangle|^2 \leq k^2\alpha \leq \alpha$. Therefor $\sup_{\|f\|<1} \sum_{i=n}^\infty |\langle f, f_i\rangle|^2 \leq \alpha$. □

**Theorem 2.5.** *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Let $\{f_i\}_{i=1}^\infty$ and $\{g_i\}_{i=1}^\infty$ be respective Bessel sequences for $\mathcal{H}$ and $\mathcal{K}$. Then $\{(f_i, g_i)\}_{i=1}^\infty$ is a an ultra Bessel sequence for $\mathcal{H} \oplus \mathcal{K}$ if and only if $\{f_i\}_{i=1}^\infty$ and $\{g_i\}_{i=1}^\infty$ are ultra Bessel sequences for $\mathcal{H}$ and $\mathcal{K}$, respectively.*

*Proof.* Let $\{(f_i, g_i)\}_{i=1}^\infty$ be a an ultra Bessel sequence for $\mathcal{H} \oplus \mathcal{K}$. Since the operator

$$T : \mathcal{H} \oplus \mathcal{K} \to \mathcal{H}, \quad T(f \oplus g) = f$$

is bounded, Lemma 2.4 of [6] implies that $\{f_i\}_{i=1}^\infty$ is an ultra Bessel sequence for $\mathcal{H}$. Similarly $\{g_i\}_{i=1}^\infty$ is an ultra Bessel sequence for $\mathcal{K}$.

For the converse implication, it is enough to show that

$$\sup_{\|(f,g)\|=1} \sum_{i=n}^\infty |\langle (f, g), (f_i, g_i)\rangle|^2 \leq 2 \sup_{\|f\|=1} \sum_{i=n}^\infty |\langle f, f_i\rangle|^2 + 2 \sup_{\|g\|=1} \sum_{i=n}^\infty |\langle g, g_i\rangle|^2.$$

Let $(f, g) \in \mathcal{H} \oplus \mathcal{K}$ and $\|(f, g)\| = 1$. If $f = 0$ then $\|g\| = 1$ and

$$\sum_{i=n}^\infty |\langle (f, g), (f_i, g_i)\rangle|^2 = \sum_{i=n}^\infty |\langle g, g_i\rangle|^2 \leq \sup_{\|g\|=1} \sum_{i=n}^\infty |\langle g, g_i\rangle|^2.$$

If $\|f\| = 1$ then $g = 0$ and

$$\sum_{i=n}^\infty |\langle (f, g), (f_i, g_i)\rangle|^2 = \sum_{i=n}^\infty |\langle f, f_i\rangle|^2 \leq \sup_{\|f\|=1} \sum_{i=n}^\infty |\langle f, f_i\rangle|^2.$$

When $0 < \|f\| < 1$, by using Lemma 2.4 we have

$$\begin{aligned}\sum_{i=n}^{\infty} |\langle (f,g),(f_i,g_i)\rangle|^2 &= \sum_{i=n}^{\infty} |\langle f,f_i\rangle + \langle g,g_i\rangle|^2 \\ &\leq 2\sum_{i=n}^{\infty} |\langle f,f_i\rangle|^2 + 2\sum_{i=n}^{\infty} |\langle g,g_i\rangle|^2 \\ &\leq 2\sup_{\|f\|<1}\sum_{i=n}^{\infty} |\langle f,f_i\rangle|^2 + 2\sup_{\|g\|<1}\sum_{i=n}^{\infty} |\langle g,g_i\rangle|^2 \\ &\leq 2\sup_{\|f\|=1}\sum_{i=n}^{\infty} |\langle f,f_i\rangle|^2 + 2\sup_{\|g\|=1}\sum_{i=n}^{\infty} |\langle g,g_i\rangle|^2.\end{aligned}$$

□

**Corollary 2.6.** *Let $\mathcal{H}_j$ be a Hilbert space and $\{f_{ij}\}_{i=1}^{\infty}$ be a Bessel sequence for $\mathcal{H}_j$, for $j = 1,2,...,M$. Then $\{F_i\}_{i=1}^{\infty}$ is a an ultra Bessel sequence for $\bigoplus_{j=1}^{M}\mathcal{H}_j$ if and only if $\{f_{ij}\}_{i=1}^{\infty}$ is an ultra Bessel sequence for $\mathcal{H}_j$, for $j = 1,2,...,M$, where $F_i = \{f_{ij}\}_{j=1}^{M}$.*

