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G-dual function-valued frames in $L_2(0,\infty)$

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Abstract

In this paper, g-dual function-valued frames in $L_2(0,\infty)$ are introduced. We can achieve more reconstruction formulas to obtain signals in $L_2(0,\infty)$ by applying g-dual function-valued frames in $L_2(0,\infty)$.

1. Introduction

Given a separable Hilbert space $\mathcal{H}$ with inner product $\langle .,. \rangle$, a sequence $\{f_k\}_{k=1}^{\infty}$ is called a frame for $\mathcal{H}$ if there exist constants $A>0$, $B<\infty$ such that for all $f\in\mathcal{H}$,

$$A\|f\|^2 \le \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \le B\|f\|^2, \tag{1.1}$$

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where A and B are the lower and upper frame bounds, respectively. The second inequality of the frame condition (1.1) is also known as the Bessel condition for $\{f_k\}_{k=1}^{\infty}$. For more information concerning frames refer to [1, 2, 4, 11].
We consider three classes of operators on $L_2(\mathbb{R})$. Their definitions are as follows:
Translation by $a \in \mathbb{R}$, $T_a : L_2(\mathbb{R}) \to L_2(\mathbb{R})$, $(T_a g)(x) = g(x - a)$,
Modulation by $b \in \mathbb{R}$, $E_b : L_2(\mathbb{R}) \to L_2(\mathbb{R})$, $(E_b g)(x) = e^{2\pi ibx} g(x)$,
Dilation by $c \neq 0$, $D_c : L_2(\mathbb{R}) \to L_2(\mathbb{R})$, $(D_c g)(x) = \frac{1}{\sqrt{|c|}} g(\frac{x}{c})$.

A Gabor frame is a frame for $L_2(\mathbb{R})$ of the form $\{E_{mb} T_{na} g\}_{m,n\in\mathbb{Z}}$ where $a, b > 0$, g is a fixed function in $L_2(\mathbb{R})$. This frame is special case of shift-invariant systems (up to an irrelevant complex factor). Casazza and Lammers define a function-valued inner product of two function $f, g \in L_2(\mathbb{R})$ as

$$\langle f, g\rangle_a (x) = \sum_{n\in\mathbb{Z}} f(x - na)\overline{g(x - na)}, \quad \forall x \in \mathbb{R},$$

where a is a fixed positive real number [3]. They called it as a-inner product and used it in the study of Gabor frames.

The dilation-invariant system generated by the sequence $\{g_k\}_{k\in\mathbb{Z}}$ in $L_2(\mathbb{R})$ and $a > 1$ is the sequence $\{D_{a^j} g_k\}_{j,k\in\mathbb{Z}}$, where D_{a^j} is the dilation operator by a^j. Dilation-invariant systems contain wavelet frames and hence they will play an important role in the analysis of wavelet frames.
A function-valued inner product on $L_2(0, \infty)$ by using of the dilation operator has been introduced in [8]. The authors use of a function-valued inner product in the study of the dilation-invariant systems:
Fix $a > 1$. For each pair $f, g \in L_2(0, \infty)$, the function $\langle f, g\rangle_a$ on $(0, \infty)$ is defined by

$$\langle f, g\rangle_a (x) := \sum_{j\in\mathbb{Z}} a^j f(a^j x)\overline{g(a^j x)}$$

and is called function-valued inner product on $L_2(0, \infty)$ with respect to a. It is easy to show that $\langle f, g\rangle = \int_1^a \langle f, g\rangle_a (x)dx$, where $\langle ., .\rangle$ is the original inner product in $L_2(0, \infty)$. Also, the function-valued norm on $L_2(0, \infty)$ with respect to a is defined by

$$\|f\|_a(x) := \sqrt{\langle f, f\rangle_a (x)}, \quad \forall f \in L_2(0, \infty) \quad and \quad \forall x \in (0, \infty).$$

The function ϕ on $(0, \infty)$ is called dilation periodic function with period a if $\phi(ax) = \phi(x)$ for all $x \in (0, \infty)$. The set of bounded dilation periodic functions on $(0, \infty)$ is denoted by B_a.

