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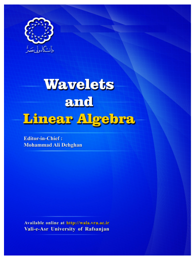

Schur multiplier norm of product of matrices

M. Khosravi[a,*], A. Sheikhhosseini[a]

[a] *Department of Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Islamic Republic of Iran*

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Abstract

For $A \in \mathbb{M}_n$, the Schur multiplier of A is defined as $S_A(X) = A \circ X$ for all $X \in \mathbb{M}_n$ and the spectral norm of S_A can be state as $\|S_A\| = \sup_{X \neq 0} \frac{\|A \circ X\|}{\|X\|}$. The other norm on S_A can be defined as $\|S_A\|_\omega = \sup_{X \neq 0} \frac{\omega(S_A(X))}{\omega(X)} = \sup_{X \neq 0} \frac{\omega(A \circ X)}{\omega(X)}$, where $\omega(A)$ stands for the numerical radius of A. In this paper, we focus on the relation between the norm of Schur multiplier of product of matrices and the product of norm of those matrices. This relation is proved for Schur product and geometric product and some applications are given. Also we show that there is no such relation for operator product of matrices. Furthermore, for positive definite matrices A and B with $\|S_A\|_\omega \leqslant 1$ and $\|S_B\|_\omega \leqslant 1$, we show that $A \sharp B = n(I - Z)^{1/2} C (I + Z)^{1/2}$, for some contraction C and Hermitian contraction Z.

*Corresponding author
Email addresses: khosravi−m@uk.ac.ir (M. Khosravi), sheikhhosseini@uk.ac.ir (A. Sheikhhosseini)

1. Introduction

Let $\mathbb{M}_n$ be the algebra of all $n \times n$ complex matrices. A Hermitian matrix $A \in \mathbb{M}_n$ is called positive semidefinite and is denoted by $A \geq 0$, if $x^*Ax \geqslant 0$ for all $x \in \mathbb{C}^n$. For two Hermitian matrices A and B, the relation $A \geq B$ or $B \leq A$ means that $A - B \geq 0$.

A Hermitian matrix A is called positive definite if $x^*Ax > 0$ for all non-zero $x \in \mathbb{C}^n$. It is easy to show that A is positive definite if and only if A is invertible positive semidefinite matrix.

Throughout this paper, the symbol $\|.\|$ is used for usual spectral norm.

The numerical radius of $A \in \mathbb{M}_n$, denoted by $\omega(A)$, is defined by

$$\omega(A) = \max\{|x^*Ax| : x \in \mathbb{C}^n, x^*x = 1\}.$$

One should note that the numerical radius defines a norm on $\mathbb{M}_n$ which is not unitarily invariant.

In $\mathbb{M}_n$, beside the usual matrix product, the entrywise product is quite important and interesting. The entrywise product of two matrices A and B is called their Schur (or Hadamard) product and denoted by $A \circ B$. With this multiplication $\mathbb{M}_n$ becomes a commutative algebra, for which the matrix with all entries equal to one is the unit. The linear operator S_A on $\mathbb{M}_n$, called the Schur multiplier operator, defined by $S_A(X) := A \circ X$.
The induced norm of S_A with respect to spectral norm will be denoted by

$$\|S_A\| = \sup_{X \neq 0} \frac{\|S_A(X)\|}{\|X\|} = \sup_{X \neq 0} \frac{\|A \circ X\|}{\|X\|},$$

and the induced norm of S_A with respect to numerical radius norm will be denoted by

$$\|S_A\|_\omega = \sup_{X \neq 0} \frac{\omega(S_A(X))}{\omega(X)} = \sup_{X \neq 0} \frac{\omega(A \circ X)}{\omega(X)}.$$

This operator and its norm is an interesting topic in matrix theory and combinatorics. In [15], Ong states the necessary and sufficient conditions under which $\|S_A\| = \|A\|$. He also proved a similar result for infinite matrices as operators on ℓ_2 in [7]. A comparison of this norm with some other norms of $\mathbb{M}_n$ is given in [8].