*Proof.* If $\{F_i\}_{i=1}^{\infty}$ is a an ultra Bessel sequence for $\bigoplus_{j=1}^{M}\mathcal{H}_j$, then it is clear that $\{f_{ij}\}_{i=1}^{\infty}$ is an ultra Bessel sequence for $\mathcal{H}_j$ for $j = 1,2,...,M$. The converse concluded by

$$\sup_{\|\{g_j\}_{j=1}^{M}\|=1}\sum_{i=n}^{\infty} |\langle \{g_j\}_{j=1}^{M},\{f_{ij}\}_{j=1}^{M}\rangle|^2 \leq M\sum_{j=1}^{M}\sup_{\|g_j\|=1,g_j\in\mathcal{H}_j}\sum_{i=n}^{\infty} |\langle g_j,f_{ij}\rangle|^2.$$

□

**Example 2.7.** Let $\{e_i\}_{i=1}^{\infty}$ and $\{h_i\}_{i=1}^{\infty}$ be orthonormal basis for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Then

(1) $\{(\frac{e_i}{i},\frac{h_i}{i})\}_{i=1}^{\infty}$ is an ultra Bessel sequence for $\mathcal{H}\oplus\mathcal{K}$.
(2) $\{(e_i,h_i)\}_{i=1}^{\infty}$ is a Bessel sequence for $\mathcal{H}\oplus\mathcal{K}$ but it is not an ultra Bessel sequence for $\mathcal{H}\oplus\mathcal{K}$.

*Proof.* (1) Let $\|(f,g)\| = 1$. Then

$$\begin{aligned}\sum_{i=n}^{\infty}\left|\left\langle (f,g),(\frac{e_i}{i},\frac{h_i}{i})\right\rangle\right|^2 &= \sum_{i=n}^{\infty}\frac{1}{i^2}|\langle f,e_i\rangle + \langle g,h_i\rangle|^2 \\ &\leqslant 2\sum_{i=n}^{\infty}\frac{1}{i^2}|\langle f,e_i\rangle|^2 + 2\sum_{i=n}^{\infty}\frac{1}{i^2}|\langle g,h_i\rangle|^2 \\ &\leqslant 2\sum_{i=n}^{\infty}\frac{\|f\|^2}{i^2} + 2\sum_{i=n}^{\infty}\frac{\|g\|^2}{i^2} \\ &\leqslant 4\sum_{i=n}^{\infty}\frac{1}{i^2}.\end{aligned}$$

(2) We have

$$\begin{aligned}\sum_{i=1}^{\infty}|\langle(f,g),(e_i,h_i)\rangle|^2 &= \sum_{i=1}^{\infty}|\langle f,e_i\rangle+\langle g,h_i\rangle|^2\\ &\leqslant 2\sum_{i=n}^{\infty}|\langle f,e_i\rangle|^2+2\sum_{i=n}^{\infty}|\langle g,h_i\rangle|^2\\ &\leqslant 2\|f\|^2+2\|g\|^2=2\|(f,g)\|^2,\end{aligned}$$

so $\{(e_i,h_i)\}_{i=1}^{\infty}$ is a Bessel sequence for $\mathcal{H}\oplus\mathcal{K}$. On the other hand,

$$\sup_{\|(f,g)\|=1}\sum_{i=n}^{\infty}|\langle(f,g),(e_i,h_i)\rangle|^2\geq|\langle\frac{1}{\sqrt{2}}(e_n,h_n),(e_n,h_n)\rangle|^2=2,$$

hence $\{(f_i,e_i)\}_{i=1}^{\infty}$ is not an ultra Bessel sequence for $\mathcal{H}\oplus\mathcal{K}$. □

## 3. ultra Bessel sequences of subspaces

It is proved in [2], if $\{W_i\}_{i=1}^{\infty}$ is a family of closed subspaces in a Hilbert space $\mathcal{H}$ and $\{v_i\}_{i=1}^{\infty}$ is a family of weights such that $\sum_{i=1}^{\infty}v_i^2<\infty$, then $\{W_i\}_{i=1}^{\infty}$ is an ultra Bessel sequence of subspaces in $\mathcal{H}$. Here, we give another sufficient condition on a sequence of subspaces to be an ultra Bessel sequence of subspaces.