Example 1.1. Let f be a bounded function on $(0, \infty)$ and let $(G, +)$ be a finite group (for example $G = \mathbb{Z}_n, n \in \mathbb{N}, n \geq 2$). Then the function ϕ defined by $\phi(x) = \sum_{j\in G} f(a^j x)$, for all $x \in (0, \infty)$ is in B_a.

For any function ϕ on $[1, a]$, the function $\widetilde{\phi}$ defined by $\widetilde{\phi}(a^j x) = \phi(x)$, for all $j \in \mathbb{Z}$ and $x \in [1, a]$ is dilation periodic. Throughout this paper, let $\widetilde{\phi}$ be the dilation periodic function defined as above for any complex function ϕ on $[1, a]$.

Proposition 1.2. *[8] Let $f, g \in L_2(0,\infty)$ and $\phi \in B_a$. Then*

$$\langle \phi f, g\rangle_a = \phi \langle f, g\rangle_a \quad \textit{and} \quad \langle f, \phi g\rangle_a = \bar{\phi} \langle f, g\rangle_a$$

For any $f, g \in L_2(0,\infty)$, f and g are function-valued orthogonal with respect to a, or simply function-valued orthogonal if $\langle f, g\rangle_a = 0$ a.e. on $[1, a]$.
A sequence $\{e_n\}_{n\in\mathbb{Z}}$ in $L_2(0,\infty)$ is called function-valued orthogonal with respect to a if $e_n \perp_a e_m$, for all $n \neq m \in \mathbb{Z}$. If also $\|e_n\|_a = 1$ a.e. on $[1, a]$, then $\{e_n\}_{n\in\mathbb{Z}}$ is called a function-valued orthonormal sequence with respect to a, or simply function-valued orthonormal sequence, in $L_2(0,\infty)$.
A sequence $\{e_n\}_{n\in\mathbb{Z}}$ is called function-valued orthonormal basis with respect to a, or simply function-valued orthonormal basis, for $L_2(0,\infty)$ if it is a function-valued orthonormal sequence and $\overline{span}\{\widetilde{\psi_m e_n}\}_{m,n\in\mathbb{Z}} = L_2(0,\infty)$, where ψ_m is defined by $\psi_m(x) = \frac{1}{\sqrt{a-1}} e^{2\pi i \frac{m}{a-1}(a-x)}$ for all $m \in \mathbb{Z}$ and $x \in [1, a]$.

Proposition 1.3. *[8] If $\{e_n\}_{n\in\mathbb{Z}}$ is a function-valued orthonormal basis in $L_2(0,\infty)$, then $\{\widetilde{\psi_m e_n}\}_{m,n\in\mathbb{Z}}$ is an orthonormal basis in $L_2(0,\infty)$ and $f = \sum_{n\in\mathbb{Z}} \widetilde{\langle f, e_n\rangle_a} e_n$ on $(0,\infty)$.*

Let E be a measurable subset of $(0,\infty)$ and $1 \leq p \leq \infty$. A linear operator $L : L_2(0,\infty) \to L_p(E)$, is called a function-valued factorable operator with respect to a, or simply function-valued factorable operator if $L(\phi f) = \phi L(f)$ for all $f \in L_2(0,\infty)$ and $\phi \in B_a$.

Proposition 1.4. *[8] If $L : L_2(0,\infty) \to L_2(0,\infty)$ is a bounded function-valued factorable operator, then for all $f, g \in L_2(0,\infty)$ we have*

$$\langle L(f), g\rangle_a (x) = \langle f, L^*(g)\rangle_a (x), \quad \textit{for all } x \in (0,\infty),$$

where L^* is the adjoint operator of L.