In addition, Mathias [14] showed that for $(x_1, \cdots, x_n) \in \mathbb{R}_n^+$ if the (i, j)th entries of A, G and H are arithmetic, geometric and harmonic means of x_i and x_j, respectively, then $\|S_A\| \geq \|S_G\| \geq \|S_H\|$.

In [12] L. Livshits studied a generalized Schur product of matrices whose entries are bounded linear operators on a Hilbert space. The product on the entries is the usual operator multiplication and Chaisuriya [6] obtained some results about Schur multiplier in the frame work of C^*-algebras.

Also the idempotent Schur multiplier operator that is the Schur multiplier of $\{0, 1\}$-matrices and its relation with combinatorics and graph theory were studied in [11, 13].

The upper bounds on the Schur multiplier norm were obtained by some authors. For more details see [3, 4, 9].

2. Main results

The following facts are some basic theorems in Matrix theory.

Theorem 2.1. *[2, Haagerup theorem and Corollary 3] For $A \in \mathbb{M}_n$ the following assertions are equivalent:*
(i) $\|S_A\| \le 1$.
(ii) There is $0 \le R_1, R_2 \in \mathbb{M}_n$ *such that*

$$\begin{bmatrix} R_1 & A \\ A^* & R_2 \end{bmatrix} \ge 0, \qquad R_1 \circ I \le I \qquad \text{and} \qquad R_2 \circ I \le I.$$

Moreover, if A is Hermitian, then $\|S_A\| = \|S_A\|_\omega$.

Ando and Okubo state a similar results for $\|S_A\|_\omega$ as follows:

Theorem 2.2. *[2, Ando-Okubo's theorem] For $A \in \mathbb{M}_n$ the following assertions are equivalent:*
(i) $\|S_A\|_\omega \le 1$.
(ii) There is $0 \le R \in \mathbb{M}_n$ *such that*

$$\begin{bmatrix} R & A \\ A^* & R \end{bmatrix} \ge 0, \qquad R \circ I \le I.$$

Proposition 2.3. *For $A, B \in \mathbb{M}_n$, we have*

$$\|S_{A\circ B}\| \le \|S_A\|\|S_B\| \qquad \text{and} \qquad \|S_{A\circ B}\|_\omega \le \|S_A\|_\omega\|S_B\|_\omega.$$

Proof. Let $\|S_A\| = \|S_B\| = 1$. By Haagerup's theorem, there exists positive semidefinite matrices R_1, R_2, R_3, R_4 with

$$\begin{bmatrix} R_1 & A \\ A^* & R_2 \end{bmatrix} \ge 0 \qquad \text{and} \qquad \begin{bmatrix} R_3 & B \\ B^* & R_4 \end{bmatrix} \ge 0,$$

where $R_i \circ I \le I$. Since the Schur product of two positive semidefinite matrices is a positive semidefinite, we can conclude that

$$\begin{bmatrix} R_1 \circ R_3 & A \circ B \\ A^* \circ B^* & R_2 \circ R_4 \end{bmatrix} = \begin{bmatrix} R_1 & A \\ A^* & R_2 \end{bmatrix} \circ \begin{bmatrix} R_3 & B \\ B^* & R_4 \end{bmatrix} \ge 0.$$

In addition $(R_i \circ R_j) \circ I \le I$. Thus $\|S_{A\circ B}\| \le \|S_A\|\|S_B\|$.
A similar argument using Ando-Okubo's theorem leads to the second inequality. □

For positive definite matrices $A, B \in \mathbb{M}_n$, the matrix geometric mean is defined by

$$A\sharp B \equiv A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

The matrix geometric mean has the symmetric property: $A\sharp B = B\sharp A$. In the case that A and B commute, we have $A\sharp B = (AB)^{1/2}$. It is easy to check that $(\alpha A)\sharp B = \alpha^{1/2}(A\sharp B)$ for all $\alpha \ge 0$.

Lemma 2.4. *[2, Corollary 3 and Corollary 4] If $A = [a_{ij}] \in \mathbb{M}_n$ is positive semidefinite, then* $\|S_A\|_\omega = \|S_A\| = \max a_{ii}$.