**Proposition 3.1.** *Let $\{W_i\}_{i=1}^{\infty}$ be a sequence of closed subspaces of $\mathcal{H}$ and $\{v_i\}_{i=1}^{\infty}$ be a sequence of weights. If the operators*

$$T_{n,v}:\Big(\sum_{i=1}^{\infty}\oplus W_i\Big)_{\ell^2}\to\mathcal{H},\quad T_{n,v}(\{f_i\}_{i=1}^{\infty})=\sum_{i=n}^{\infty}v_if_i$$

*are bounded for all $n\in\mathbb{N}$ and $\|T_{n,v}\|\to 0$ then $\{W_i\}_{i=1}^{\infty}$ is an ultra Bessel sequence of subspaces in $\mathcal{H}$.*

*Proof.* It is clear that $\{W_i\}_{i=1}^{\infty}$ is Bessel sequence of subspaces for $\mathcal{H}$ with respect to $\{v_i\}_{i=1}^{\infty}$. Let $f\in\mathcal{H}$ with $\|f\|=1$, then we have

$$\begin{aligned}\sum_{i=n}^{\infty}v_i^2\|\pi_{W_i}(f)\|^2 &= \Big\langle\sum_{i=n}^{\infty}v_i^2\pi_{W_i}(f),f\Big\rangle\\ &= \Big\langle T_{n,v}(\{v_i\pi_{W_i}(f)\}_{i=1}^{\infty}),f\Big\rangle\\ &= \Big\langle T_{n,v}T_{W,v}^{*}(f),f\Big\rangle\\ &\leq \|T_{n,v}\|\|T_{W,v}^{*}\|,\end{aligned}$$

for all $n\in\mathbb{N}$. □

**Corollary 3.2.** *Let $\{W_i\}_{i=1}^{\infty}$ be a Bessel sequence of subspaces in $\mathcal{H}$ with respect to $\{v_i\}_{i=1}^{\infty}$. Let*

$$T_{n,v} : \Big( \sum_{i=1}^{\infty} \oplus W_i \Big)_{\ell^2} \to \mathcal{H}, \quad T_{n,v}(\{f_i\}_{i=1}^{\infty}) = \sum_{i=n}^{\infty} v_i f_i$$

*be given for all $n \in \mathbb{N}$. If $\|T_{n,v}\| \to 0$ then $\{W_i\}_{i=1}^{\infty}$ is an ultra Bessel sequence of subspaces in $\mathcal{H}$.*

Now, we give an example of an ultra Bessel sequence of subspaces.

**Example 3.3.** If $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for a Hilbert space $\mathcal{H}$ and

$$W_i = \text{span}\{\frac{e_i}{\sqrt{i}}\}, \quad i \in \mathbb{N},$$

then $\{W_i\}_{i=1}^{\infty}$ is an ultra Bessel sequence of subspaces in $\mathcal{H}$ with respect to $v_i = \frac{1}{\sqrt{i}}$, for all $i \in I$. In fact,

$$\begin{aligned} \sup_{\|f\|=1} \sum_{i=n}^{\infty} \|\pi_{W_i}(f)\|^2 &= \sup_{\|f\|=1} \sum_{i=n}^{\infty} \frac{1}{i} |\langle f, e_i \rangle|^2 \\ &\leq \sup_{\|f\|=1} \Big( \sum_{i=n}^{\infty} \frac{|\langle f, e_i \rangle|^2}{i^2} \Big)^{\frac{1}{2}} \Big( \sum_{i=n}^{\infty} |\langle f, e_i \rangle|^2 \Big)^{\frac{1}{2}} \\ &\leq \Big( \sum_{i=n}^{\infty} \frac{1}{i^2} \Big)^{\frac{1}{2}} \to 0, \end{aligned}$$

as $n \to \infty$.