A sequence $\{f_n\}_{n\in\mathbb{Z}}$ in $L_2(0,\infty)$ is called a function-valued frame with respect to a for $L_2(0,\infty)$, or simply function-valued frame for $L_2(0,\infty)$ if there exist constants $A > 0, B < \infty$ such that

$$A\|f\|_a^2(x) \leq \sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle_a (x)|^2 \leq B\|f\|_a^2(x),$$

for a.e. $x \in [1, a]$ and for all $f \in L_2(0,\infty)$.
Let $\{f_n\}_{n\in\mathbb{Z}}$ be a function-valued frame for $L_2(0,\infty)$. A function-valued frame $\{g_n\}_{n\in\mathbb{Z}}$ is called a **dual function-valued frame** of $\{f_n\}_{n\in\mathbb{Z}}$ with respect to a, or simply dual function-valued frame of $\{f_n\}_{n\in\mathbb{Z}}$ for $L_2(0,\infty)$ if for all $f \in L_2(0,\infty)$

$$f = \sum_{n\in\mathbb{Z}} \widetilde{\langle f, g_n\rangle_a} f_n. \tag{1.2}$$

Theorem 1.5. *[8] Let $\{f_n\}_{n\in\mathbb{Z}}$ be a sequence in $L_2(0,\infty)$. The following statements are equivalent:*
1) $\{f_n\}_{n\in\mathbb{Z}}$ is a function-valued frame for $L_2(0,\infty)$.
2) $\{\widetilde{\psi_m f_n}\}_{m,n\in\mathbb{Z}}$ is a frame for $L_2(0,\infty)$.

Two frames $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are dual frames for $\mathcal{H}$ if

$$f = \sum_{k=1}^{\infty} < f, g_k > f_k, \ \forall f \in \mathcal{H}.$$

Dual frames have an important role for the reconstruction of signals. From this point of view, dual frames have been generalized. Pseudo duals [10], Oblique dual frames [5, 9] and approximately dual frames [6] are some generalizations of dual frames.
G-duals of a frame in a separable Hilbert space $\mathcal{H}$ are introduced in [7].
Let $\{f_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$. A frame $\{g_k\}_{k=1}^{\infty}$ is called a generalized dual frame or g-dual frame of $\{f_k\}_{k=1}^{\infty}$ for $\mathcal{H}$ if there exists an invertible operator $A \in B(\mathcal{H})$ such that for all $f \in \mathcal{H}$,

$$f = \sum_{k=1}^{\infty} < Af, g_k > f_k.$$

In this paper, g-dual function-valued frames in $L_2(0,\infty)$ are introduced. Also an application of g-dual function-valued frames in $L_2(0,\infty)$ for characterizing g-dual frame of a dilation-invariant system in $L_2(0,\infty)$ is given.

2. G-dual function-valued frames in $L_2(0,\infty)$

Definition 2.1. Let $\{f_n\}_{n\in\mathbb{Z}}$ be a function-valued frame for $L_2(0,\infty)$. A function-valued frame $\{g_n\}_{n\in\mathbb{Z}}$ is called a **g-dual function-valued frame** of $\{f_n\}_{n\in\mathbb{Z}}$ with respect to a, or simply g-dual function-valued frame of $\{f_n\}_{n\in\mathbb{Z}}$ for $L_2(0,\infty)$ if there exists a bounded invertible function-valued factorable operator L on $L_2(0,\infty)$ such that for all $f \in L_2(0,\infty)$

$$f = \sum_{n\in\mathbb{Z}} \widetilde{\langle Lf, g_n\rangle}_a f_n. \tag{2.1}$$

The function-valued factorable operator L in (2.1) is unique. Indeed, if $L_\natural$ and $L_\flat$ are two bounded invertible function-valued factorable operators which satisfy in (2.1), then for all $f \in L_2(0,\infty)$

$$L_\natural^{-1} f = \sum_{n\in\mathbb{Z}} \widetilde{\langle f, g_n\rangle}_a f_n = L_\flat^{-1} f.$$

Also, we say the function-valued frame $\{g_n\}_{n\in\mathbb{Z}}$ is a g-dual function-valued frame of $\{f_n\}_{n\in\mathbb{Z}}$ with the corresponding bounded invertible function-valued factorable operator L (or with bounded invertible function-valued factorable operator L).