Lemma 2.5. *[10, Theorem 7.7.9] For positive definite matrices A and B and any matrix C, the block matrix $\begin{bmatrix} A & C \\ C^* & B \end{bmatrix}$ is positive semidefinite if and only if $B \geqslant C^*A^{-1}C$.*

Theorem 2.6. *Let A and B be positive definite matrices. Then*

$$\|S_{A\sharp B}\| \leqslant \|S_A\|^{\frac{1}{2}}\|S_B\|^{\frac{1}{2}} = \|S_A\|\sharp\|S_B\|.$$

Proof. Since $\|S_{\alpha A}\| = \alpha\|S_A\|$ for all $\alpha \geqslant 0$, without loss of generality, we show that $\|S_{A\sharp B}\| \leqslant 1$, when $\|S_A\|^{\frac{1}{2}} = \|S_B\|^{\frac{1}{2}} = 1$.

From $A, B \geqslant 0$ and by Lemma 2.4, it follows that $\|S_A\| = \max a_{ii} = 1$ and $\|S_B\| = \max b_{ii} = 1$. Hence $A \circ I \leqslant I$, $B \circ I \leqslant I$. In addition, by Lemma 2.5, from

$$(A\sharp B)A^{-1}(A\sharp B) = A^{1/2}(A^{-1/2}BA^{-1/2})A^{1/2} = B,$$

it follows that the matrix $\begin{bmatrix} A & A\sharp B \\ A\sharp B & B \end{bmatrix}$ is positive semidefinite. Thus by Haagerup's theorem, we have $\|S_{A\sharp B}\| \leqslant 1$.

□

Corollary 2.7. *Let A and B be positive definite matrices such that $AB = BA$. Then*

$$\|S_{(AB)^{\frac{1}{2}}}\| \leqslant \|S_A\|^{\frac{1}{2}}\|S_B\|^{\frac{1}{2}}.$$

One should note that by [10, Problem 7.7.P27], we have

$$\|(S_A - S_B)(X)\| = \|A \circ X - B \circ X\| = \|(A - B) \circ X\| = \|S_{A-B}(X)\| \leq \|A - B\|,$$

for all matrix X with $\|X\| = 1$.

In addition, using triangle inequality, $|\,\|S_{A+\varepsilon I}\| - \|S_A\|\,| \leqslant \|S_{A+\varepsilon I} - S_A\| = \|S_{\varepsilon I}\| \leq |\varepsilon|$, it is easy to conclude that the the Schur multiplier norm is continuous.

In the following two corollaries, we use the continuity of Schur multiplier.

Corollary 2.8. *Let A be a positive semidefinite matrix. Then*

$$\|S_{A^{\frac{1}{2}}}\| \leq \|S_A\|^{\frac{1}{2}}.$$

Proof. First, let A be a positive definite matrix. Then by Corollary 2.7, it can be easily seen that

$$\|S_{A^{\frac{1}{2}}}\| \leq \|S_A\|^{\frac{1}{2}}\|S_I\|^{\frac{1}{2}}.$$

But from Lemma 2.4, we have $\|S_I\| = 1$ and the result follows. For the case that A is positive semidefinite matrices consider $A + \varepsilon I$. Now, by the continuity of Schur multiplier norm when ε tends to zero one can get the result. □

Example 2.9. It is clear that when A is a diagonal matrix, then $\|S_{A^{\frac{1}{2}}}\| = \|S_A\|^{\frac{1}{2}}$. But it is not true for general case, for example, let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then $A^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$. By Lemma 2.4,

$$\|S_A\| = 1 < \|S_{A^2}\|^{\frac{1}{2}} = \sqrt{2}.$$

Corollary 2.10. *Let A be a positive semidefinite matrix. Then*

$$\|S_{A^\alpha}\| \leq \|S_A\|^\alpha,$$

when $0 \leq \alpha \leq 1$. *In particular,*

$$\|S_{A^{\frac{1}{n}}}\| \leq \|S_A\|^{\frac{1}{n}}.$$

Proof. By continuity of Schur multiplier, it is sufficient to show that the statement is true for all rational number of the form $\frac{m}{2^k}$ when $0 \leq m \leq 2^k$. We prove it by induction on k.