**Proposition 3.4.** *Let $\{W_i\}_{i=1}^{\infty}$ and $\{Z_i\}_{i=1}^{\infty}$ be two ultra Bessel sequences of subspaces in Hilbert space $\mathcal{H}$ with respect to $\{v_i\}_{i=1}^{\infty}$ for which $W_i \perp Z_i$, for all $i \in \mathbb{N}$. Then $\{\overline{W_i + Z_i}\}_{i=1}^{\infty}$ is an ultra Bessel sequence of subspaces in $\mathcal{H}$ with respect to $\{v_i\}_{i=1}^{\infty}$.*

*Proof.* Let $f \in \mathcal{H}$ and $\|f\| = 1$. Then for each $n \in \mathbb{N}$ we have

$$\begin{aligned} \sum_{i=n}^{\infty} v_i^2 \|\pi_{\overline{W_i + Z_i}}(f)\|^2 &= \sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i}(f) + \pi_{Z_i}(f)\|^2 \\ &= \sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 + \sum_{i=n}^{\infty} v_i^2 \|\pi_{Z_i}(f)\|^2 \\ &\leq \sup_{\|f\|=1} \sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 + \sup_{\|f\|=1} \sum_{i=n}^{\infty} v_i^2 \|\pi_{Z_i}(f)\|^2. \end{aligned}$$

□

**Corollary 3.5.** *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Let $\{W_i\}_{i=1}^{\infty}$ and $\{Z_i\}_{i=1}^{\infty}$ be two ultra Bessel sequences of subspaces in $\mathcal{H}$ with respect to $\{v_i\}_{i=1}^{\infty}$. Let $T_1, T_2 \in B(\mathcal{H},\mathcal{K})$ be invertible operators such that $T_1(W_i) \perp T_2(Z_i)$, for all $i \in \mathbb{N}$. Then $\{\overline{T_1(W_i) + T_2(Z_i)}\}_{i=1}^{\infty}$ is an ultra Bessel sequence of subspaces in $\mathcal{K}$ with respect to $\{v_i\}_{i=1}^{\infty}$.*

**Corollary 3.6.** *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Let $\{W_i\}_{i=1}^{\infty}$ and $\{Z_i\}_{i=1}^{\infty}$ be two ultra Bessel sequences of subspaces in $\mathcal{H}$ with respect to $\{v_i\}_{i=1}^{\infty}$, for which $W_i \perp Z_i$, for all $i \in \mathbb{N}$. If $T \in B(\mathcal{H},\mathcal{K})$ is an unitary operator then $\{\overline{T(W_i + Z_i)}\}_{i=1}^{\infty}$ is an ultra Bessel sequence of subspaces in $\mathcal{K}$ with respect to $\{v_i\}_{i=1}^{\infty}$.*

**Proposition 3.7.** *Let $\{W_i\}_{i=1}^{\infty}$ and $\{W_i'\}_{i=1}^{\infty}$ be ultra Bessel sequences of subspaces in Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ with respect to $\{v_i\}_{i=1}^{\infty}$, respectively. Then $\{W_i \oplus W_i'\}_{i=1}^{\infty}$ is an ultra Bessel sequence of subspaces in Hilbert space $\mathcal{H} \oplus \mathcal{K}$ with respect to $\{v_i\}_{i=1}^{\infty}$.*

*Proof.* Let $\pi_{W_i \oplus W_i'}$ be orthogonal projection from $\mathcal{H} \oplus \mathcal{K}$ onto $W_i \oplus W_i'$. It clear that

$$\pi_{W_i \oplus W_i'}(f,g) = (\pi_{W_i} f, \pi_{W_i'} g), \quad (f,g) \in \mathcal{H} \oplus \mathcal{K}.$$

Let $(f,g) \in \mathcal{H} \oplus \mathcal{K}$ and $\|(f,g)\| = 1$. For each $n \in \mathbb{N}$ we have

$$\begin{aligned}
\sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i \oplus W_i'}(f,g)\|^2 &= \sum_{i=n}^{\infty} v_i^2 \|(\pi_{W_i} f, \pi_{W_i'} g)\|^2 \\
&= \sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 + \sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i'}(g)\|^2 \\
&\leq \sup_{\|f\| \leq 1} \sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 + \sup_{\|g\| \leq 1} \sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i'}(g)\|^2 \\
&\leq \sup_{\|f\| = 1} \sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 + \sup_{\|g\| = 1} \sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i'}(g)\|^2
\end{aligned}$$