Proposition 2.2. *Let $\{f_n\}_{n\in\mathbb{Z}}$ and $\{g_n\}_{n\in\mathbb{Z}}$ be function-valued frames for $L_2(0,\infty)$. Then $\{g_n\}_{n\in\mathbb{Z}}$ is a g-dual function-valued frame of $\{f_n\}_{n\in\mathbb{Z}}$ for $L_2(0,\infty)$ with bounded invertible function-valued factorable operator L if and only if $\{f_n\}_{n\in\mathbb{Z}}$ is a g-dual function-valued frame of $\{g_n\}_{n\in\mathbb{Z}}$ for $L_2(0,\infty)$ with bounded invertible function-valued factorable operator L^*.*

Proof. We first assume that $\{g_n\}_{n\in\mathbb{Z}}$ is a g-dual function-valued frame of $\{f_n\}_{n\in\mathbb{Z}}$ for $L_2(0,\infty)$ with bounded invertible function-valued factorable operator L and hence $f = \sum_{n\in\mathbb{Z}} \widetilde{\langle Lf, g_n\rangle_a} f_n$ for all $f \in L_2(0,\infty)$. For all $h, k \in L_2(0,\infty)$, $\widetilde{\langle h,k\rangle_a} = \langle h,k\rangle_a$, on $[1,a]$. Thus by Proposition 1.2 we have

$$\begin{aligned}
\left\langle f, (L^{-1})^* g\right\rangle_a &= \left\langle L^{-1} f, g\right\rangle_a \\
&= \left\langle \sum_{n\in\mathbb{Z}} \widetilde{\langle f, g_n\rangle_a} f_n, g\right\rangle_a \\
&= \sum_{n\in\mathbb{Z}} \widetilde{\langle f, g_n\rangle_a} \langle f_n, g\rangle_a \\
&= \sum_{n\in\mathbb{Z}} \langle f, g_n\rangle_a \widetilde{\langle f_n, g\rangle_a} \\
&= \overline{\sum_{n\in\mathbb{Z}} \widetilde{\langle g, f_n\rangle_a} \langle g_n, f\rangle_a} \\
&= \overline{\left\langle \sum_{n\in\mathbb{Z}} \widetilde{\langle g, f_n\rangle_a} g_n, f\right\rangle_a} \\
&= \left\langle f, \sum_{n\in\mathbb{Z}} \widetilde{\langle g, f_n\rangle_a} g_n\right\rangle_a
\end{aligned}$$

on $[1,a]$ for all $g \in L_2(0,\infty)$. Therefore

$$\left\langle f, (L^{-1})^* g - \sum_{n\in\mathbb{Z}} \widetilde{\langle g, f_n\rangle_a} g_n\right\rangle = \int_1^a \left\langle f, (L^{-1})^* g - \sum_{n\in\mathbb{Z}} \widetilde{\langle g, f_n\rangle_a} g_n\right\rangle_a (x) dx = 0$$

Thus $g = \sum_{n\in\mathbb{Z}} \widetilde{\langle L^* g, f_n\rangle_a} g_n$. The converse is obtained by $(L^*)^* = L$. □

Example 2.3. Assume that $\{g_n\}_{n\in\mathbb{Z}}$ is a dual function-valued frame of $\{f_n\}_{n\in\mathbb{Z}}$ for $L_2(0,\infty)$(for example $\{S^{-1} f_n\}_{n\in\mathbb{Z}}$ is a dual function-valued frame of $\{f_n\}_{n\in\mathbb{Z}}$ for $L_2(0,\infty)$ by Theorem 2.4) and assume that ϕ is a non zero constant function on $(0,\infty)$. Then $\{\phi g_n\}_{n\in\mathbb{Z}}$ is a g-dual function-valued frame of $\{f_n\}_{n\in\mathbb{Z}}$ for $L_2(0,\infty)$ with bounded invertible function-valued factorable operator L defined on $L_2(0,\infty)$ by $Lf = \frac{1}{\phi} f$, for all $f \in L_2(0,\infty)$.

Every function-valued frame is a g-dual function-valued frame of itself.

Theorem 2.4. *Let $\{f_n\}_{n\in\mathbb{Z}}$ be a function-valued frame for $L_2(0,\infty)$ and*

$$Sf = \sum_{n\in\mathbb{Z}} \widetilde{\langle f, f_n\rangle_a} f_n, \quad \forall\, f \in L_2(0,\infty). \tag{2.2}$$

1) S is a well define bounded invertible function-valued factorable operator on $L_2(0,\infty)$.
2) $\{f_n\}_{n\in\mathbb{Z}}$ is a g-dual function-valued frame of itself with bounded invertible function-valued factorable operator S^{-1}.