It is easy to see that for $k = 1, 2$ the inequality holds. Let the relation is true for k. Without loss of generality, let $0 < 2m + 1 < 2^{k+1}$. By Theorem 2.6 and Corollary 2.8 and the hypothesis of induction, we have

$$\begin{aligned} \|S_{A^{\frac{2m+1}{2^{k+1}}}}\| &= \|S_{A^{\frac{m}{2^k}} \sharp A^{\frac{m+1}{2^k}}}\| \leq \|S_{A^{\frac{m}{2^k}}}\| \sharp \|S_{A^{\frac{m+1}{2^k}}}\| \\ &\leq \|S_A\|^{\frac{m}{2^k}} \sharp \|S_A\|^{\frac{m+1}{2^k}} = \|S_A\|^{\frac{2m+1}{2^{k+1}}}. \end{aligned}$$

Thus the inequality is true for all $\frac{m}{2^k}$. □

Corollary 2.11. *For positive semidefinite matrix A, the inequality*

$$\|S_{A^m}\|^n \leq \|S_{A^n}\|^m,$$

holds for all positive numbers $m \leq n$.

Proof. If $m \leq n$, then $\frac{m}{n} \leq 1$. So by the previous corollary

$$\|S_{(A^n)^{\frac{m}{n}}}\| \leq \|S_{A^n}\|^{\frac{m}{n}}.$$

That is

$$\|S_{A^m}\| \leq \|S_{A^n}\|^{\frac{m}{n}},$$

or

$$\|S_{A^m}\|^n \leq \|S_{A^n}\|^m.$$

□

Remark 2.12. By Corollaries 2.7 and 2.8, we find two bound for $\|S_{(AB)^{\frac{1}{2}}}\|$. One is $\|S_A\|^{\frac{1}{2}}\|S_B\|^{\frac{1}{2}}$ and another is $\|S_{AB}\|^{1/2}$. The question is, which one is lower? That is $\|S_{AB}\| \leq \|S_A\|\|S_B\|$ or $\|S_{AB}\| \geq \|S_A\|\|S_B\|$. Example 2.9 shows that the first one is false.

On the other hand if we consider $A = diag(1, 2)$ and $B = diag(2, 1)$, then

$$\|S_{AB}\| = 2 < 4 = \|S_A\|\|S_B\|.$$

That is these two bound are independent.

In addition, it can be conclude that although the Schur multiplier norm of Schur product and geometric mean of matrices have some relation with Schur multiplier norm of matrices, there is no relation between Schur multiplier norm of operator product of matrices and Schur multiplier norm of matrices, even if A and B are positive and commutes with each other.

The following proposition states an upper bound for Schur multiplier norm of Schur product of matrices.

Lemma 2.13. *[1, Theorem 1] $\omega(A) \leq 1$ if and only if A admits the factorization $A = (I-Z)^{1/2}C(I+Z)^{1/2}$ for some contraction C and Hermitian contraction Z.*

Lemma 2.14. *Let $\|S_F\|_\omega \leq 1$. Then F admits the factorization $F = n(I-Z)^{1/2}C(I+Z)^{1/2}$ for some contraction C and Hermitian contraction Z.*

Proof. Since $\|S_F\|_\omega \leq 1$, by definition, we have $\omega(F \circ X) \leq \omega(X)$. In particular $\omega(F) \leq \omega(J) = n$, where $J = [1] \in \mathbb{M}_n$, this means $\omega(F/n) \leq 1$. Then by Lemma 2.13, the matrix F admits the factorization $F = n(I-Z)^{1/2}C(I+Z)^{1/2}$ for some contraction C and Hermitian contraction Z. □

Proposition 2.15. *Let A and B be positive definite matrices such that $\|S_A\|_\omega \leqslant 1$ and $\|S_B\|_\omega \leqslant 1$. Then*

$$A\sharp B = n(I-Z)^{1/2}C(I+Z)^{1/2},$$

for some contraction C and Hermitian contraction Z.

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