□

**Corollary 3.8.** *Let $\{W_{ij}\}_{i=1}^{\infty}$ be an ultra Bessel sequence of subspaces in Hilbert spaces $\mathcal{H}_j$ with respect to $\{v_i\}_{i=1}^{\infty}$, for $j = 1, 2, ..., M$ . Then $\{\oplus_{j=1}^{M} W_{ij}\}_{i=1}^{\infty}$ is an ultra Bessel sequence of subspaces in Hilbert space $\oplus_{j=1}^{m} \mathcal{H}_j$ with respect to $\{v_i\}_{i=1}^{\infty}$.*

It is proved in [2], if $\mathcal{H}$ is a finite dimensional Hilbert space and $\{W_i\}_{i=1}^{\infty}$ is a frame of subspace for $\mathcal{H}$ with respect to $\{v_i\}_{i=1}^{\infty}$, then there exists $N_0 \in \mathbb{N}$ such that $\{W_i\}_{i=1}^{n}$ is a frame of subspaces for $\mathcal{H}$ with respect to $\{v_i\}_{i=1}^{n}$, for all $n \geq N_0$. We call the integer $N_0$ the frameness bound of $\{W_i\}_{i=1}^{\infty}$.
On the other hand, if $\{W_i\}_{i=1}^{\infty}$ and $\{W_i'\}_{i=1}^{\infty}$ be frames of subspaces for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ with respect to $\{v_i\}_{i=1}^{\infty}$, respectively, then $\{W_i \oplus W_i'\}_{i=1}^{\infty}$ is a frame of subspaces for $\mathcal{H} \oplus \mathcal{K}$ with respect to $\{v_i\}_{i=1}^{\infty}$ [7]. Let $\mathcal{H}_i$ and $\mathcal{K}_i$ be Hilbert spaces and $T_i \in B(\mathcal{H}_i, \mathcal{K}_i)$ for $i = 1, 2$. We define the operator $T_1 \oplus T_2 \in B(\mathcal{H}_1 \oplus \mathcal{H}_2, \mathcal{K}_1 \oplus \mathcal{K}_2)$ by

$$(T_1 \oplus T_2)(f,g) = (T_1 f, T_2 g).$$

**Proposition 3.9.** *Let $\mathcal{H}$ and $\mathcal{K}$ be finite dimensional Hilbert spaces and $\{W_i\}_{i=1}^{\infty}$ and $\{W_i'\}_{i=1}^{\infty}$ be frames of subspaces for $\mathcal{H}$ and $\mathcal{K}$ with respect to $\{v_i\}_{i=1}^{\infty}$, respectively. Let $N_0$ and $N_0'$ be respective frameness bound of $\{W_i\}_{i=1}^{\infty}$ and $\{W_i\}_{i=1}^{\infty}$. Let $n \geqslant \max\{N_0, N_0'\}$ and $S_n$ and $T_n$ be the frame operator and synthesis operator of $\{W_i \oplus W_i'\}_{i=1}^{n}$, respectively. Then*

*(i)* $S_n \to S_{W,v} \oplus S_{W',v}$ *in* $B(\mathcal{H} \oplus \mathcal{K})$,

*(ii)* $T_n \to T_{W\oplus W',v}$ *in* $B\Big((\sum_{i=1}^{\infty} \oplus W_i \oplus W_i')_{\ell^2}, \mathcal{H}\Big)$.

*Proof.* By Theorem 2.4 of [3], $S_n \to S_{W\oplus W',v}$ and $T_n \to T_{W\oplus W',v}$, where $T_{W\oplus W',v}$ and $S_{W\oplus W',v}$ are the synthesis and frame operator of $\{W_i \oplus W_i'\}_{i=1}^{\infty}$, respectively. But

$$\begin{aligned} S_{W\oplus W',v}(f,g) &= \sum_{i=1}^{\infty} v_i^2 \pi_{W_i\oplus W_i'}(f,g) = \sum_{i=1}^{\infty} v_i^2 \Big(\pi_{W_i}(f), \pi_{W_i'}(g)\Big) \\ &= \Big(\sum_{i=1}^{\infty} v_i^2 \pi_{W_i}(f), \sum_{i=1}^{\infty} v_i^2 \pi_{W_i'}(g)\Big) \\ &= \Big(S_{W,v}(f), S_{W',v}(g)\Big) = (S_{W,v} \oplus S_{W',v})(f,g), \end{aligned}$$

for all $(f,g) \in \mathcal{H} \oplus \mathcal{K}$. □

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