Proof. Let $\{f_n\}_{n\in\mathbb{Z}}$ be a function-valued frame for $L_2(0,\infty)$ with bounds A and B. Then for all $f \in L_2(0,\infty)$,

$$A\|f\|_a^2(x) \leq \sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle_a (x)|^2 \leq B\|f\|_a^2(x), \quad for\ a.e.\ x \in [1,a]. \tag{2.3}$$

Thus

$$\begin{aligned}
\|\sum_{|n|=N}^{M} \widetilde{\langle f, f_n\rangle_a} f_n\|_{L_2(0,\infty)}^2 &= \int_1^a \|\sum_{|n|=N}^{M} \widetilde{\langle f, f_n\rangle_a} f_n\|_a^2(x)dx \\
&= \int_1^a \sup_{\substack{\|g\|_a=1 \\ on[1,a]}} |\left\langle \sum_{|n|=N}^{M} \widetilde{\langle f, f_n\rangle_a} f_n, g\right\rangle_a (x)|^2 dx \\
&= \int_1^a \sup_{\substack{\|g\|_a=1 \\ on[1,a]}} |\sum_{|n|=N}^{M} \widetilde{\langle f, f_n\rangle_a}(x) \langle f_n, g\rangle_a (x)|^2 dx \\
&\leq \int_1^a \sup_{\substack{\|g\|_a=1 \\ on[1,a]}} \sum_{|n|=N}^{M} |\widetilde{\langle f, f_n\rangle_a}(x)|^2 \sum_{|n|=N}^{M} |\langle f_n, g\rangle_a (x)|^2 dx \\
&\leq B \int_1^a \sum_{|n|=N}^{M} |\widetilde{\langle f, f_n\rangle_a}(x)|^2 dx \to 0,
\end{aligned}$$

as $M, N \to \infty$, since, the second inequality in (2.3) and Monotone Convergence Theorem imply that $\sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle_a|^2$ converges in $L_1[1,a]$. Thus S is well define and $\|S\| < B$. It is easy to show that S is linear. If $\phi \in B_a$, then for all $f \in L_2(0,\infty)$

$$\begin{aligned}
S(\phi f) &= \sum_{n\in\mathbb{Z}} \widetilde{\langle \phi f, f_n\rangle_a} f_n \\
&= \phi \sum_{n\in\mathbb{Z}} \widetilde{\langle f, f_n\rangle_a} f_n \\
&= \phi S(f)
\end{aligned}$$

by Proposition 1.2. Also inequality (2.3) shows that

$$A\|f\|_a^2(x) \leq \langle Sf, f\rangle_a (x) \leq B\|f\|_a^2(x), \quad for\ a.e.\ x \in [1,a].$$

By integration of above inequality on $[1,a]$ we have $A\|f\|^2 \leq \langle Sf, f\rangle \leq B\|f\|^2$. Thus $\|I - B^{-1}S\| < 1$ and so S is invertible.
Also by replacing f with $S^{-1}f$ in (2.2),

$$f = \sum_{n\in\mathbb{Z}} \widetilde{\langle S^{-1}f, f_n\rangle_a} f_n, \quad \forall\ f \in L_2(0,\infty).$$

Thus $\{f_n\}_{n\in\mathbb{Z}}$ is a g-dual function-valued frame of itself with invertible function-valued factorable operator S^{-1} □

The function-valued factorable operator S defined by (2.2) is called **function-valued frame operator** of $\{f_n\}_{n\in\mathbb{Z}}$. Now we are going to give a simple way for construction of infinitly many g-dual function-valued frames of a given function-valued frame (with common bounded invertible function-valued factorable operator).

Proposition 2.5. *Assume that $\{g_n\}_{n\in\mathbb{Z}}$ is a g-dual function-valued frame of $\{f_n\}_{n\in\mathbb{Z}}$ for $L_2(0,\infty)$ with bounded invertible function-valued factorable operator L and $\phi \in B_a$. Then the sequence $\{h_n\}_{n\in\mathbb{Z}}$ defined by $h_n = \phi g_n + (1-\phi)(L^{-1})^* S^{-1} f_n$, is a g-dual function-valued frame of $\{f_n\}_{n\in\mathbb{Z}}$ for $L_2(0,\infty)$ with bounded invertible function-valued factorable operator L, where S is the function-valued frame operator of $\{f_n\}_{n\in\mathbb{Z}}$.*

Proof. For all $f \in L_2(0,\infty)$ we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \widetilde{\langle Lf, h_n\rangle}_a f_n &= \bar{\phi} \sum_{n\in\mathbb{Z}} \widetilde{\langle Lf, g_n\rangle}_a f_n + (1-\bar{\phi}) \sum_{n\in\mathbb{Z}} \widetilde{\langle f, S^{-1} f_n\rangle}_a f_n \\ &= \bar{\phi} f + (1-\bar{\phi}) f = f\end{aligned}$$

□

Example 2.6. If $\{g_n\}_{n\in\mathbb{Z}}$ is a dual function-valued frame of $\{f_n\}_{n\in\mathbb{Z}}$, then $\frac{1}{2}g_n + \frac{1}{2}S^{-1}f_n$, is a g-dual function-valued frame of $\{f_n\}_{n\in\mathbb{Z}}$ for $L_2(0,\infty)$ with bounded invertible function-valued factorable operator I, where S is the function-valued frame operator of $\{f_n\}_{n\in\mathbb{Z}}$ and I is the identity operator on $L_2(0,\infty)$.

Definition 2.7. A sequence $\{g_n\}_{n\in\mathbb{Z}}$ is called a function-valued Riesz basis with respect to a, or simply function-valued Riesz basis for $L_2(0,\infty)$ if there exist function-valued orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$ and bounded invertible function-valued factorable operator L on $L_2(0,\infty)$ such that $g_n = Le_n$, for all $n \in \mathbb{Z}$.

Not only function-valued orthonormal bases, but also function-valued Riesz bases are g-dual function-valued frames.

Proposition 2.8. *Every two function-valued Riesz bases are g-dual function-valued frames.*

Proof. Let $\{g_n\}_{n\in\mathbb{Z}}$ and $\{h_n\}_{n\in\mathbb{Z}}$ be two function-valued Riesz bases for $L_2(0,\infty)$. There exist function-valued orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$ and bounded invertible function-valued factorable operators $L_\natural$ and $L_\flat$ on $L_2(0,\infty)$ such that $g_n = L_\natural e_k$ and $h_n = L_\flat e_k$. Since $L_\natural$ and $L_\flat$ are invertible, there exists a bounded invertible function-valued factorable operator L on $L_2(0,\infty)$ such that $L_\flat L_\natural^* L = I$ and hence for all $f \in L_2(0,\infty)$ we have

$$\begin{aligned} f = L_\flat L_\natural^* L f = L_\flat\Big(\sum_{n\in\mathbb{Z}} \widetilde{\left\langle L_\natural^* L f, e_n\right\rangle}_a e_n\Big) &= \sum_{n\in\mathbb{Z}} \widetilde{\left\langle Lf, L_\natural e_n\right\rangle}_a L_\flat e_n \\ &= \sum_{n\in\mathbb{Z}} \widetilde{\langle Lf, g_n\rangle}_a h_n. \end{aligned}$$

□

The relation between g-dual frames and g-dual function-valued frames for $L_2(0,\infty)$ is given in the next theorem.

Theorem 2.9. *Let $\{f_n\}_{n\in\mathbb{Z}}$ and $\{g_n\}_{n\in\mathbb{Z}}$ be function-valued frames in $L_2(0,\infty)$. The following statements are equivalent:*
1)$\{g_n\}_{n\in\mathbb{Z}}$ is a g-dual function-valued frame of $\{f_n\}_{n\in\mathbb{Z}}$ with bounded invertible function-valued factorable operator L.
2)$\{\widetilde{\psi_m g_n}\}_{m,n\in\mathbb{Z}}$ is a g-dual frame of $\{\widetilde{\psi_m f_n}\}_{m,n\in\mathbb{Z}}$ with bounded invertible operator L.

Proof. The sequences $\{\widetilde{\psi_m g_n}\}_{m,n\in\mathbb{Z}}$ and $\{\widetilde{\psi_m f_n}\}_{m,n\in\mathbb{Z}}$ are frames in $L_2(0,\infty)$ by Theorem 1.5.
Let $\{e_n\}_{n\in\mathbb{Z}}$ be a function-valued orthonormal basis for $L_2(0,\infty)$. A similar argument as the proof of Theorem 2.4 shows that the operators $T_1 : L_2(0,\infty) \to L_2(0,\infty)$ defined by $T_1 f = \sum_{n\in\mathbb{Z}} \widetilde{\langle f, e_n\rangle}_a g_n$ and $T_2 : L_2(0,\infty) \to L_2(0,\infty)$ defined by $T_2 f = \sum_{n\in\mathbb{Z}} \widetilde{\langle f, e_n\rangle}_a f_n$ are well define bounded function-valued factorable operators. Also $T_1 e_n = g_n$, $T_2 e_n = f_n$, $T_1(\widetilde{\psi_m e_n}) = \widetilde{\psi_m g_n}$ and $T_2(\widetilde{\psi_m e_n}) = \widetilde{\psi_m f_n}$.
Now let $\{g_n\}_{n\in\mathbb{Z}}$ is a g-dual function-valued frame of $\{f_n\}_{n\in\mathbb{Z}}$ with bounded invertible function-valued factorable operator L. Then for all $f \in L_2(0,\infty)$

$$\begin{aligned} f = \sum_{n\in\mathbb{Z}} \widetilde{\langle Lf, g_n\rangle}_a f_n &= \sum_{n\in\mathbb{Z}} \widetilde{\langle Lf, T_1 e_n\rangle}_a T_2 e_n \\ &= T_2(\sum_{n\in\mathbb{Z}} \widetilde{\left\langle T_1^* Lf, e_n\right\rangle}_a e_n) \\ &= T_2 T_1^* Lf, \end{aligned}$$

by Proposition 1.3. Now $\{\widetilde{\psi_m e_n}\}_{m,n\in\mathbb{Z}}$ is an orthonormal basis in $L_2(0,\infty)$ by Proposition 1.3 and hence for all $f \in L_2(0,\infty)$

$$\begin{aligned} \sum_{n\in\mathbb{Z}} \left\langle Lf, \widetilde{\psi_m g_n}\right\rangle \widetilde{\psi_m f_n} &= \sum_{n\in\mathbb{Z}} \left\langle Lf, T_1(\widetilde{\psi_m e_n})\right\rangle T_2(\widetilde{\psi_m e_n}) \\ &= T_2(\sum_{n\in\mathbb{Z}} \left\langle T_1^* Lf, \widetilde{\psi_m e_n}\right\rangle \widetilde{\psi_m e_n}) \\ &= T_2 T_1^* Lf = f. \end{aligned}$$

Therefore $\{\widetilde{\psi_m g_n}\}_{m,n\in\mathbb{Z}}$ is a g-dual frame of $\{\widetilde{\psi_m f_n}\}_{m,n\in\mathbb{Z}}$ with bounded invertible operator L.
Conversely let $\{\widetilde{\psi_m g_n}\}_{m,n\in\mathbb{Z}}$ be a g-dual frame of $\{\widetilde{\psi_m f_n}\}_{m,n\in\mathbb{Z}}$ with bounded invertible operator L. Then $L = (T_2 T_1^*)^{-1}$ is bounded invertible function-valued factorable operator, since T_1 and T_2 are function-valued factorable operator. Also for all $f \in L_2(0,\infty)$

$$\begin{aligned} \sum_{n\in\mathbb{Z}} \widetilde{\langle Lf, g_n\rangle}_a f_n &= \sum_{n\in\mathbb{Z}} \widetilde{\langle Lf, T_1 e_n\rangle}_a T_2 e_n \\ &= T_2(\sum_{n\in\mathbb{Z}} \widetilde{\left\langle T_1^* Lf, e_n\right\rangle}_a e_n) \\ &= T_2 T_1^* Lf = f, \end{aligned}$$

by Proposition 1.3. □

Example 2.10. Let $\{e_n\}_{n\in\mathbb{Z}}$ be an orthonormal basis for $L_2(0,\infty)$ and $\lambda \neq 0$. Then $\{\lambda e_n\}_{n\in\mathbb{Z}}$ is a g-dual frame of $\{e_n\}_{n\in\mathbb{Z}}$ with bounded invertible operator L defined on $L_2(0,\infty)$ by $Lf = \frac{1}{\lambda}$, for all $f \in L_2(0,\infty)$. Therefore $\{\lambda\widetilde{\psi_m}e_n\}_{m,n\in\mathbb{Z}}$ is a g-dual function-valued frame of $\{\widetilde{\psi_m}e_n\}_{m,n\in\mathbb{Z}}$ with bounded invertible function-valued factorable operator L.

Let $\phi \in L_2(0,\infty)$. Then for all $x \in (0,\infty)$ we have

$$\begin{aligned}\widetilde{\psi_k}D_{a^j}\phi(x) &= \widetilde{\psi_k}(x)D_{a^j}\phi(x)\\ &= \frac{1}{\sqrt{a^j}}\widetilde{\psi_k}(x)\phi(a^{-j}x)\\ &= \frac{1}{\sqrt{a^j}}\widetilde{\psi_k}(a^{-j}x)\phi(a^{-j}x)\\ &= D_{a^j}\widetilde{\psi_k}\phi(x)\end{aligned}$$

and hence $\widetilde{\psi_k}$ commute with D_{a^j}. Thus the following corollary is immediate from Theorem 2.9.

Corollary 2.11. *Let $\{D_{a^j}\phi_1\}_{j\in\mathbb{Z}}$ and $\{D_{a^j}\phi_2\}_{j\in\mathbb{Z}}$ be function-valued frames for $L_2(0,\infty)$, where $\phi_1,\phi_2 \in L_2(0,\infty)$. The following are equivalent.*
1)$\{D_{a^j}\phi_1\}_{j\in\mathbb{Z}}$ is a g-dual function-valued frame of $\{D_{a^j}\phi_2\}_{j\in\mathbb{Z}}$ with bounded invertible function-valued factorable operator L.
2)The dilation invariant system generated by $\{\widetilde{\psi_k}\phi_1\}_{k\in\mathbb{Z}}$ and a is a g-dual frame of the dilation invariant system generated by $\{\widetilde{\psi_k}\phi_2\}_{k\in\mathbb{Z}}$ and a with bounded invertible operator L.

References

[1] P.G. Casazza, The Art of Frame Theory, *Taiwanese J. Math,* **4** (2000), 129-201.
[2] P. G. Casazza, and G. Kutyniok, Frames of Subspaces, *Contemporary Math,* **345** (2004), 87-114.
[3] P. G. Casazza and M. C. Lammers, Bracket Products for Weyl-Heisenberg Frames, In: Advances in Gabor Analysis (eds) Feichtinger H G and Strohmer T (2003) (Boston-MA. Birkhäuser).
[4] O. Christensen, An Introduction to Frames and Riesz Bases, *Birkhäuser, Boston, Basel, Berlin*, 2002.
[5] O. Christensen and Y. Eldar, Oblique dual frames and shift-invariant spaces, *Appl. Comp. Harm. Anal.*, **17** (2004) 48 - 68.
[6] O. Christensen and R. S. Laugesen, Approximately dual frames in Hilbert spaces and application to Gabor frames, Sampl. *Theory Signal Image Process*. **9** (2011), 77-90.
[7] M. A. Dehghan and M. A. Hasankhani Fard, G-dual frames in Hilbert spaces, *U.P.B. Sci. Bull., Series A*, **75**(1) (2013), 129-140.
[8] M. A. Hasankhani Fard and M. A. Dehghan, A new function-valued inner product and corresponding function-valued frames in $L_2(0,\infty)$, *Linear Multilinear Algebra*, (Published online: 01 Jul 2013).
[9] A. A. Hemmat and J. P. Gabardo, Properties of oblique dual frames in shift-invariant systems, *J. Math. Anal. Appl.*, **356** (2009) 346-354.
[10] S. Li and H. Ogawa, Pseudo duals of frames with applications, *Appl. Comput. Harmon. Anal.,* **11** (2001) 289-304.
[11] R. Young, An Introduction to Nonharmonic Fourier Series, *Academic Press, New York*, 1980.